*J. of Ramanujan Society of Mathematics and Mathematical Sciences*
*Vol. 7, No. 2 (2020), pp. 109-118*

**ISSN (Online): 2582-5461**

**ISSN (Print): 2319-1023**

# ON NEW IDENTITIES INVOLVING CHROMATIC OVERPARTITIONS

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(**Received:** *Jan. 01, 2020* **Accepted:** *Apr. 27, 2020* **Published:** *Jun. 30, 2020)*

**Abstract:** Our basic aim is to provide two new identities involving $q$-hypergeometric series inspired in some Euler's partitions identities. These are obtained making use of the new concept of chromatic overpartitions explored in this article.

**Keywords and Phrases:** Integer Partitions; Overpartitions; Chromatic Partitions; Chromatic Overpartitions; Hypergeometric Series.

**2010 Mathematics Subject Classification:** 05A17, 11P82, 11P84.

## 1. Introduction

In [16], Schneider and Sills gave new insights about what the called a norm of an integer partition $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_k$ of $n$, denoted by $N(\lambda)$. The norm function is defined by
$N(\lambda) = \lambda_1.\lambda_2 \cdots \lambda_k$ and it is present in several works in combinatorial number theory and additive number theory as in MacMahon [14] dated of 1917 as shows the next result.

For $q \in \mathbb{C}$, and $|q| < 1$,

$$\prod_{k=1}^{n} \frac{1}{1-q^k} = \sum_{\lambda \vdash n} \frac{1}{N(\lambda) m_1! m_2! \ldots (1-q)^{m_1}(1-q^2)^{m_2} \ldots},$$

where $\lambda = \langle 1^{m_1} 2^{m_2} 3^{m_3} \ldots \rangle$ and $\lambda \vdash n$ means that $\lambda$ is a partition of $n$.

In Alegri [1], the author, gave bijective proofs for some classes of partitions called $k$-chromatic overpartitions and super-chromatic overpartitions. Our work here have relationship to the norm function as displayed in the next equation.

$$\prod_{n=1}^{\infty} \frac{1+nq^n}{1-nq^n} = \sum_{n\geq 1} \left( \sum_{\lambda \vdash \overline{n}} N(\lambda) \right) q^n, \tag{1}$$

where $\lambda \vdash \overline{n}$ means that $\lambda$ is an overpartition of $n$.

In order to give combinatorial interpretation for some equations as in the previous one, we provide mathematical definitions for the new class of integer partitions, the chromatic partitions of an integer. This is a generalization of a well-know concept of a ordinary partition of a integer $n$ as defined and used in Euler [10] and developed in works as Andrews [2, 3, 4, 5]. A chromatic partition of an integer $n$ is a partition of $n$ in which every part $m$ appear colored in one of $m$ available colors and the order of presentation of equal parts matters.

A chromatic partition of $n$ presents a mathematical similarity to the $n$-colour partitions, an object of study of Agarwal and Andrews in [7] and Agarwal [6]. An $n$-colour partition of a positive integer $n$ is a partition in which a part of size $m$, can come in $m$ different colours which is presented in subscripts satisfying the lexicographical order. There are thirteen 4-colour partitions: $4_4, 4_3, 4_2, 4_1, 3_3 + 1_1, 3_2 + 1_1, 3_1 + 1_1, 2_2 + 2_2, 2_2 + 2_1, 2_2 + 1_1 + 1_1, 2_1 + 1_1 + 1_1, 1_1 + 1_1 + 1_1 + 1_1$. One may note that there are a bijective association with $n$-colour partitions and plane partitions as explained in the introduction section of [7]. The class of partitions discussed in this paper is different because the order of the colours matter, and , for $n > 3$, the number of chromatic partitions of $n$ is greater the $n$-colour partitions. For example, $2_1 + 2_2$ is one of the fourteen chromatic partition of $n = 4$.

In an attempt to prove the identities of this paper we use the concept of chromatic overpartition, a generalization of the class of overpartitions as instituted by Lovejoy and Corteel in [11] and widely used ever since. Works as Lovejoy [12, 13], making use of the overpartition concept as well as the more recent work of Corteel, Savelief and Vuletíc in [9]. An overpartition of an integer $n$ is a partition of $n$ in which the first occurrence of a part can appear overlined. For instance, $\overline{4} + 4 + 4 + \overline{3} + 2 + \overline{1} + 1$ is a overpartition of 19. Let $\overline{p}(n)$ denote by the number of overpartitions of $n$. The generating function of the number of overpartitions, is given by the next equations.

