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ON SOME GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS ON THE BASIS OF THEIR GENERALIZED RELATIVE ORDER (α, β)

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Abstract: In this paper we wish to investigate some interesting results associated with the comparative growth properties of composite entire functions using generalized relative order (α, β) and generalized relative lower order (α, β), where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

Keywords and Phrases: Entire function, growth, composition, generalized relative order (α, β), generalized relative lower order (α, β).

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1. Introduction, Definitions and Notations

Let $\mathbb{C}$ be the set of all finite complex numbers and f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ and the maximum term $\mu_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ are defined as $M_f = \max_{|z|=r} |f(z)|$ and $\mu_f(r) = \max_{n \geq 0} (|a_n| r^n)$ respectively. Since $M_f(r)$ is strictly increasing and continuous, therefore there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s \to \infty} M_f^{-1}(s) = \infty$. We use the standard notations and definitions of the theory of entire functions which are available in [10] and [11], and therefore we do not explain those in details.

Let L is a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$. Moreover we assume that throughout the present paper α, α_1, α_2, β, β_1 and β_2 always denote the functions belonging to L^0 unless otherwise specifically stated. The value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} (\alpha \in L, \beta \in L)$$

is called [7] generalized order (α, β) of f. For details about generalized order (α, β) one may see [7]. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α, β) in some different direction. For the purpose of further applications, Biswas et al. [4, 5] rewrite the definition of the generalized order (α, β) of entire function in the following way after giving a minor modification to the original definition (e.g. see, [7]).

Definition 1. [4, 5] *The generalized order (α, β) and generalized lower order (α, β) of an entire function f, denoted by $\rho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[f]$ respectively, are defined as:*

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

Since for $0 \leq r < R$,

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R)\{cf.[9]\},$$

it is easy to see that

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\mu_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\mu_f(r))}{\beta(r)}.$$

Mainly the growth investigation of entire functions has usually been done through their maximum moduli function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. Now in order to make some progress in the study of relative order, Biswas et al. [3] have introduced the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of an entire function with respect to another entire function in the following way:

Definition 2. [3] *Let $\alpha,\ \beta \in L^0$. The generalized relative order (α, β) and generalized relative lower order (α, β) of an entire function f with respect to an entire function g denoted by $\varrho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:*

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}.$$

In terms of maximum terms of entire functions, Definition 2 can be reformulated as:

Definition 3. *Let $\alpha,\ \beta \in L^0$. The growth indicators $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to another entire function g are defined as:*

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\alpha(\mu_g^{-1}(\mu_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\alpha(\mu_g^{-1}(\mu_f(r)))}{\beta(r)}.$$

In fact, Lemma 5 states the equivalence of Definition 2 and Definition 3.

Here, in this paper, we investigate some interesting results associated with the comparative growth properties of composite entire functions using generalized relative order (α, β) and generalized relative lower order (α, β).

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [6] *Let f and g are any two entire functions with $g(0) = 0$. Also let b satisfy $0 < b < 1$ and $c(b) = \frac{(1-b)^2}{4b}$. Then for all sufficiently large values of r,*

$$M_f(c(b)M_g(br)) \leq M_{f\circ g}(r) \leq M_f(M_g(r)).$$

In addition if $b = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \geq M_f\Big(\frac{1}{8}M_g\Big(\frac{r}{2}\Big)\Big).$$

Lemma 2. [8] *Let f and g are entire functions. Then for every $\delta > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \leq \frac{\delta}{\delta - 1}\mu_f\Big(\frac{\delta R}{R - r}\mu_g(R)\Big).$$

Lemma 3. [8] *If f and g are any two entire functions. Then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\Big(\frac{1}{16}\mu_g\Big(\frac{r}{4}\Big)\Big).$$

