J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 9, No. 1 (2021), pp. 75-82

ISSN (Online): 2582-5461

ISSN (Print): 2319-1023

EXPLICIT EVALUATION OF RATIOS OF THETA FUNCTIONS

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(**Received:** *Sep. 05, 2021* **Accepted:** *Nov. 08, 2021* **Published:** *Dec. 30, 2021)*

Abstract: In the literature one can find evaluation of ratios of theta function $\frac{f(-q)}{q^{\frac{n-1}{24}}f(-q^n)}$ for $n = 2, 4, 5, 7, 9, 25$. The purpose of this article is to obtain evaluation of $\frac{f(-q)}{q^{\frac{6}{24}}f(-q^6)}$ for certain rational k with $q = e^{-2\pi\sqrt{k}}$.

Keywords and Phrases: Theta functions, Continued fraction.

2020 Mathematics Subject Classification: 11J70, 14K25.

1. Introduction

For any complex numbers a and q with $|q| < 1$, we define

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n).$$

Ramanujan general theta-function $f(a, b)$, [6, p. 197], is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab,ab)_\infty, \quad |ab| < 1. \tag{1.1}$$

He also defines [6, p. 197],

$$f(-q) = f(-q,-q^2) = \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}} = (q;q)_\infty. \tag{1.2}$$

For Im$\tau > 0$, let $q = e^{2\pi i\tau}$. Then

$$f(-q) = e^{\frac{-\pi i\tau}{12}}\eta(\tau), \tag{1.3}$$

where $\eta(\tau)$ denote the classical dedekind eta-function [4, p. 134].

The theta-function $f(-q)$ has got nice connection with Rogers-Ramanujan's continued fraction [7, p. 365]

$$R(q) = \frac{q^{1/5}}{1}\,_{+}\frac{q}{1}\,_{+}\frac{q^2}{1}\,_{+}\frac{q^3}{1}\,_{+}\frac{q^4}{1}\,_{+\dots}, \quad |q| < 1,$$

Ramanujan cubic continued fraction [7, p. 366]

$$G(q) = \frac{q^{1/3}}{1}\,_{+}\frac{q+q^2}{1}\,_{+}\frac{q^2+q^4}{1}\,_{+}\frac{q^3+q^6}{1}\,_{+\dots}, \quad |q| < 1,$$

and Ramanujan-Göllnitz-Gordon continued fraction [6, p. 229]

$$E(q) = \frac{q^{1/2}}{1-q}\,_{+}\frac{q^2}{1-q^3}\,_{+}\frac{q^4}{1-q^5}\,_{+\dots}, \quad |q| < 1.$$

For example

$$\frac{1}{R(q)} - R(q) + 1 = \frac{f(-q^{\frac{1}{5}})}{qf(-q^5)},$$

$$\frac{1}{R^5(q)} - R^5(q) - 11 = \frac{f^6(-q)}{qf^6(-q^5)},$$

$$\left[\frac{1}{G(q)} + 4G^2(q)\right]^3 = 27 + \frac{f^{12}(-q)}{qf^{12}(-q^3)},$$

$$\left[\frac{1}{E(q)} + E(q)\right]^4 - 16 = \frac{f^8(-q^2)}{q^2f^8(-q^8)}.$$

Thus, evaluating $\dfrac{f(-q)}{q^{\frac{n-1}{24}}f(-q^n)}$ for $n \geq 2$ plays an important role in evaluating continued fraction at $q = e^{-2\pi\sqrt{k}}$, for k rational. In the literature can found the evaluating $\dfrac{f(-q)}{q^{\frac{n-1}{24}}f(-q^n)}$ for $n = 2, 3, 4, 5, 9, 25$.

The purpose of the article is to evaluate $\dfrac{f(-q)}{q^{\frac{5}{24}}f(-q^6)}$ for certain rational k, with

$q = e^{-2\pi\sqrt{k}}$.

2. Main Result

Theorem 2.1. *For* $q = e^{-\pi\sqrt{\frac{2n}{3}}}$, *let*

$$A_n = \frac{f(-q)}{6^{\frac{1}{4}} q^{\frac{5}{24}} f(-q^6)},$$

then

$$A_n A_{\frac{1}{n}} = 1.$$

Proof. From [2, p. 43], we have if $\alpha\beta = \pi^2$, then

$$e^{\pi\left(\frac{\beta-\alpha}{12}\right)} \frac{f(-e^{-2\alpha})}{f(-e^{-2\beta})} = \sqrt[4]{\frac{\beta}{\alpha}}. \tag{2.1}$$

Consider

$$A_n A_{\frac{1}{n}} = \frac{1}{\sqrt{6}} e^{\left(\frac{25\pi}{144}\right)} \left\{ \frac{f\left(-e^{-\pi\sqrt{\frac{2n}{3}}}\right)}{f\left(-e^{-6\pi\sqrt{\frac{2n}{3}}}\right)} \frac{f\left(-e^{-\pi\sqrt{\frac{2}{3n}}}\right)}{f\left(-e^{-6\pi\sqrt{\frac{2}{3n}}}\right)} \right\} \tag{2.2}$$

$$= \frac{1}{\sqrt{6}} e^{\left(\frac{25\pi}{144}\right)} \left\{ \frac{f\left(-e^{-2\pi\sqrt{\frac{n}{6}}}\right)}{f\left(-e^{-2\pi\sqrt{\frac{6}{n}}}\right)} \frac{f\left(-e^{\frac{-2\pi}{\sqrt{6n}}}\right)}{f\left(-e^{-2\pi\sqrt{6n}}\right)} \right\}. \tag{2.3}$$