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \sum_{n=0}^{\infty} \frac{(-1;q)_n q^n}{(q;q)_n} = \frac{(-q;q)_\infty}{(q;q)_\infty}, \tag{2}$$

where $(a;q)_n = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1})$ is the $q$-shifted factorial, and $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$.

The identity in the second equation can be obtained by use the $q$-binomial theorem, which can be found in Andrews [4], by the use of $x = q$ and $a = -1$. In this work we will obtain an another identity resembles this one, but in terms of chromatic overpartitions.

As in Alegri [1], using heavy combinatorial arguments, we will give bijective proofs for the theorems 1 and 2 inspired in two Euler partition identities whose the function form is given next.

$$\sum_{n=0}^{\infty} \frac{q^n}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

$$\prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} = \prod_{n=1}^{\infty}(1+q^n).$$

The last equation establish that the number of partitions of $n$ in odd parts is equal to the number of partitions of $n$ into distinct parts.

As explained in Alegri [1], the importance of bijective proofs is due to the fact of making a bijective association between one class of partition to another, using simple invertible operation, without the need to establish a generating function for both. From this, we will find equations involving hypergeometric series that translate the combinatorial meaning of the partition identities we found. Several bijective proofs as the two presented here can be found in Pak [15] and Bressoud [8], for example.

## 2. Definitions and generating functions

**Definition 2.1.** *A chromatic partition of $n$ is an integer partition of $n$ in which every part $m$ receives an index, ranging from $1$ to $m$, referring to colors. The order of appearance of colored parts should be considered, i.e., equal parts associated to different colors are considered distinct.*

If we designated the colors red, yellow, green, gray, blue and violet by the numbers 1 to 6, respectively, the partition $6_6+6_5+5_4+5_3+4_2+3_1+3_2$ represents $6+6+5+5+4+3+3$.

**Example 2.2.** $7_7+7_4+5_3+5_4+2_2+2_1+1_1+1_1$ *is a chromatic partition of* $30$ *different from* $7_4+7_7+5_3+5_4+2_2+2_1+1_1+1_1$. *For* $n=6$ *we have* $56$ *chromatic partitions.*

**Definition 2.3.** *A chromatic overpartition of an integer $n$ is a chromatic partition of $n$ where the first occurrence of a colored part may appear overlined.*

**Example 2.4.** $\overline{7_6} + 7_6 + 6_4 + 6_5 + 4_2 + \overline{3_2} + \overline{3_0} + 1_0$ *is a chromatic overpartition of* 37*, while* $\overline{7_6} + \overline{7_6} + 6_4 + 6_5 + 4_2 + \overline{3_2} + \overline{3_0} + 1_0$ *is not an overpartition of* 37*.*

Denote $hp(n)$ by the number of chromatic partitions of an integer $n$. For $|q| < 1$, the product below is the generating function for the sequence $(hp(n))$.

$$\sum_{n=0}^{\infty} hp(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1 - nq^n)}. \tag{3}$$

Denote $h\bar{p}(n)$ by the number of chromatic overpartitions of an integer $n$. We are interested in find the generating function for the sequence $(h\bar{p}(n))_{n\geq 0} =$ (1, 2, 6, 16, 38, 88, 200, 428, ...). From the equation (3) it is easy to obtain the generating function for the sequence of the number of chromatic partitions into distinct parts as the following product.

$$\prod_{n=1}^{\infty}(1 + nq^n). \tag{4}$$

In a chromatic overpartition, the first occurrence of a colored part can be marked, so these parts can be obtained from the product in equation (4), while unmarked parts are generated by (3), thus, the generating function for the sequence $(h\bar{p}(n))_{n\geq 0}$ is given by the next product.

**Proposition 2.5.** *For* $|q| < 1$*,*

$$\sum_{n=0}^{\infty} h\bar{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1 + nq^n}{1 - nq^n}$$

In order to prove the second theorem of this paper, is convenient to define the set of chromatic partitions $\overline{A_n}$, for all positive integer $n$ as given below.