Lemma 4. [2] *Suppose f is an entire function and $A > 1$, $0 < B < A$. Then for all sufficiently large r,*

$$M_f(Ar) \geq BM_f(r).$$

Lemma 5. *Definition 2 and Definition 3 are equivalent.*

Proof. Taking $R = ar$ for any $a > 1$ in the inequalities $\mu_g(r) \leq M_g(r) \leq \frac{R}{R-r}\mu_g(R)$ $\{cf.$ [9]$\}$, for $0 \leq r < R$ we obtain that

$$M_g^{-1}(r) \leq \mu_g^{-1}(r)$$

and

$$\mu_g^{-1}(r) \leq aM_g^{-1}\Big(\frac{ar}{(a-1)}\Big).$$

Since $M_g^{-1}(r)$ and $\mu_g^{-1}(r)$ are increasing functions of r, then for any $a > 1$ it follows from the above, Lemma 4 and the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{a}{a-1}\mu_f(ar)$ {cf. [9]} that

$$M_g^{-1}\Big(M_f\Big(\frac{(a-1)r}{(2a-1)a}\Big)\Big) \leq \mu_g^{-1}(\mu_f(r)) \tag{2.1}$$

and

$$\mu_g^{-1}\left(\mu_f\left(r\right)\right) \leq aM_g^{-1}\Big(M_f\Big(\frac{(2a-1)}{(a-1)}r\Big)\Big). \tag{2.2}$$

Therefore the lemma follows from (2.1) and (2.2).

3. Main Results

In this section we present the main results of the paper.

Theorem 1. *Let f, g and h are any three entire functions such that $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < +\infty$.*
(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant or $\lim\limits_{r\to+\infty}\frac{\exp(\alpha_2(r))}{\beta_1(r)} = +\infty$, then

$$\lim_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = 0.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = 0.$$

Proof. Since $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]_h$ we can choose $\varepsilon(>0)$ is such a way that

$$\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon < \lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon. \tag{3.1}$$

In view of Lemma 1, we get for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(M_g(\beta_2^{-1}(\log r))). \tag{3.2}$$

Now the following three cases may arise .

Case I. Let $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant. Then we have from (3.2) for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant B(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\exp(\alpha_2(M_g(\beta_2^{-1}(\log r))))$$

$$i.e.,\ \alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant B(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.3}$$

Case II. Let $\lim\limits_{r\to+\infty} \frac{\exp(\alpha_2(r))}{\beta_1(r)} = +\infty$. Then for all sufficiently large positive numbers of r we get from (3.2) that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))) < (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.4}$$

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$. Then for all sufficiently large positive numbers of r we get from (3.2) that

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))))) \leq (1+o(1))\alpha_2(M_g(\beta_2^{-1}(\log r)))$$

$$i.e.,\ \alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))))) \leqslant r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.5}$$

Also from the definition of $\lambda_{(\alpha_1,\beta_1)}[f]_h$, we get for all sufficiently large positive numbers of r that

$$\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r))))) \geqslant r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)}. \tag{3.6}$$

Now combining (3.3) of Case I and (3.6) we get for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} \leq \frac{B(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)}}. \tag{3.7}$$

Therefore in view of (3.1) it follows from (3.7) that

$$\lim_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = 0.$$

Similar conclusion can also be derived from (3.4) of Case II and (3.6).

Hence the first part of the theorem follows.

Further combining (3.5) of Case III and (3.6) we obtain for all sufficiently large positive numbers of r that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} \leq \frac{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)}}. \tag{3.8}$$

Therefore in view of (3.1) we get from above that

$$\lim_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = 0.$$

Hence the second part of the theorem follows from above.

Thus the theorem follows.

Theorem 2. *Let f, g and h are any three entire functions such that $\lambda_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < +\infty$.*
(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant or $\lim_{r\to+\infty} \frac{\exp(\alpha_2(r))}{\beta_1(r)} = +\infty$, then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = 0.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then

$$\liminf_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = 0.$$

The proof of Theorem 2 is omitted as it can be carried out in the line of Theorem 1.

Theorem 3. *Let f, g and h are any three entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \lambda_{(\alpha_2,\beta_2)}[g] < +\infty$.*
(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant or $\lim_{r\to+\infty} \frac{\exp(\alpha_2(r))}{\beta_1(r)} = 0$, then

$$\lim_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = +\infty.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = +\infty.$$

Proof. Let us choose $0 < \varepsilon < \lambda_{(\alpha_1,\beta_1)}[f]_h$. Now for all sufficiently large positive numbers of r we get from Lemma 1 that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))) \geqslant (1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1\Big(M_g\Big(\frac{\beta_2^{-1}(\log r)}{2}\Big)\Big). \tag{3.9}$$

Now the following three cases may arise.