Employing (2.1) twice in the above, we obtain

$$A_n A_{\frac{1}{n}} = 1. \tag{2.4}$$

Theorem 2.2. *We have*

$$e^{\left(\frac{5\pi}{24}\sqrt{\frac{2}{3}}\right)} \frac{f(-e^{-\pi\sqrt{\frac{2}{3}}})}{f(-e^{-2\pi\sqrt{6}})} = \sqrt{6}.$$

Proof. Setting $n = 1$ in Theorem 2.1, we obtain required result.

Theorem 2.3. *We have*

$$(i) \qquad \frac{f(-e^{\frac{-2\pi}{\sqrt{3}}})}{e^{\frac{-5\pi}{12\sqrt{3}}} f(-e^{-4\pi\sqrt{3}})} = \left(\frac{3}{2}\right)^{\frac{1}{8}} (3+\sqrt{3})^{\frac{1}{4}}$$

and

$$(ii) \qquad \frac{f(-e^{\frac{-\pi}{\sqrt{3}}})}{e^{\frac{-5\pi}{24\sqrt{3}}} f(-e^{-2\pi\sqrt{3}})} = \left(\frac{2}{3}\right)^{\frac{1}{8}} \left[6(3-\sqrt{3})\right]^{\frac{1}{4}}.$$

Proof. From [8, p. 74], we have

$$e^{\frac{\pi}{3\sqrt{3}}}\frac{f(-e^{\frac{-2\pi}{\sqrt{3}}})f(-e^{-2\pi\sqrt{3}})}{f(-e^{-4\pi\sqrt{3}})f(-e^{\frac{-4\pi}{\sqrt{3}}})}=(2)^{\frac{5}{6}}. \tag{2.5}$$

Also from [9, p. 55], we have

$$e^{\frac{\pi}{2\sqrt{3}}}\frac{f(-e^{\frac{-2\pi}{\sqrt{3}}})f(-e^{\frac{-4\pi}{\sqrt{3}}})}{f(-e^{-2\pi\sqrt{3}})f(-e^{-4\pi\sqrt{3}})}=\frac{(3)^{\frac{1}{4}}}{(2)^{\frac{1}{4}}}(\sqrt{3+\sqrt{3}}). \tag{2.6}$$

Multiplying the above two identities, we obtain (i). Setting $n=2$ in Theorem 2.1 and then using Theorem 2.3 (i), we obtain (ii).

Theorem 2.4. *We have*

$$(i) \qquad e^{\frac{5\pi\sqrt{3}}{24}}\frac{f(-e^{-\pi\sqrt{3}})}{f(-e^{-6\pi\sqrt{3}})}=\sqrt{\sqrt{3}nm}$$

and

$$(ii) \qquad e^{\frac{5\pi}{36\sqrt{3}}}\frac{f(-e^{\frac{-2\pi}{3\sqrt{3}}})}{f(-e^{\frac{-4\pi}{\sqrt{3}}})}=\frac{1}{\sqrt{\sqrt{3}nm}}.$$

Where

$$n=\frac{2^{\frac{1}{24}}\sqrt[4]{1+\sqrt{3}}\left[2+\sqrt{3+(80)\times 2^{\frac{2}{3}}-(100)\times 2^{\frac{1}{3}}}\right]^{\frac{1}{8}}}{\left[\sqrt[3]{2}-1\right]^{\frac{1}{3}}},$$

$$m=\frac{1}{\sqrt[4]{26+15\sqrt{3}}}+\frac{1}{\left(6\left(2/\left(a+\sqrt{4b^3+a^2}\right)\right)^{\frac{1}{3}}\right)}-\frac{b}{\left(3\times 2^{\frac{2}{3}}\left(a+\sqrt{4b^3+a^2}\right)^{\frac{1}{3}}\right)},$$

with

$$a=648\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{\frac{1}{2}}+648\frac{(\sqrt{3}-1)}{\sqrt{2}\left(\frac{5+3\sqrt{3}}{\sqrt{2}}\right)^{\frac{1}{2}}}+\frac{432}{\left(\frac{5+3\sqrt{3}}{\sqrt{2}}\right)^{\frac{3}{2}}}$$

and

$$b=36\sqrt{2}\left[3-2\sqrt{3}\right].$$

Proof. Let G_n and g_n be Ramanujan class invariants [3, p. 187], then we have

$$(G_ng_n)^8\left(G_n^8-g_n^8\right)=\frac{1}{4}. \tag{2.7}$$

Also, Ramanujan recorded $G_3 = 2^{\frac{1}{12}}$ and $G_{27} = \frac{2^{\frac{1}{12}}}{(\sqrt[3]{2}-1)^{\frac{1}{3}}}$.
Using value of G_3 and G_{27} in (2.7), then solving the g_3 and g_{27}, we find that

$$g_3 = \frac{\sqrt[4]{1+\sqrt{3}}}{2^{\frac{7}{24}}}$$

and

$$g_{27} = \frac{\left[2+\sqrt{3+(80)\times 2^{\frac{2}{3}}-(100)\times 2^{\frac{1}{3}}}\right]^{\frac{1}{3}}}{2^{\frac{1}{6}}\left[\sqrt[3]{2}-1\right]^{\frac{1}{3}}}.$$

