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COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS IN DISLOCATED METRIC SPACE

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Dedicated to Prof. K. Srinivasa Rao on his 75^{th} Birth Anniversary

Abstract: In this paper, we discuss the existence and uniqueness of common fixed point and some new common fixed point theorems for two pairs of weakly compatible mappings in a dislocated metric space, our results generalizes and improves many fixed point results in the present literature of fixed point theory in dislocated metric spaces.

Keywords and Phrases: Fixed point, common fixed point, dislocated metric space, weakly compatible maps.

2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and Preliminaries

In 2000, Hitzler, P. and Seda, A. K. [5], introduced the concept of dislocated topology where the initiation of dislocated metric space is appeared. After the concept of dislocated metric space many authors have established fixed point theorem in dislocated metric space, one can see many results in the field of dislocated metric space [4-12]. Hitzler, P. and Seda, A. K. [5], generalized the famous Banach contraction principle [3] in this space. Aage, C.T. and Salunke, J. N. [1] and Isufati, A. [7], established some important fixed point theorems for single and pair of mappings in dislocated metric space. Jungck, G. and Rhoades B.E. [12], introduced the concept of weak compatibility then many interesting fixed point theorems of compatible and weakly compatible maps under various contractive conditions have been obtained by a number of authors. In 2012, Jha, K. and Panthi, D. [8, 9 & 11] have established a common fixed point theorem for two pairs of weakly compatible mappings in dislocated metric space. In 2015 Bennani, et al. [4], established some common fixed point theorems in dislocated metric spaces. Our result generalizes and improves the result of fixed point theorem established by Bennani, et al. [4].

Definition 1.1. [**13**] Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions
1. $d(x, y) = d(y, x)$
2. $d(x, y) = d(y, x) = 0$ implies $x = y$
3. $d(x, y) = d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called dislocated metric (or simply d-metric) on X.

Definition 1.2. [**5**] A sequence $\{x_n\}$ in a d-metric space (X, d) is called a Cauchy sequence if for given $\epsilon > 0$ there exists $n_0 \in N$ such that for all $m, n \geq n_0$ we have $d(x_m, x_n) < \epsilon$.

Definition 1.3. [**5**] A sequence in a d-metric space converges with respect to d (or in d) if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case x is called limit point of $\{x_n\}$ (in d) and we write $x_n \to x$.

Definition 1.4. [**5**] A d-metric space (X, d) is called complete if every Cauchy sequence is convergent.

Definition 1.5. [**12**] Let A and S be two self-mappings of a d-metric space (X, d). A and S are said to be weakly compatible if they commute at their coincident point; that is, $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.

Definition 1.6. [**6**] Let (X, d) be a d-metric space. A map $T : X \to X$ is called contraction mapping if there exists a number λ with $0 \leq \lambda < 1$ such that $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$.

Remark 1.1. It is easy to verify that in a dislocated metric space, we have the following technical properties:

- A subsequence of a Cauchy sequence in d-metric space is a Cauchy sequence.
- A Cauchy sequence in d-metric space which possesses a convergent subsequence, converges.
- Limits in a d-metric space are unique.

Theorem 1.1. [**8**] Let A, B, T and S be four continuous self-mappings of a complete d-metric space (X, d) such that

1. $TX \subset AX$ and $SX \subset BX$;
2. The pairs (S, A) and (T, B) are weakly compatible and
3. $d(Sx, Ty) \leq \alpha d(Ax, Ty) + \beta d(By, Sx) + \gamma d(Ax, By)$ for all $x, y \in X$, where α,β, $\gamma \geq 0$ satisfying $\alpha + \beta + \gamma < \frac{1}{2}$.

Then A, B, T and S have a unique common fixed point in X.

Theorem 1.2. [**11**] Let A, B, T and S be four continuous self-mappings of a complete d-metric space (X, d) such that
1. $TX \subset AX$ and $SX \subset BX$;
2. The pairs (S, A) and (T, B) are weakly compatible;
3. $d(Sx, Ty) \leq \alpha[d(Ax, Ty) + d(By, Sx)] + \beta[d(By, Ty) + d(Ax, Sx)] + \gamma d(Ax, By)$ for all $x, y \in X$, where α,β, $\gamma \geq 0$ satisfying $\alpha + \beta + \gamma < \frac{1}{4}$.

Then A, B, T and S have a unique common fixed point in X.

Theorem 1.3. [4] Let A, B, T and S be four self-mappings of a complete d-metric space (X, d) such that
1. $TX \subset AX$ and $SX \subset BX$;
2. The pairs (S, A) and (T, B) are weakly compatible;
3. $d(Sx, Ty) \leq \alpha d(Ax, Ty) + \beta d(By, Sx)] + \gamma d(Ax, By)$ for all $x, y \in X$, where α,β, $\gamma \geq 0$ satisfying $\alpha + \beta + \gamma < \frac{1}{2}$.
4. The range of one of the mapping A, B, T or S is a complete subspace of X

Then A, B, T and S have a unique common fixed point in X.