**Definition 2.6.** *Let* $\overline{A_n}$ *the set of partitions of* $n$ *whose the elements is partitions in which*

- *all parts are odd.*
- *a non-overlined colored part* $s_a$ *appears repeatedly in a power of two times, say* $2^l$*, including* $l = 0$*. The color of this sum of parts is taken in the set* $\{1, 2, \ldots, 2^l s\}$*.*
- *an overlined part appears as in the same way as in the previous item, but as an ovelined sum of parts.*

**Example 2.7.** *The partition* $\overline{7_{26}+7_{26}+7_{26}+7_{26}}+7_7+5_9+5_9+3_2+1_1+1_1$ *is a partition of* $\overline{A_{50}}$. *There are fifty partitions in* $\overline{A_{50}}$: $5_a, \overline{5_a}, 1_b+1_b+1_b+1_b+1_1, \overline{1_b+1_b+1_b+1_b}+1_1, 1_b+1_b+1_b+1_b+\overline{1_1}, \overline{1_b+1_b+1_b+1_b}+\overline{1_1}, 3_c+1_d+1_d, \overline{3_c}+1_d+1_d, 3_c+\overline{1_d+1_d}, \overline{3_c}+\overline{1_d+1_d}$, *where* $a\in\{1,2,3,4,5\}, b\in\{1,2,3,4\}, c\in\{1,2,3\}$ *and* $d\in\{1,2\}$.

**Lemma 2.8.** *The generating function of the number of partitions in* $\overline{A_n}$ *is*

$$\prod_{j\ odd}\left(\sum_{n=0}^{\infty}c_j(n)q^n\right),$$

*where*

$$c_1(r)=\begin{cases}1 & ,\ if\ r=0\\ 2^{n_1+n_2+\ldots+n_s+1} & ,\ if\ r=2^{n_1}+2^{n_2}+\cdots+2^{n_s}\end{cases}$$

*and for* $j\geq 2$,

$$c_j(r)=\begin{cases}1 & ,\ if\ r=0\\ j^s\left(2^{n_1+n_2+\ldots+n_s+1}\right) & ,\ if\ j|r, \frac{r}{j}=2^{n_1}+2^{n_2}+\cdots+2^{n_s}\\ 0 & ,\ if\ j\nmid r\end{cases}$$

**Proof.** For instance, the sum of ones non-overlined as given next:

$\ldots+(1_a+1_a+1_a+1_a+1_a+1_a+1_a+1_a)+(1_b+1_b+1_b+1_b)+(1_c+1_c)+1_0$, is generated by the sum

$$1+q+2q^2+2q^3+4q^4+4q^5+8q^6+8q^7+8q^9+16q^{10}\ldots.$$

Since a sum of ones in each parenthesis may appear overlined, this sum is generated by

$$1+2q+4q^2+4q^3+8q^4+8q^5+16q^6+16q^7+16q^9+32q^{10}\ldots=\sum_{n=0}^{\infty}c_1(n)q^n.$$

Since for the sum of threes, as $3_a$, $(3_b+3_b)$, $(3_b+3_b)+3_a$, $(3_c+3_c+3_c+3_c)$, $(3_c+3_c+3_c+3_c)+3_a$, and so on, the number of such partitions is generated by the next series.

$$1+3q^3+6q^6+12q^9+12q^{12}+36q^{15}+72q^{18}+216q^{21}+24q^{24}+72q^{27}+\ldots$$

Considering the overlined and non-overlined multiple parts of 3, the next series generates them.

$$1+6q^3+12q^6+24q^9+24q^{12}+72q^{15}+144q^{18}+432q^{21}+48q^{24}+144q^{27}+\ldots=\sum_{n=0}^{\infty}c_3(n)q^n$$

In general, for an odd number $j$, the next series generates the multiple parts of $j$.

$$\sum_{n=0}^{\infty}c_j(n)q^n$$

So, in general, by the multiplicative principle, for all partitions in $\overline{A_n}$, the coefficients of $q^n$ in the product

$$\prod_{j\ odd}\left(\sum_{n=0}^{\infty}c_j(n)q^n\right),$$

generate the number of such partitions.

