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CANONICAL ELEMENT CONJECTURE AND COHEN-MACAULAY RINGS

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Abstract: Let M be a module over a commutative Noetherian ring A and J be an ideal of A. In this short note, it is proved, by an elementary argument, that if $x_1, ..., x_n$ is an M-sequence contained in J, then $Hom_A(A/J, H^n_J(M)) \cong (x_1, ..., x_n)M :_M J/(x_1, ..., x_n)M$. As an application of this result, the canonical element conjecture is established in a certain case.

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1. Introduction

Throughout this short note A is a commutative Noetherian ring. For an A-module M and a sequence $x_1, ..., x_n$ of element of A, we say that $x_1, ..., x_n$ is an M-sequence if $(x_1, ..., x_n)M \neq M$ and x_i is a non-zero divisor on $M/(x_1, ..., x_{i-1})M$ for all $i = 1, ..., n$. Also, for an ideal I of A, we use notation $H^t_I(M)$ to denote the t-th local cohomology module of M with support in I. Finally, for ideals I and J of A, the submodule $\{\alpha \in H^t_I(M) : J\alpha = 0\}$ of $H^t_I(M)$ is denoted by $Ann_{H^t_I(M)}(J)$. Note that $Ann_{H^t_I(M)}(J) \cong Hom_A(A/J, H^t_I(M))$. Let J be an ideal of A and Let $x_1, ..., x_n$

be an M-sequence in J. In this note, we provide an elementary argument to show that $Ann_{H^t_I(M)}(J) \cong (x_1, ..., x_t)M :_M J/(x_1, ..., x_t)M$. Using this result, we prove that if (A, m) is a local Cohen-Macaulay ring of dimension n, then the natural map $Ext^n_A(A/m, Syz_n(A/m)) \longrightarrow H^n_m(Syz_n(A/m))$ is a monomorphism, where $Syz_n(A/m)$ denoted the n-th syzygy of A/m . Note that this result established the canonical element conjecture in a certain case.

2. Preliminaries

In this section, we will give two lemmas which they are tools that we use later.

Lemma 2.1. [2, Exercise 6.5.2] *let A be a ring, $x_1, ..., x_n$ an M-Sequence, $\alpha_1, ..., \alpha_n$, $\beta_1, ..., \beta_n \in M$ and $r \in A$. If $x_1\beta_1 + \cdots + x_n\beta_n = x_1 r\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$ then $\beta_1 \in (r, x_2, ..., x_n)M$.*

Lemma 2.2. *Let $x_1, ..., x_n$ be an M-sequnce, $t_1, ..., t_n, l \in N$, $\beta \in M$ and $1 \leq i \leq n$. if*

$$x_1^{t_1} \cdots x_i^{t_i}\beta \in (x_1^{t_1+l}, ..., x_n^{t_n+l})M$$

then

$$\beta \in (x_1^l, ..., x_i^l, x_{i+1}^{t_{i+1}+l}, ..., x_n^{t_n+l})M.$$

Proof. Argue by way of induction on i. if $i = 1$, than the result follow from lemma 2.1. Assume that $1 \leq i < n$ and the result settled for i. Suppose $x_1^{t_1} \cdots x_i^{t_i} x_{i+1}^{t_{i+1}}\beta \in (x_1^{t_1+l}, ..., x_n^{t_n+l})M$. From induction hypothesis we have

$$x_{i+1}^{t_{i+1}}\beta \in (x_1^l, ..., x_i^l, x_{i+1}^{t_{i+1}+l}, ..., x_n^{t_n+l})M. \tag{1}$$

Now, put $\mathfrak{a} = (x_1^l, ..., x_i^l)$ and assume $- : M \longrightarrow M/(\mathfrak{a}M)$ is the natural map. so by Lemma 2.1,

$$x_{i+1}^{t_{i+1}}\overline{\beta} \in (x_{i+1}^{t_{i+1}+l}, ..., x_n^{t_n+l})\overline{M}$$

Hence from lemma 2.1, we have $\overline{\beta} \in (x_{i+1}^l, x_{i+2}^{t_{i+2}+l}, ..., x_n^{t_n+l})\overline{M}$. we deduce that

$$\beta \in (x_1^l, ..., x_{i+1}^l, x_{i+2}^{t_{i+2}+l}, ..., x_n^{t_n+l})M.$$

3. local Cohomology and Canonical Element Conjecture

Definition 3.1. *Let A be a Noetherian ring, M an A-module and I an ideal of A. We define the i-th local cohomology of M with respect to I by*

$$H^i_I(M) = \lim_{\overrightarrow{t}} Ext^i_A(A/I^t, M)$$

Remark 3.2. *Let M an A-module and $x_1, ..., x_n$ be an M-sequence. put $I^\alpha = (x_1^\alpha, ..., x_n^\alpha)$ and $I^\beta = (x_1^\beta, ..., x_n^\beta)$ for all positive integer with $\alpha \leq \beta$. Then map*

