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ON THE GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS

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Abstract: The main aim of this paper is to study some growth properties of entire functions on the basis of generalized relative order (α, β) where α and β are continuous non-negative functions on $(-\infty, +\infty)$.

Keywords and Phrases: Entire function, growth, composition, generalized relative order (α, β), generalized relative lower order (α, β).

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1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ and the maximum term $\mu_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ are defined as $M_f = \max_{|z|=r} |f(z)|$ and $\mu_f(r) = \max_{n\geq 0} (|a_n| r^n)$ respectively. Moreover, if f is non-constant entire function then $M_f(r)$ is also strictly increasing and continuous function of r. Therefore, its inverse $M_f^{-1} : (M_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to+\infty} M_f^{-1}(s) = \infty$. We use the standard notations and definitions of the theory of entire functions which are available in [23] and [24], and therefore we do not explain those in details.

Let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$. The value

$$\varrho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} (\alpha \in L,\ \beta \in L)$$

is called [21] generalized order (α, β) of f. For details about generalized order (α, β) one may see [21]. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α, β) in some different direction. For the purpose of further applications, Biswas et al. [3, 5] have given the definitions of the generalized order (α, β) and generalized lower order (α, β) of an entire function after giving a minor modification to the original definition of generalized order (α, β) of an entire function (e.g. see, [21]).

Definition 1. [3, 5] *Let α, $\beta \in L^0$. The generalized order (α, β) and generalized lower order (α, β) denoted by $\varrho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[f]$ respectively of an entire function f are defined as:*

$$\varrho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

Mainly the growth investigation of entire functions has usually been done through their maximum moduli function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. Now in order to make some progresses in the study of relative order, Biswas et al. [9] have introduced the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of an entire function with respect to another entire function in the following way:

Definition 2. [9] *Let α, $\beta \in L^0$. The generalized relative order (α, β) and generalized relative lower order (α, β) of an entire function f with respect to an entire function g denoted by $\varrho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:*

$$\begin{matrix}\varrho_{(\alpha,\beta)}[f]_g \\ \lambda_{(\alpha,\beta)}[f]_g\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}.$$

In terms of maximum terms of entire functions, Definition 2 can be reformulated as:

Definition 3. *Let α, $\beta \in L^0$. The growth indicators $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to another entire function g are defined as:*

$$\begin{matrix} \varrho_{(\alpha,\beta)}[f]_g \\ \lambda_{(\alpha,\beta)}[f]_g \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\mu_g^{-1}(\mu_f(r)))}{\beta(r)}.$$

In fact, the Definition 2 and Definition 3 are equivalent {cf. [17]}.

The main aim of this paper is to establish some newly developed results related to the growth rates of composition of two entire functions on the basis of generalized relative order (α,β) and generalized relative lower order (α,β) of entire function with respect to another entire function which extend some earlier results (see, e.g., [22]). In fact some works in this direction have already been explored in [3] to [16].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [18] *Let f and g are any two entire functions with $g(0) = 0$. Also let B satisfy $0 < B < 1$ and $c(B) = \frac{(1-B)^2}{4B}$. Then for all sufficiently large values of r,*

$$M_f(c(B)M_g(Br)) \le M_{f(g)}(r) \le M_f(M_g(r)).$$

In addition if $B = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f(g)}(r) \ge M_f\Big(\frac{1}{8}M_g\Big(\frac{r}{2}\Big)\Big).$$

Lemma 2. [19] *Let f and g be entire functions. Then for every $\delta > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \le \frac{\delta}{\delta-1}\mu_f\Big(\frac{\delta R}{R-r}\mu_g(R)\Big).$$

Lemma 3. [19] *If f and g are any two entire functions. Then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \ge \frac{1}{2}\mu_f\Big(\frac{1}{16}\mu_g\Big(\frac{r}{4}\Big)\Big).$$

3. Main Results

In this section we present the main results of the paper. Below we suppose that functions α_1, α_2, α_3, β_1, β_2 and β_3 belong to the class L^0.

Theorem 1. *Let f, g and h be any three entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \le$*

$\varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$. and $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. Also let γ be a positive continuous function on $[0, +\infty)$ increasing to $+\infty$ and $A \geq 0$ be any number.
(i) If $\beta_1(\alpha_2^{-1}(\log r)) \geq r$ and $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\log r} = 0$, then

$$\lim_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))}{\{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))\}^{1+A}} = +\infty \text{ and} \tag{3.1}$$

(ii) if either $\beta_1(r) = B(\alpha_2(r))$ where B is any positive constant and $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\log r} = 0$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = 0$, then

$$\lim_{r\to+\infty} \frac{\exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))}{\{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))\}^{1+A}} = +\infty. \tag{3.2}$$

Proof. From the definition of $\varrho_{(\alpha_1,\beta_1)}[f]_h$, it follows for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r))))) \leq (\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\gamma(r). \tag{3.3}$$

