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# A DIRECT PROOF OF THE AAB-BAILEY LATTICE

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***Dedicated to Prof. K. Srinivasa Rao on his $75^{th}$ Birth Anniversary***

**Abstract:** The purpose of this paper is to give a direct proof of AAB-Bailey lattice.

**Keywords and Phrases:** Bailey pair, identity, AAB Bailey lattice.

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## 1. Introduction

First recall some standard basic hypergeometric notation [8]. For two indeterminate $q$ and $x$ with $|q| < 1$, let

$$(x;q)_\infty == \prod_{n=1}^{\infty}(1-xq^{n-1}),$$

which can be used to define the following shifted factorial:

$$(x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty}.$$

The multiple parameter form is abbreviated as

$$(x_1,x_2,\cdots,x_k;q)_n = (x_1;q)_n(x_2;q)\cdots(x_k;q)_n.$$

The basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left[\begin{array}{ccc}\alpha_1, & \ldots, & \alpha_r\\ \beta_1, & \ldots, & \beta_s\end{array}\bigg| q, z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1,\alpha_2,\cdots\alpha_r;q)_n}{(q,\beta_1,\cdots,\beta_s;q)_n}\{(-1)^n q^{\binom{n}{2}}\}^{1+s-r}z^n.$$

One of the most important summation formula is the sum of a very-well-poised ${}_6\phi_5$ series

$$ {}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & aq^{n+1} \end{matrix}\,\Big|\, q, \frac{aq^{n+1}}{bc}\right] = \frac{(qa, qa/bc; q)_n}{(qa/b, qa/c; q)_n}. \quad (1) $$

The Bailey transform and Bailey lemma play a very important role in the theory and applications of the basic hypergeometric series [3, 8, 12]. Many important identities can be proved by using the Bailey lemma [9, 13, 14]. The Bailey transform was first discovered by Bailey [5]. Slater [11] utilized it to obtain many Rogers-Ramanujan type identities. Subsequently Andrews [2] established the iterative “Bailey chain” concept which led to a wide range of applications. We first give the concept of the Bailey pair in the following.

**Definition 1.1** Let $\alpha = (\alpha_0, \alpha_1, \dots)$ and $\beta = (\beta_0, \beta_1, \dots)$. a pair of sequences $(\alpha, \beta)$ is called a Bailey pair with parameters $a$ if $\alpha_0 = 1$ and

$$ \beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \quad (2) $$

for all $n \geq 0$.

In [2, (4.1)], Andrews gave the following inversion relation:

$$ \alpha_n = (1 - aq^{2n}) \sum_{k=0}^{n} \frac{(aq;q)_{n+k-1}(-1)^{n-k} q^{\binom{n-k}{2}}}{(q;q)_{n-k}} \beta_n. \quad (3) $$

and the following Bailey lemma.

**Lemma 1.2** (Bailey lemma [2]) If $(\alpha, \beta)$ is a Bailey pair relative to a, then so is the new pair$(\alpha', \beta')$ given by

$$ \alpha'_n = \frac{(\rho, \sigma; q)_n (aq/\rho\sigma)^n}{(aq/\rho, aq/\sigma; q)_n} \alpha_n $$

and

$$ \beta'_n = \sum_{r=0}^{n} \frac{(\rho, \sigma; q)_r (aq/\rho\sigma; q)_{n-r} (aq/\rho\sigma)^r}{(q;q)_{n-r}(aq/\rho, aq/\sigma; q)_n} \beta_r, $$

In [1], Agarwal, Andrews and Brewwoud also shown the successive Bailey pairs are necessarily linearly arranged, but that even within the constraints of fixed $\rho$ and $\sigma$ we have several ways of defining a new Bailey pair, giving rise to what they termed a Bailey lattice.

**Theorem 1.3** (AAB Bailey lattice, [1, Lemma 1.2]) Let $(\alpha, \beta)$ be a Bailey pair relative to $a$, and set $\alpha'_{-1} := 0$. If we define $(\alpha', \beta')$ by

$$\alpha'_n = (1-a)(\frac{a}{\rho\sigma})^n \frac{(\sigma, \rho; q)_n}{(a/\rho, a/\sigma; q)_n} \left[\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right] \tag{4}$$

and

$$\beta'_n = \sum_{r=0}^{n} \frac{(\sigma, \rho; q)_r (a/\rho\sigma; q)_{n-r}}{(q;q)_{n-r}(a/\rho, a/\sigma)_n} (\frac{a}{\rho\sigma})^r \beta_r \tag{5}$$

then $(\alpha', \beta')$ is a Bailey pair relative to $aq^{-1}$.

The AAB Bailey lattice plays an important role in the theory of Bailey pair [4, 6, 7, 10]. In [14], Zhang and Huang gave a WP-Bailey lattice similar to that of the AAB lattice. In [15], Zhang and Wu established a $U(n+1)$ extension of the AAB Bailey lattice. The purpose of this note is to give a direct proof of the AAB Bailey lattice.