$\varphi_{\alpha\beta} : M/(I^\alpha M) \longrightarrow M/(I^\beta M)$ *given by*
$\varphi_{\alpha\beta}(I^\alpha M + y) = I^\beta M + (x_1^{\beta-\alpha} \cdots x_n^{\beta-\alpha})y$ *for all* $y \in M$ *is well-defined. Now it is easy to check that* $\Big(M/(I^\alpha M), \varphi_{\alpha\beta}\Big)_{\alpha \leq \beta}$ *is a direct system of A-modules. Assume* $\Big(X, \varphi_\alpha\Big)_{\alpha \in N}$ *is the direct limit of this direct system. Then by [2, 5.2.9] we have* $X = H^n_{(x_1,\dots,x_n)}(M)$.

Proposition 3.3. *With the hypothesis of the above remark, the following hold:*
(1) φ_α *is a monomorphism for all positive integer* α.
(2) *Suppose* J *is an ideal of* A *such that* $x_1, \dots, x_n$ *is in* J*. then the map*

$$\psi : \frac{(x_1, \dots, x_n)M :_M J}{(x_1, \dots, x_n)M} \longrightarrow Ann_{H^n_I(M)}(J)$$

given by $\psi((x_1, \dots, x_n)M + \xi) = \varphi_1((x_1, \dots, x_n)M + \xi)$ *is an isomorphism.*

Proof. Assume $\varphi_\alpha(I^\alpha M + x) = 0$. Then there exists β bigger than α such that $\varphi_{\alpha\beta}(I^\alpha M + x) = 0$. Hence $(x_1^{\beta-\alpha} \cdots x_n^{\beta-\alpha})x \in I^\beta M$. So by lemma 2.2 we have $x \in (x_1^\alpha, \dots, x_n^\alpha)M$. We deduce that φ_α is a monomorphism. Now assume $y \in Ann_{H^n_I(M)}(J)$ is arbitrary. Then there exist positive integer α and $\xi \in M$ such that $y = \varphi_\alpha(I^\alpha M+\xi)$. Since $x_1 \in J$, then $x_1 y = 0$. On other hand $0 = x_1\varphi_\alpha(I^\alpha M+\xi) = \varphi_\alpha(I^\alpha M + x_1\xi)$. Hence $x_1\xi + I^\alpha M = 0$. Therefore there exist $\eta_1, \dots, \eta_n \in M$ such that $x_1\xi = x_1^\alpha\beta_1 + \cdots + x_n^\alpha\beta_n$. So by Lemma 2.1 we have $\xi \in (x_1^{\alpha-1}, x_2^\alpha, \dots, x_n^\alpha)M$. So that there exist $\tau_1, \dots, \tau_n \in M$ such that $\xi = x_1^{\alpha-1}\tau_1 + x_2^\alpha\tau_2 + \cdots + x_n^\alpha\tau_n$. Hence we have
$y = \varphi_\alpha(I^\alpha M + x_1^{\alpha-1}\tau_1)$. Now assume inductively that there exists τ_i in M with $0 \leq i < n$ such that

$$y = \varphi_\alpha(I^\alpha M + x_1^{\alpha-1}x_2^{\alpha-1}...x_i^{\alpha-1}\tau_i). \tag{2}$$

we might find an element $\tau_{i+1} \in M$ such that

$$\varphi_\alpha(I^\alpha M + x_1^{\alpha-1} \cdots x_i^{\alpha-1}x_{i+1}^{\alpha-1}\tau_{i+1}) = y$$