Since $M_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 1 and for all sufficiently large values r that

$$\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \geq \alpha_1\Big(M_h^{-1}\Big(M_f\Big(\frac{1}{8}M_g\Big(\frac{\beta_2^{-1}(\log r)}{2}\Big)\Big)\Big)\Big)$$

$$i.e., \ \alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \geq$$

$$(1 + o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1\Big(M_g\Big(\frac{\beta_2^{-1}(\log r)}{2}\Big)\Big). \tag{3.4}$$

If $\beta_1(\alpha_2^{-1}(\log r)) \geq r$, then from (3.4) it follows for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \geq$$

$$(1 + o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1\Big(\alpha_2^{-1}\Big(\alpha_2\Big(M_g\Big(\frac{\beta_2^{-1}(\log r)}{2}\Big)\Big)\Big)\Big) \tag{3.5}$$

$$i.e., \ \alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \geq$$

$$(1 + o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(\alpha_2^{-1}(\log r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)})). \tag{3.6}$$

Now combining (3.3) and (3.6) it follows for all sufficiently large values of r that

$$\frac{\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))}{\{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))\}^{1+A}} \geq$$

$$\frac{(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(\alpha_2^{-1}(\log r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}))}{(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)^{1+A}\{\gamma(r)\}^{1+A}}.$$

Since $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\log r} = 0$ and $\beta_1(\alpha_2^{-1}(\log r)) \geq r$, therefore $\frac{\beta_1(\alpha_2^{-1}(\log r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}))}{\{\gamma(r)\}^{1+A}} \to +\infty$ as $r \to +\infty$, then from above it follows that

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))}{\{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))\}^{1+A}} = +\infty,$$

i.e., (3.1) is proved.

If $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant, then from (3.4) it follows for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \geq$$

$$B(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\alpha_2\Big(M_g\Big(\frac{\beta_2^{-1}(\log r)}{2}\Big)\Big) \tag{3.7}$$

$$i.e.,\ \alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \geq \log r^{B(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}$$

$$i.e.,\ \exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))) \geq r^{B(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}. \tag{3.8}$$

Hence in view of (3.3) and (3.8), we get for all sufficiently large values of r that

$$\frac{\exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))}{\{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))\}^{1+A}} \geq \frac{r^{B(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}}{(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)^{1+A}\{\gamma(r)\}^{1+A}}.$$

As $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\log r} = 0$, so $\frac{r^{B(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}}{\{\gamma(r)\}^{1+A}} \to +\infty$ as $r \to +\infty$. Thus it follows from above that

$$\frac{\exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))}{\{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))\}^{1+A}} \to +\infty, \text{ as } r \to +\infty,$$

i.e., (3.2) is proved.

Finally if $\beta_1(\alpha_2^{-1}(r)) \in L^0$. Then from (3.5) we obtain for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \geqslant (1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(\alpha_2^{-1}(\log r))$$

$$i.e.,\ \exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))) \geqslant$$
$$\exp((1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(\alpha_2^{-1}(\log r))),$$

whence in view of (3.3) and the condition $\lim_{r\to+\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = 0$ we get from above that

$$\frac{\exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))}{\{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))\}^{1+A}}$$
$$\geq \frac{\exp((1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(\alpha_2^{-1}(\log r)))}{(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)^{1+A}\{\gamma(r)\}^{1+A}} \to +\infty \text{ as } r \to +\infty,$$

i.e., (3.2) is proved again. Thus the proof of Theorem 1 is completed.

Remark 1. *Theorem 1 is still valid with "limit superior" instead of "limit" if we replace the condition "*$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$*" by "* $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < +\infty$*".*

In the line of Theorem 1 one may state the following theorem without proof:

Theorem 2. *Let f, g, h and k be any four entire functions such that $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$, $\varrho_{(\alpha_3,\beta_3)}[g]_k < +\infty$ and $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. Also let γ be a positive continuous function on $[0,+\infty)$ increasing to $+\infty$. For any number $A \geq 0$,*
(i) if $\beta_1(\alpha_2^{-1}(\log r)) \geq r$ and $\lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = 0$, then

$$\lim_{r\to+\infty} \frac{\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))}{\{\alpha_3(M_k^{-1}(M_g(\beta_3^{-1}(\gamma(r)))))\}^{1+A}} = +\infty \ and$$

(ii) if either $\beta_1(r) = B(\alpha_2(r))$ where B is any positive constant and $\lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = 0$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r\to+\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = 0$, then

$$\lim_{r\to+\infty} \frac{\exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))}{\{\alpha_3(M_k^{-1}(M_g(\beta_3^{-1}(\gamma(r)))))\}^{1+A}} = +\infty,$$

Remark 2. *In Theorem 2 if we take the condition "$\lambda_{(\alpha_3,\beta_3)}[g]_k < +\infty$" instead of "$\varrho_{(\alpha_3,\beta_3)}[g]_k < +\infty$", then also Theorem 2 remains true with "limit superior" in* *place of " limit ".*