Case I. Let $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant. Then from (3.9) we obtain for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))) \geqslant B(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}. \tag{3.10}$$

Case II. Let $\lim\limits_{r\to+\infty} \frac{\exp(\alpha_2(r))}{\beta_1(r)} = 0$. Then from (3.9) it follows for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))) > (1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}. \tag{3.11}$$

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$. Then from (3.9) it follows for all sufficiently large positive numbers of r that

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))))) \geq (1+o(1))\alpha_2\Big(M_g\Big(\frac{\beta_2^{-1}(\log r)}{2}\Big)\Big)$$

$$\text{i.e.,}\ \exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))))) \geq r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}. \tag{3.12}$$

Again from the definition of $\rho_{(\alpha_1,\beta_1)}[f]_h$ we get for all sufficiently large positive numbers of r that

$$\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r))))) \leq r^{(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)}. \tag{3.13}$$

Now combining (3.10) of Case I and (3.13) we get for all sufficiently large positive numbers of r that

$$\begin{aligned} &\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} \\ \geq\ & \frac{B(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}}{r^{(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)}}. \end{aligned}$$

Since $\rho_{(\alpha_1,\beta_1)}[f]_h < \lambda_{(\alpha_2,\beta_2)}[g]$, it follows from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = +\infty.$$

Similar conclusion can also be derived from (3.11) of Case II and (3.13).

Therefore the first part of the theorem follows.

Again combining (3.12) of Case III and (3.13) we obtain for all sufficiently large positive numbers of r that

$$\begin{aligned}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} &\geq \frac{r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}}{r^{(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)}}\\ i.e., \lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} &= +\infty,\end{aligned}$$

Therefore the second part of the theorem follows from above.

Hence the theorem follows.

Theorem 4. *Let f, g and h are any three entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \lambda_{(\alpha_2,\beta_2)}[g] < +\infty$.*
(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant or $\lim_{r\to+\infty}\frac{\exp(\alpha_2(r))}{\beta_1(r)} = 0$, then

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = +\infty.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then

$$\limsup_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))} = +\infty.$$

The proof of Theorem 4 is omitted as it can be carried out in the line of Theorem 3.

Theorem 5. *Let f, g and h are any three entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < +\infty$ and $0 < \lambda_{(\alpha_2,\beta_2)}[g] \leq \rho_{(\alpha_2,\beta_2)}[g] < +\infty$.*
(i) If $\beta_1(r) = \alpha_2(r)$, then

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h} &\leq \liminf_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))}\\ &\leqslant \min\left\{\rho_{(\alpha_2,\beta_2)}[g], \frac{\rho_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}\right\}\end{aligned}$$

and

$$\max\left\{\lambda_{(\alpha_2,\beta_2)}[g], \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h \cdot \rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h}\right\} \leq \limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]_h \cdot \rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

(ii) If $\beta_1(\alpha_2^{-1}(r)) \in L^0$, *then*

$$\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_1,\beta_1)}[f]_h} \leq \liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\alpha_2^{-1}(\beta_2(r)))))} \leqslant 1 \leqslant \limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\alpha_2^{-1}(\beta_2(r)))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

(iii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *then*

$$\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h} \leq \liminf_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \min\left\{\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}, \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h}\right\} \leq \max\left\{\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}, \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h}\right\} \leq \limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Proof. From the definitions of generalized relative order (α_1, β_1) and generalized relative lower order (α_1, β_1) of f with respect to h, we have for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_f(r))) \leq (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(r), \tag{3.14}$$

$$\alpha_1(M_h^{-1}(M_f(r))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(r) \tag{3.15}$$

and also for a sequence of positive numbers of r tending to infinity we get that

$$\alpha_1(M_h^{-1}(M_f(r))) \geq (\rho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(r). \tag{3.16}$$

Similarly for a sequence of positive numbers of r tending to infinity we obtain that

$$\alpha_1(M_h^{-1}(M_f(r))) \leq (\lambda_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(r). \tag{3.17}$$

Now in view of Lemma 1, we have for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \leqslant (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(M_g(r)) \tag{3.18}$$

and also we get for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \leqslant (\lambda_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(M_g(r)). \tag{3.19}$$

Similarly, in view of Lemma 1, it follows for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \geq (1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1\Big(M_g\Big(\frac{r}{2}\Big)\Big) \tag{3.20}$$

and also we obtain for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \geq (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1\Big(M_g\Big(\frac{r}{2}\Big)\Big). \tag{3.21}$$

Now the following two cases may arise:
Case I. Let $\beta_1(r) = \alpha_2(r)$.