## 3. Main Results

**Theorem 3.1.** *For any complex number $q$, where $|q|<1$,*

$$\prod_{n=1}^{\infty}\frac{1+nq^n}{1-nq^n}=1+\sum_{n=0}^{\infty}\frac{2(1+q)(1+2q^2)\ldots(1+(n-1)q^{(n-1)})nq^n}{(1-q)(1-2q^2)\cdots(1-nq^n)} \tag{5}$$

**Proof.** One may note that

$$\frac{2(1+q)(1+2q^2)\cdots(1-(n-1)q^{n-1})nq^n}{(1-q)(1-2q)\cdots(1-nq^n)}=\frac{2nq^n}{1-nq^n}\prod_{k=1}^{n-1}\frac{1+kq^k}{1-kq^k} \tag{6}$$

is the generating function for chromatic overpartitions with the largest part being $n$ in the sense that

$$\frac{2nq^n}{1-nq^n}$$

generates parts of size $n$ whereas

$$\prod_{k=1}^{\infty}\frac{1+kq^k}{1-kq^k}$$

generates parts of size $k$, for $k$ ranging from 1 to $n-1$. The role of the coefficient 2 in the r.h.s. of equation (6) is due to the fact that the first occurrence of the greatest part $n$ could appear (or not) overlined.

Hence, taking the limit $n \to \infty$, the right-hand side of (6) is also the generating function for chromatic overpartitions.

**Theorem 3.2.** *For any complex number $q$, where $|q| < 1$,*

$$\prod_{n=1}^{\infty}(1+2nq^n) = \prod_{j\ odd}\left(\sum_{n=0}^{\infty} c_j(n)q^n\right). \tag{7}$$

**Proof.** In order to proof the identity of the theorem we will establish a bijection from partitions in the set $\{\overline{A_n}|n \in \mathbb{N}\}$ to the set of chromatic partition into distinct parts.

From a non-overlined chromatic overpartition into distinct parts, we can split an even part in a power of two odd parts as done in the classical bijective proof of the Euler identity in which it can be found in the chapter 2 of Andrews [2]. Consider a partition in $\overline{A_n}$, as $\lambda_{1,k_1} + \lambda_{2,k_2} + \ldots + \lambda_{j,k_j}$, and denote a colored part by $\lambda_{l,k_l}$. Since every pair of equal parts non-overlined appear in the same color, and these are taken to one part with twice the size, each partition of $\overline{A_n}$ corresponds to a chromatic partition into distinct non-overlined parts . The overlined equal parts, as $\overline{\lambda_{1,k_1} + \lambda_{2,k_2} + \ldots + \lambda_{j,k_j}}$, with $\lambda_{1,k_1} + \lambda_{2,k_2} + \ldots + \lambda_{j,k_j} = m = 2^l$, are mapped in an overlined part $\overline{m}$. Clearly this operation is invertible, thus so characterizing the bijection between the set chromatic partitions in distinct parts and chromatic partitions in odd parts such as those described here.

The generating function for the number of th chromatic partitions into distinct parts is given by the product

$$\prod_{n=1}^{\infty}(1+2nq^n),$$

and the generating function for the number of overpartitions in $\overline{A_n}$ in given by the product as exhibit in lemma 2.8. Since we had proved that the number of overpartitions into distinct parts of $n$ is equal to the number of overpartitions in $\overline{A_n}$, the equation (7) is valid.

The next table exhibit an example of bijective association among all chromatic overpartitions of 5 into distintic parts and partitions in $\overline{A_5}$ as described in the previous theorem.

| distinct parts | partitions in $\overline{A_5}$ | colors |
|---|---|---|
| $5_a$ | $5_a$ | $0 \leq a \leq 4$ |
| $\overline{5_a}$ | $\overline{5_a}$ | $1 \leq a \leq 5$ |
| $4_a + 1_0$ | $1_a + 1_a + 1_a + 1_a + 1_0$ | $1 \leq a \leq 4$ |
| $4_a + \overline{1_0}$ | $1_a + 1_a + 1_a + 1_a + \overline{1_0}$ | $1 \leq a \leq 4$ |
| $\overline{4_a} + 1_0$ | $\overline{1_a + 1_a + 1_a + 1_a} + 1_0$ | $1 \leq a \leq 4$ |
| $\overline{4_a} + \overline{1_0}$ | $\overline{1_a + 1_a + 1_a + 1_a} + \overline{1_0}$ | $1 \leq a \leq 4$ |
| $3_a + 2_b$ | $3_a + 1_b + 1_b$ | $1 \leq a \leq 3, 1 \leq b \leq 2$ |
| $3_a + \overline{2_b}$ | $3_a + \overline{1_b + 1_b}$ | $1 \leq a \leq 3, 1 \leq b \leq 2$ |
| $\overline{3_a} + 2_b$ | $\overline{3_a} + 1_b + 1_b$ | $1 \leq a \leq 3, 1 \leq b \leq 2$ |
| $\overline{3_a} + \overline{2_b}$ | $\overline{3_a} + \overline{1_b + 1_b}$ | $1 \leq a \leq 3, 1 \leq b \leq 2$ |

Table 1: Bijective association between chromatic partitions into distinct parts and partitions in $\overline{A_5}$

Analogous to the proof of the equation (5), we can find many new identities as follows.