To this end note that $0 = x_{i+1}y = \varphi_\alpha(I^\alpha M + x_1^{\alpha-1}...x_i^{\alpha-1}x_{i+1}\tau_i)$. Since φ_α is monomorphism, we have $x_1^{\alpha-1}...x_i^{\alpha-1}x_{i+1}\tau_i \in I^\alpha M$. Hence by lemma 2.1, we have $x_{i+1}\tau_i \in (x_1, \dots, x_i, x_{i+1}^\alpha, \dots, x_n^\alpha)M$. Now assume $- : M \longrightarrow M/((x_1, \dots x_i)M)$ is the natural map. Thus $x_{i+1}\overline{\tau_i} \in (x_{i+1}^\alpha, \dots, x_n^\alpha)\overline{M}$. So by lemma 2.1, we have $\overline{\tau_i} \in (x_{i+1}^{\alpha-1}, x_{i+2}^\alpha, \dots, x_n^\alpha)\overline{M}$. Therefore there exist $\eta_{i+1}, \dots, \eta_n$ in M such that $\overline{\tau_i} = x_{i+1}^{\alpha-1}\overline{\eta_{i+1}} + x_{i+2}^\alpha\overline{\eta_{i+2}} + \cdots + x_n^\alpha\overline{\eta_n}$. In other words

$$\tau_i - (x_{i+1}^{\alpha-1}\eta_{i+1} + x_{i+2}^\alpha\eta_{i+2} + \cdots + x_n^\alpha\eta_n) \in (x_1, \dots, x_i)M$$

We deduce that there exist $\eta_1, ..., \eta_n \in M$ such that

$$\tau_i = x_1\eta_1 + \cdots + x_i\eta_i + x_{i+1}^{\alpha-1}\eta_{i+1} + x_{i+2}^{\alpha}\eta_{i+2} + \cdots + x_n^{\alpha}\eta_n.$$

Now with substituted τ_i in (2), we see that $y = \varphi_\alpha(I^\alpha M + x_1^{\alpha-1} \cdots x_{i+1}^{\alpha-1}\eta_{i+1})$. Hence we have

$$y = \varphi_\alpha(I^\alpha M + x_1^{\alpha-1} \cdots x_{i+1}^{\alpha-1}\eta_{i+1}) = \varphi_\alpha(\varphi_{1\alpha}(IM + \eta_{i+1})) = \varphi_1(IM + \eta_{i+1}).$$

Corollary 3.4. *Let M be an A-module and let J be an ideal of A. Suppose that $x_1, ..., x_t$ is an M-sequence in J. Then*

$$Ann_{H_J^t(M)}(J) \cong ((IM :_M J)/(IM))$$

where I is the ideal of A generated by $x_1, ..., x_t$.
Proof. We have $H_J^t(M) \cong H_J^0(H_I^t(M))$. Hence

$$\begin{aligned} Ann_{H_J^t(M)}(J) &\cong Hom_R(\frac{R}{J}, H_J^t(M)) \\ &\cong Hom_R(\frac{R}{J}, \varinjlim_t Hom_R(\frac{R}{J^\alpha}, H_I^t(M))) \\ &\cong \varinjlim_t Hom_R(\frac{R}{J}, Hom_R(\frac{R}{J^\alpha}, H_I^t(M))) \\ &\cong \varinjlim_t Hom_R(\frac{R}{J} \otimes_R \frac{R}{J^\alpha}, H_I^t(M)) \\ &\cong \varinjlim_t Hom_R(\frac{R}{J}, H_I^t(M)) \\ &\cong Ann_{H_I^t(M)}(J) \end{aligned}$$

Remark 3.5. [1, Exercise 2.1.26] *Let $(A, \mathfrak{m})$ be a local cohen macaulay ring of dimension n and M be a finitely generated A-module. Then the n-th syzygy of M is either 0 or a maximal cohen macaulay module.*

Theorem 3.6. *Let $(A, \mathfrak{m})$ be a cohen-macaulay local ring. Then the Canonical map $Ext_A^n(A/\mathfrak{m}, Syz_n(A/\mathfrak{m}) \longrightarrow H_\mathfrak{m}^n(Syz_n(A/\mathfrak{m}))$ is nonzero.*
Proof. let $N = Syz_n(A/\mathfrak{m})$. Then, by the above Remark, N is a maximal cohen-macaulay $A-module$. Now let $x_1, ..., x_n$ be a system of parametry for A. Then $x_1, ..., x_n$ is an $N-sequence$. Hence, by proposition 3.3,

$$\frac{(x_1, ..., x_n)N :_N J}{(x_1, ..., x_n)N} \cong Ann_{H_I^n(N)}(J)$$

On the other hand

$$Ext^n_A(A/\mathfrak{m}, N) \cong \frac{(x_1, ..., x_n)N :_N J}{(x_1, ..., x_n)N}.$$

It therefore followed that the canonical map of statement is non-zero.

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