Now we have from (3.18) for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \leqslant (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r), \tag{3.22}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \leqslant (\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r). \tag{3.23}$$

Also we obtain from (3.19) for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \leqslant (\lambda_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r). \tag{3.24}$$

Further it follows from (3.20) for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \geq (1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon)\beta_2(r), \tag{3.25}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \geq (1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\rho_{(\alpha_2,\beta_2)}[g] - \varepsilon)\beta_2(r). \tag{3.26}$$

Moreover, we obtain from (3.21) for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r))) \geq (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon)\beta_2(r). \qquad (3.27)$$

Therefore from (3.15) and (3.22), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r)}{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_2(r)}$$

$$i.e., \ \limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{\rho_{(\alpha_1,\beta_1)}[f]_h \cdot \rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}. \qquad (3.28)$$

Similarly from (3.16) and (3.22), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r)}{(\rho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_2(r)}$$

$$i.e., \ \liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \rho_{(\alpha_2,\beta_2)}[g]. \qquad (3.29)$$

In the same way also from (3.15) and (3.23), we obtain that

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{\rho_{(\alpha_1,\beta_1)}[f]_h \cdot \lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}. \qquad (3.30)$$

Similarly from (3.15) and (3.24), we get that

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \rho_{(\alpha_2,\beta_2)}[g]. \qquad (3.31)$$

Thus from (3.29), (3.30) and (3.31), it follows that

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \min\left\{\rho_{(\alpha_2,\beta_2)}[g], \frac{\rho_{(\alpha_1,\beta_1)}[f]_h \cdot \lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}\right\}. \qquad (3.32)$$

Further from (3.14) and (3.25), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r))(M_h^{-1}}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon)\beta_2(r)}{(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_2(r)}$$

$$i.e., \liminf_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))}\geq\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h}. \tag{3.33}$$

Similarly, from (3.17) and (3.25) we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))}\geq\lambda_{(\alpha_2,\beta_2)}[g]. \tag{3.34}$$

Likewise from (3.14) and (3.26), we get that

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))}\geq\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h}, \tag{3.35}$$

Similarly from (3.14) and (3.27), we have that

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))}\geq\lambda_{(\alpha_2,\beta_2)}[g]. \tag{3.36}$$

Thus from (3.34), (3.35) and (3.27) it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))}\geq\max\left\{\lambda_{(\alpha_2,\beta_2)}[g],\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h}\right\}. \tag{3.37}$$

Therefore the first part of the theorem follows from (3.28), (3.32), (3.33) and (3.37).
Case II. Let $\beta_1(\alpha_2^{-1}(r))\in L^0$.

Now we have from (3.18) for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r)))\leqslant(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}((\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r)))$$
$$i.e., \alpha_1(M_h^{-1}(M_{f\circ g}(r)))\leqslant(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r))) \tag{3.38}$$

Further from (3.20), it follows for all sufficiently large positive numbers of r that

$$\alpha_1(M_h^{-1}(M_{f\circ g}(r)))\geq(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r))). \tag{3.39}$$

Now from (3.15) and (3.38), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\alpha_2^{-1}(\beta_2(r)))))}\leqslant\frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e., \limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\alpha_2^{-1}(\beta_2(r)))))} \leqslant \frac{\rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_1,\beta_1)}[f]_h}. \tag{3.40}$$

Also from (3.16) and (3.38), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\alpha_2^{-1}(\beta_2(r)))))} \leqslant \frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\rho_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e., \liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\alpha_2^{-1}(\beta_2(r)))))} \leqslant 1. \tag{3.41}$$

Further from (3.14) and (3.39), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\alpha_2^{-1}(\beta_2(r)))))} \geq \frac{(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e., \liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\alpha_2^{-1}(\beta_2(r)))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_1,\beta_1)}[f]_h}. \tag{3.42}$$

Also from (3.17) and (3.39) it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\alpha_2^{-1}(\beta_2(r)))))} \geq \frac{(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\lambda_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e., \limsup_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_1(M_h^{-1}(M_f(\alpha_2^{-1}(\beta_2(r)))))} \geq 1. \tag{3.43}$$