Theorem 3. *Let f, g and h be any three entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$ and $\varrho_{(\alpha_2,\beta_2)}[g] < +\infty$. Also let γ be a positive continuous function on $[0,+\infty)$ increasing to $+\infty$ and $A \geq 0$ be any number.*
(i) If $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\log r} = +\infty$, then

$$\lim_{r\to+\infty} \frac{\{\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))} = 0 \text{ and} \tag{3.9}$$

(ii) if either $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant and $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\log r}$ $= +\infty$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$, then

$$\lim_{r\to+\infty} \frac{\{\exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))\}^{1+A}}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))} = 0. \tag{3.10}$$

Proof. From the definition of $\lambda_{(\alpha_1,\beta_1)}[f]_h$, we get for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r))))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\gamma(r). \tag{3.11}$$

Since $M_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 1 for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant$$

$$(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(M_g(\beta_2^{-1}(\log r))). \tag{3.12}$$

If $\beta_1(\alpha_2^{-1}(\log r)) \leq r$, then we get from (3.12) for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant$$

$$(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(\alpha_2^{-1}(\alpha_2(M_g(\beta_2^{-1}(\log r))))) \tag{3.13}$$

$$i.e., \ \alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant$$

$$(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(\alpha_2^{-1}(\log r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)})). \tag{3.14}$$

Since $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim_{r\to+\infty} \frac{\log\gamma(r)}{\log r} = +\infty$, so we obtain from (3.11) and (3.12) for all sufficiently large values of r that

$$\frac{\{\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))} \leqslant$$

$$\frac{(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)^{1+A}[\beta_1(\alpha_2^{-1}(\log r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}))]^{1+A}}{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\gamma(r)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\{\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))} = 0,$$

i.e., (3.9) is proved.

If $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant, then get from (3.12) for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant B(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\alpha_2(M_g(\beta_2^{-1}(\log r)))$$

$$i.e.,\ \alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant B(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\log r$$

$$i.e.,\ \exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))) \leqslant r^{B(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \qquad (3.15)$$

So combining (3.11) and (3.15), we obtain for all sufficiently large values of r that

$$\frac{\{\exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))\}^{1+A}}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))} \leqslant \frac{r^{B(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)(1+A)}}{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\gamma(r)}.$$

As $\lim_{r\to+\infty}\frac{\log\gamma(r)}{\log r} = +\infty$, so $\frac{r^{B(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)(1+A)}}{\gamma(r)} \to 0$ as $r \to +\infty$. Thus it follows from above that

$$\lim_{r\to+\infty}\frac{\{\exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))\}^{1+A}}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))} = 0,$$

i.e., (3.10) is proved.

Finally if $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r\to+\infty}\frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$, then we have from (3.13) for all sufficiently large values of r that

$$\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\log r))$$

$$i.e., \ \exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))$$

$$\leqslant \exp((1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\log r))), \quad (3.16)$$

whence in view of (3.11) and the condition $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))}=+\infty$ we get from above that

$$\frac{\{\exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))\}^{1+A}}{\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\gamma(r)))))}$$

$$\leqslant \frac{[\exp((1+o(1))(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\log r)))]^{1+A}}{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\gamma(r)} \to 0 \text{ as } r\to+\infty,$$

i.e., (3.10) is proved again. Thus the theorem follows.

Remark 3. *In Theorem . if we take the condition "$\varrho_{(\alpha_1,\beta_1)}[f]_h>0$" instead of "$0<\lambda_{(\alpha_1,\beta_1)}[f]_h\le\varrho_{(\alpha_1,\beta_1)}[f]_h<+\infty$", the theorem remains true with "limit inferior" in place of "limit".*

Theorem 4. *Let f, g, h and k be any four entire functions such that $\varrho_{(\alpha_1,\beta_1)}[f]_h<+\infty$, $\lambda_{(\alpha_3,\beta_3)}[g]_k>0$ and $\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$. Also let γ be a positive continuous function on $[0,+\infty)$ increasing to $+\infty$ and $A\ge0$ be any number.*
(i) If $\beta_1(\alpha_2^{-1}(\log r))\le r$ and $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty$, then

$$\lim_{r\to+\infty}\frac{\{\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\gamma(r)))))}=0 \ and$$

(ii) if either $\beta_1(r)=B\alpha_2(r)$ where B is any positive constant and $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty$ or $\beta_1(\alpha_2^{-1}(r))\in L^0$ and $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))}=+\infty$, then

$$\lim_{r\to+\infty}\frac{\{\exp(\alpha_1(M_h^{-1}(M_{f(g)}(\beta_2^{-1}(\log r)))))\}^{1+A}}{\alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(\gamma(r)))))}=0.$$

The proof of Theorem 4 would run parallel to that of Theorem 3. We omit the details.

Remark 4. *In Theorem 4, if we take the condition "$\varrho_{(\alpha_3,\beta_3)}[g]_k>0$" instead of "$\lambda_{(\alpha_3,\beta_3)}[g]_k>0$", the theorem remains true with "limit" replaced by "limit inferior".*

Remark 5. *In view of Definition 2 and with the help of Lemma 2 and Lemma 3, the same results of above theorems and remarks can also be deduced with maximum terms of entire functions.*

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