2. Main Results

Theorem 2.1. Let A, B, T and $S : X \times X$ be four self-mappings of a complete d-metric space (X, d) such that
1. $TX \subset AX$ and $SX \subset BX$
2. The pairs (S, A) and (T, B) are weakly compatible;
3. $d(Sx, Ty) \leq \alpha d(Ax, Ty) + \beta d(By, Sx) + \gamma d(Ax, By) + \eta d(By, Ty)$ (2.1)

for all $x, y \in X$, where α,β, $\gamma, \eta \geq 0$ satisfying $\alpha + \beta + \gamma + \eta < \frac{1}{2}$.

4. The range of one of the mapping A, B, T or S is a complete subspace of X.

Then A, B, T and S have a unique common fixed point in X.

Proof.
Using condition (i), we define sequences $\{x_n\}$ and $\{y_n\}$ in X by the rule,

$$y_{2n} = Bx_{2n+1} = Sx_{2n} \text{ and } y_{2n+1} = Ax_{2n+2} = Tx_{2n+1};\ n = 0, 1, 2, ...$$

If $y_{2n} = y_{2n+1}$ for some n, then $Bx_{2n+1} = Tx_{2n+1}$. Therefore x_{2n+1} is coincidence

point of B and T. Also, if $y_{2n+1} = y_{2n+2}$ for some n, then $Ax_{2n+2} = Sx_{2n+2}$. Hence x_{2n+2} is coincidence point of A and S. Assume that $y_{2n} \neq y_{2n+1}$ for all n. Then, we have from condition (2.1)
$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})$

$$\leq \alpha d(Ax_{2n}, Tx_{2n+1}) + \beta d(Bx_{2n+1}, Sx_{2n}) + \gamma d(Ax_{2n}, Bx_{2n+1}) + \eta d(Bx_{2n+1}, Tx_{2n+1}) \tag{2.2}$$

$$\begin{aligned}
&\leq \alpha d(y_{2n-1}, y_{2n+1}) + \beta d(y_{2n}, y_{2n}) + \gamma d(y_{2n-1}, y_{2n}) + \eta d(y_{2n}, y_{2n+1})\\
&\leq \alpha[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + \beta[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n})]\\
&\quad + \gamma d(y_{2n-1}, y_{2n}) + \eta d(y_{2n}, y_{2n+1})\\
&\leq (\alpha+\gamma) d(y_{2n-1}, y_{2n}) + (\alpha + 2\beta + \eta) d(y_{2n}, y_{2n+1})
\end{aligned}$$

Therefore,
$$d(y_{2n}, y_{2n+1}) \leq \frac{(\alpha+\gamma)}{(1-\alpha-2\beta-\eta)} d(y_{2n-1}, y_{2n})$$
$= hd(y_{2n-1}, y_{2n})$
Where $h = \dfrac{(\alpha+\gamma)}{(1-\alpha-2\beta-\eta)} < 1$
$d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n})$
This shows that

$$d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n) \leq h^2 d(y_{n-2}, y_{n-1}) \leq h^3 d(y_{n-3}, y_{n-2}) \leq ... \leq h^n d(y_0, y_1)$$

Thus for every integer $q > 0$, we have

$$\begin{aligned}
d(y_n, y_{n+q}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + ... + d(y_{n+q-1}, y_{n+q})\\
&\leq h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + h^{n+2} d(y_0, y_1) + ... + h^{n+q-1} d(y_0, y_1)\\
&\leq h^n [1 + h^1 + h^2 + h^3 + ... + h^{q-1}] d(y_0, y_1)\\
&\leq \frac{h^n}{1-h} d(y_0, y_1)
\end{aligned}$$

Since, $0 < h < 1$, $h^n \to 0$ as $n \to \infty$.
So we get $d(y_n, y_{n+q}) \to 0$. This implies that $\{y_n\}$ is a Cauchy sequence in a complete dislocated metric space, there exists a point $z \in X$ such that $\{y_n\} \to z$. Therefore, according to Remarks 1.1, the sub sequences, $\{Bx_{2n+1}\} \to z$, $\{Sx_{2n}\} \to z$, $\{Ax_{2n+1}\} \to z$ and $\{Tx_{2n+1}\} \to z$.
Since $TX \subset AX$, there exists a point $u \in X$ such that $z = Au$.
Now consider,

$$
\begin{aligned}
d(Su,z) &= d(Su,Tx_{2n+1})\\
&\leq \alpha d(Au,Tx_{2n+1}) + \beta d(Bx_{2n+1},Su) + \gamma d(Au,Bx_{2n+1}) + \eta d(Bx_{2n+1},Tx_{2n+1}) \qquad (2.3)\\
&= \alpha d(z,Tx_{2n+1}) + \beta d(z,Su) + \gamma d(z,Bx_{2n+1}) + \eta d(z,z)\\
&= \alpha d(z,z) + \beta d(z,Su) + \gamma d(z,z) + \eta d(z,z)\\
&= \beta d(z,Su)
\end{aligned}
$$