**Corollary 3.3.** *For a complex number $q$, where $|q| < 1$ and $b \in \mathbb{N} \cup \{1/2\}$,*

$$\prod_{n=1}^{\infty} \frac{(1 + 2nbq^{2n})}{(1 - 2nbq^{2n})} = 1 + \sum_{n=0}^{\infty} \frac{4nb(1 + 2bq^2)(1 + 4bq^4) \cdots (1 + (2b(n-1))q^{2(n-1)})q^{2n}}{(1 - 2bq^2)(1 - 4bq^4) \cdots (1 - 2nbq^{2n})}. \tag{8}$$

For $b = 1$ the equation (8) refers to the generating function for the number of chromatic overpartitions into even parts. In the next result we use chromatic overpartitions in odd parts.

**Corollary 3.4.** *For all complex number $q$, where $|q| < 1$,*

$$\prod_{n=1}^{\infty} \frac{(1 + (2n+1)q^{2n+1})}{(1 - (2n+1)q^{2n+1})} = 1 + \sum_{n=0}^{\infty} \frac{(4n+2)(1 + q) \cdots (1 + (2n-1)q^{2(n-1)})q^{2n+1}}{(1 - q) \cdots (1 - (2n+1)q^{2n+1})}.$$

**4. Acknowledgements** The author thank the anonymous referee for their valuable suggestions.

## References

[1] Alegri, M., Bijective Proofs Involving Chromatic Overpartitions, Notes on Number Theory and Discrete Mathematics, vol. 25, no.1, 2019, 128-136.

[2] Andrews, G.E., Eriksson, K., Integer Partitions, Cambridge University Press, 2004.

[3] Andrews, G.E., Partitions: yesterday and today, New Zealand Math. Soc., Wellington, 1979.

[4] Andrews, G.E., Askey, R., Roy, R., Special Functions, Cambridge University Press, 1999.

[5] Andrews, G. E., Partitions identities, Advances in Math. 9, 1972, 10–51.

[6] Agarwal, A. K., n-color partitions, Number Theory and Discrete Mathematics (2002), 301–314.

[7] Agarwal, A. K., Andrews, G. E., Rogers-Ramanujan identities for partitions with “N copies of N”, J. Combin. Theory Ser. A 45, (1987), 40–49.

[8] Bressoud, D. M., A new family of partition identities, Pacific J. Math., (1978) 77, 71–74.

[9] Corteel, S., Savelief, C., Vuletíc, M., Plane overpartitions and cylindric partitions, Journal of Combinatorial Theory, Series A Vol. 118, 139-169, 2009.

[10] Euler, L., De partitione numerorum, Introductio in Analysin Infinitorum, Caput XVI (1753), in: Leonardi Euleri Opera Omnia, A. Kratzer, F. Rudio (Eds.), Teubner, Leipzig, 1911.

[11] Lovejoy, J., Corteel, S., Overpartitions, Trans. Amer. Math. Soc. 356, 2004, 1623-1635.

[12] Lovejoy, J., Overpartitions Pairs, Ann. Inst. Fourier (Grenoble), 56 (2006), 781–794.

[13] Lovejoy, J., Partitions and overpartitions with attached parts, Arch. Math., 88(2007), 316–322.

[14] MacMahon, P. A., Seventh Memoir on the Partition of Numbers: A Detailed Study of the Enumeration of the Partitions of Multipartite Numbers, Phil. Trans. R. Soc. London, A CCXVII (1917), 81–113.

[15] Pak, I., Partition Bijections, a Survey, Ramanujan Journal, 12(2006), 5–75.

[16] Schneider, R., Sills, A.V., The Product of Parts or “Norm” of a Partition, arXiv, (2019) 15p.