Hence the second part of the theorem follows from (3.40), (3.41), (3.42) and (3.43).
Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$.
Then we have from (3.18) for all sufficiently large positive numbers of r that

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r))))) \leqslant (1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r), \tag{3.44}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r))))) \leqslant (1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r). \tag{3.45}$$

Further, it follows from (3.20) for all sufficiently large positive numbers of r that

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r))))) \geq (1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r), \tag{3.46}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r))))) \geq (1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r). \quad (3.47)$$

Now from (3.15) and (3.44), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r)}{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\beta_2(r)}$$

$$i.e., \ \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}. \quad (3.48)$$

Also from (3.16) and (3.44), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r)}{(\rho_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\beta_2(r)}$$

$$i.e., \ \liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_f}. \quad (3.49)$$

Similarly from (3.15) and (3.45), we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_f}. \quad (3.50)$$

Thus from (3.49) and (3.50) it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \leqslant \min\left\{\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_f}, \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_f}\right\}. \quad (3.51)$$

Further from (3.14) and (3.46), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r)}{(\rho_{(\alpha_1,\beta_1)}[f]_f+\varepsilon)\beta_2(r)}$$

$$i.e., \ \liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_f}. \quad (3.52)$$

Also from (3.17) and (3.46) it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r)}{(\lambda_{(\alpha_1,\beta_1)}[f]_f+\varepsilon)\beta_2(r)}$$

$$i.e., \ \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r))))}\geq\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_f}. \tag{3.53}$$

Similarly from (3.14) and (3.47), we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r))))}\geq\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_f}. \tag{3.54}$$

Thus from (3.53) and (3.54) it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\beta_2(r))))}\geq\max\left\{\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_f},\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_f}\right\}. \tag{3.55}$$

Thus the third part of the theorem follows from (3.48), (3.51), (3.52) and (3.55).

Theorem 6. *Let f, g and h are any three entire functions such that $0<\lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h<+\infty$ and $0<\lambda_{(\alpha_2,\beta_2)}[g]\leq\rho_{(\alpha_2,\beta_2)}[g]<+\infty$. Also let $0<\lambda_{(\alpha_2,\beta_2)}[g]_k\leq\rho_{(\alpha_2,\beta_2)}[g]_k<+\infty$ where k is another entire function.*
(i) If $\beta_1(r)=\alpha_2(r)$, then

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_2,\beta_2)}[g]_k} &\leq \liminf_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_2(M_k^{-1}(M_g(r)))}\\ &\leqslant \min\left\{\frac{\rho_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k},\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k}\right\}\end{aligned}$$

and

$$\begin{aligned}&\max\left\{\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_2,\beta_2)}[g]_k},\frac{\rho_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_2,\beta_2)}[g]_k}\right\}\\ \leq\ &\limsup_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_2(M_k^{-1}(M_g(r)))}\leq\frac{\rho_{(\alpha_1,\beta_1)}[f]_h\cdot\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.\end{aligned}$$

Theorem 7. *(ii) If $\beta_1(\alpha_2^{-1}(r))\in L^0$, then*

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_2,\beta_2)}[g]_k} &\leq \liminf_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))}\\ &\leqslant \min\left\{\frac{\rho_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_2,\beta_2)}[g]_k},\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}\right\}\\ &\leq \max\left\{\frac{\rho_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_2,\beta_2)}[g]_k},\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}\right\}\\ &\leq \limsup_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(r)))}{\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))}\leq\frac{\rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.\end{aligned}$$

(iii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$*, then*

$$\begin{aligned}\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_2,\beta_2)}[g]_k} &\leq \liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_2(M_k^{-1}(M_g(r)))} \\ &\leqslant \min\left\{\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_2,\beta_2)}[g]_k}, \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k}\right\} \\ &\leqslant \max\left\{\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_2,\beta_2)}[g]_k}, \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k}\right\} \\ &\leq \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(r)))))}{\alpha_2(M_k^{-1}(M_g(r)))} \leq \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.\end{aligned}$$

The proof of Theorem 6 is omitted as it can be carried out in the line of Theorem 5.

Remark 1. *The same results of above theorems in terms of maximum terms of entire functions can also be deduced with the help of Lemma 2 and Lemma 3.*

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