From the above two identities, we obtain

$$e^{\frac{\pi\sqrt{3}}{6}} \frac{f(-e^{-\pi\sqrt{3}})f(-e^{-3\pi\sqrt{3}})}{f(-e^{-2\pi\sqrt{3}})f(-e^{-6\pi\sqrt{3}})} = \frac{2^{\frac{1}{24}}\sqrt[4]{1+\sqrt{3}}\left[2+\sqrt{3+(80)\times 2^{\frac{2}{3}}-(100)\times 2^{\frac{1}{3}}}\right]^{\frac{1}{8}}}{\left[\sqrt[3]{2}-1\right]^{\frac{1}{3}}} \tag{2.8}$$

From [1], we have if $q = e^{-2\pi\sqrt{\frac{n}{3}}}$ and $\mu_n = \frac{1}{3^{\frac{1}{4}}}\frac{f(-q)}{q^{\frac{1}{12}}f(-q^3)}$
then

$$\mu_n \mu_{\frac{1}{n}} = 1, \tag{2.9}$$

$$\mu_1 = 1 \tag{2.10}$$

and

$$\mu_4 = \left(\frac{3\sqrt{3}+5}{\sqrt{2}}\right)^{\frac{1}{6}}. \tag{2.11}$$

we have from [5, p. 35], we obtain

$$\left(\frac{\mu_{9n}\mu_{36n}}{\mu_n\mu_{4n}}\right)^2 = 3\mu_n\mu_{4n}\mu_{9n}\mu_{36n} + \frac{3}{\mu_n\mu_{4n}\mu_{9n}\mu_{36n}} + 3.$$

Setting $n = \frac{1}{4}$ in the above, then using (2.9),(2.10) and (2.11), we find that

$$x^3 - \frac{3}{2^{\frac{1}{4}}}\sqrt{3\sqrt{3}-5}\; x^2 - \frac{3}{2^{\frac{1}{6}}}\sqrt[3]{3\sqrt{3}-5}\; x - \frac{3}{2^{\frac{1}{12}}}\sqrt[6]{3\sqrt{3}-5} = 0$$

where $x = \mu_9 \mu_{\frac{9}{4}}$.
Solving the above cubic equation for real roots, we find that

$$x = \frac{1}{\sqrt[4]{26+15\sqrt{3}}} + \frac{1}{\left(6\left(2/\left(a+\sqrt{4b^3+a^2}\right)\right)^{\frac{1}{3}}\right)} - \frac{b}{\left(3\times 2^{\frac{2}{3}}\left(a+\sqrt{4b^3+a^2}\right)^{\frac{1}{3}}\right)},$$

with
$a = 648\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{\frac{1}{2}} + 648\frac{\left(\sqrt{3}-1\right)}{\sqrt{2}\left(\frac{5+3\sqrt{3}}{\sqrt{2}}\right)^{\frac{1}{2}}} + \frac{432}{\left(\frac{5+3\sqrt{3}}{\sqrt{2}}\right)^{\frac{3}{2}}}$
and
$b = 36\sqrt{2}\left[3-2\sqrt{3}\right]$.
By using the definition of μ_n, we find that

$$e^{\frac{\pi\sqrt{3}}{4}} \frac{f(-e^{-\pi\sqrt{3}})f(-e^{-2\pi\sqrt{3}})}{f(-e^{-3\pi\sqrt{3}})f(-e^{-6\pi\sqrt{3}})} = \sqrt{3}m, \tag{2.12}$$

where

$$m = \frac{1}{\sqrt[4]{26+15\sqrt{3}}} + \frac{1}{\left(6\left(2/\left(a+\sqrt{4b^3+a^2}\right)\right)^{\frac{1}{3}}\right)} - \frac{b}{\left(3\times 2^{\frac{2}{3}}\left(a+\sqrt{4b^3+a^2}\right)^{\frac{1}{3}}\right)}.$$

with
$a = 648\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{\frac{1}{2}} + 648\frac{\left(\sqrt{3}-1\right)}{\sqrt{2}\left(\frac{5+3\sqrt{3}}{\sqrt{2}}\right)^{\frac{1}{2}}} + \frac{432}{\left(\frac{5+3\sqrt{3}}{\sqrt{2}}\right)^{\frac{3}{2}}}$
and
$b = 36\sqrt{2}\left[3-2\sqrt{3}\right]$.
From (2.8) and (2.12), we obtain (i). (ii) follows from Theorem 2.1 and Theorem 2.4 (i).

Acknowledgments
The authors would like to thank referees for their useful and valuable comments.

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