## 2. A direct proof of the AAB Bailey lattice

By the definition of Bailey pair, we have

$$\begin{aligned}&\sum_{r=0}^{n} \frac{\alpha'_r}{(q;q)_{n-r}(aq;q)_{n+r}} \\ &= \sum_{r=0}^{n} \frac{(1-a)(\sigma, \rho; q)_r (\frac{a}{\rho\sigma})^r}{(q;q)_{n-r}(a;q)_{n+r}(a/\rho, a/\sigma; q)_r} \left[\frac{\alpha_r}{1-aq^{2r}} - \frac{aq^{2r-2}\alpha_{r-1}}{1-aq^{2r-2}}\right].\end{aligned} \tag{6}$$

Letting

$$\Omega = \frac{\alpha_r}{1-aq^{2r}} - \frac{aq^{2r-2}\alpha_{r-1}}{1-aq^{2r-2}},$$

from (3), we have

$$\Omega = \sum_{j=0}^{r} \frac{(aq;q)_{r+j-1}(-1)^{r-j} q^{\binom{r-j}{2}}}{(q;q)_{r-j}} \beta_j - aq^{2r-2} \sum_{j=0}^{r-1} \frac{(aq;q)_{r+j-2}(-1)^{r-j-1} q^{\binom{r-j-1}{2}}}{(q;q)_{r-j-1}} \beta_j.$$

After some simplifications, which yields

$$\Omega = \sum_{j=0}^{r} \frac{(1-aq^{2r-1})(aq;q)_{r+j-2}(-1)^{r-j} q^{\binom{r-j}{2}}}{(q;q)_{r-j}} \beta_j.$$

Then substituting $\Omega$ into the above identity, we have the following result.

$$\begin{aligned}
&\sum_{r=0}^{n}\frac{\alpha'_r}{(q;q)_{n-r}(aq;q)_{n+r}}\\
=&\sum_{r=0}^{n}\frac{(1-a)(\sigma,\rho;q)_r(\frac{a}{\rho\sigma})^r}{(q;q)_{n-r}(a;q)_{n+r}(a/\rho,a/\sigma;q)_r}\sum_{j=0}^{r}\frac{(1-aq^{2r-1})(aq;q)_{r+j-2}(-1)^{r-j}q^{\binom{r-j}{2}}}{(q;q)_{r-j}}\beta_j\\
=&\sum_{j=0}^{n}\beta_j\sum_{r=j}^{n}\frac{(1-a)(\sigma,\rho;q)_r(\frac{a}{\rho\sigma})^r}{(q;q)_{n-r}(a;q)_{n+r}(a/\rho,a/\sigma;q)_r}\frac{(1-aq^{2r-1})(aq;q)_{r+j-2}(-1)^{r-j}q^{\binom{r-j}{2}}}{(q;q)_{r-j}}\\
=&\sum_{j=0}^{n}\beta_j\sum_{r=0}^{n-j}\frac{(1-a)(\sigma,\rho;q)_{r+j}(\frac{a}{\rho\sigma})^{r+j}}{(q;q)_{n-r-j}(a;q)_{n+r+j}(a/\rho,a/\sigma;q)_{r+j}}\frac{(1-aq^{2r+2j-1})(aq;q)_{r+2j-2}(-1)^r q^{\binom{r}{2}}}{(q;q)_r},
\end{aligned}$$

and the second sum in the above identity should be

$$\begin{aligned}
&\frac{(\sigma,\rho;q)_j(aq;q)_{2j-2}(\frac{a}{\rho\sigma})^j}{(q;q)_{n-j}(a;q)_{n+j}(a/\rho,a/\sigma;q)_j}\\
&\times\sum_{r=0}^{n-j}\frac{(1-a)(q^j\sigma,q^j\rho;q)_{r+j}(\frac{a}{\rho\sigma})^r(1-aq^{2r+2j-1})(aq^{2j-1};q)_r(-1)^r q^{\binom{r}{2}}}{(q^{1+n-j};q)_{-r}(aq^{n+j};q)_r(q^ja/\rho,q^ja/\sigma;q)_r(q;q)_r}.
\end{aligned}$$

After some manipulations and by applying the very-well-poised ${}_6\phi_5$ summation formula (1), we obtain

$$\begin{aligned}
&\sum_{r=0}^{n}\frac{\alpha'_r}{(q;q)_{n-r}(aq;q)_{n+r}}\\
=&\sum_{j=0}^{n}\beta_j\frac{(\sigma,\rho;q)_j(\frac{a}{\rho\sigma})^j(a;q)_{2j}}{(q;q)_{n-j}(a;q)_{n+j}(a/\rho,a/\sigma;q)_j}\frac{(q^{2j}a,a/\rho\sigma;q)_{n-j}}{(q^ja/\rho,q^ja/\sigma)_{n-j}}\\
=&\sum_{j=0}^{n}\frac{(\sigma,\rho;q)_j(\frac{a}{\rho\sigma})^j(a/\rho\sigma;q)_{n-j}}{(q;q)_{n-j}(a/\rho,a/\sigma;q)_n}\beta_j\\
=&\beta'_n.
\end{aligned}$$

This completes the proof.

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