Now, taking limit as $n \to \infty$, we get, $d(Su,z) \leq \beta d(z,Su)$ which is a contradiction. So, we have $Su = Au = z$.
Again, since $SX \subset BX$, there exists a point $v \in X$ such that $z = Bv$. We claim that $z = Tv$. If $z \neq Tv$, then

$$
\begin{aligned}
d(z,Tv) &= d(Su,Tv)\\
&\leq \alpha d(Au,Tv) + \beta d(Bv,Su) + \gamma d(Au,Bv) + \eta d(Bv,Tv) \qquad (2.4)\\
&= \alpha d(z,Tv) + \beta d(z,z) + \gamma d(z,z) + \eta d(z,Tv)\\
&\leq \alpha d(z,Tv) + \beta[d(z,Tv) + d(Tv,z)] + \gamma[d(z,Tv) + d(Tv,z)] + \eta d(z,Tv)\\
&= (\alpha + 2\beta + 2\gamma + \eta)d(z,Tv)\\
d(z,Tv) &\leq (\alpha + 2\beta + 2\gamma + \eta)d(z,Tv)
\end{aligned}
$$

which is a contradiction.
So, we get $z = Tv$. Hence, we have $Su = Au = Tv = Bv = z$.
Since the pair (S,A) are weakly compatible so by definition $SAu = ASu$ implies $Sz = Az$. Now, we show that z is the fixed point of S. If $Sz \neq z$, then

$$
\begin{aligned}
d(Sz,z) &= d(Sz,Tv)\\
&\leq \alpha d(Az,Tv) + \beta d(Bv,Sz) + \gamma d(Az,Bv) + \eta d(Bv,Tv) \qquad (2.5)\\
&= \alpha d(Sz,z) + \beta d(z,Sz) + \gamma d(Sz,z) + \eta d(z,z)\\
&\leq (\alpha + \beta + \gamma + 2\eta)d(Sz,z)\\
d(Sz,z) &\leq (\alpha + \beta + \gamma + 2\eta)d(Sz,z)
\end{aligned}
$$

which is a contradiction. So, we have $Sz = z$. This implies that $Az = Sz = z$.
Again, the pair (T,B) are weakly compatible, so by definition $TBv = BTv$ implies

$Tz = Bz$. Now, we show that z is the fixed point of T. If $Tz \neq z$, then

$$\begin{aligned} d(z,Tz) &= d(Sz,Tz) \\ &\leq \alpha d(Az,Tz) + \beta d(Bz,Sz) + \gamma d(Az,Bz) + \eta d(Bz,Tz) \qquad (2.6) \\ &= \alpha d(z,Tz) + \beta d(Tz,z) + \gamma d(z,Tz) + \eta d(Tz,Tz) \\ &\leq (\alpha+\beta+\gamma+2\eta) d(z,Tz) \\ d(z,Tz) &\leq (\alpha+\beta+\gamma+2\eta) d(z,Tz) \end{aligned}$$

which is a contradiction. This implies that $z = Tz$. Hence, we have $Az = Bz = Sz = Tz = z$.
This shows that z is the common fixed point of the self-mappings A, B, S and T

Uniqueness:

Let $u \neq v$ be two common fixed points of the mappings A, B, S and T. Then we have,

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\leq \alpha d(Au,Tv) + \beta d(Bv,Su) + \gamma d(Au,Bv) + \eta d(Bv,Tv) \\ &= \alpha d(u,v) + \beta d(v,u) + \gamma d(u,v) + \eta d(v,v) \\ &= (\alpha+\beta+\gamma+2\eta) d(u,v) \\ d(u,v) &\leq (\alpha+\beta+\gamma+2\eta) d(u,v) \end{aligned}$$

a contradiction. This shows that $d(u,v) = 0$
Since (X,d) is a dislocated metric space, so we have $u = v$. This establishes the theorem. From above theorem we can obtain the following corollaries.

Corollary 2.1. Let (X,d) be a complete d-metric space. Let A and S be two self mappings satisfying,
1. $SX \subset AX$
2. The pairs (S,A) is weakly compatible;
3. $d(Sx,Sy) \leq \alpha d(Ax,Sy) + \beta d(Ay,Sx) + \gamma d(Ax,Ay) + \eta d(Ay,Sy)$
for all $x,y \in X$ where $\alpha,\beta,\gamma,\eta \geq 0$ satisfying $\alpha+\beta+\gamma+\eta < \frac{1}{2}$
4. The range of one of the mapping A, or S is a complete subspace of X. Then A and S have a unique common fixed point in X.
Proof: If we take $B = A$ and $S = T$ in theorem 2.1, and follow the similar proof as that in the theorem 2.1, we can establish this corollary.

Corollary 2.2. Let (X,d) be a complete d-metric space. Let S and $T : X \to X$ be two self mappings satisfying,

1. $d(Sx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y) + \eta d(y, Ty)$
for all $x, y \in X$ where $\alpha, \beta, \gamma, \eta \geq 0$ satisfying $\alpha + \beta + \gamma + \eta < \frac{1}{2}$
2. The range of one of the mapping S or T is a complete subspace of X. Then S and T have a unique common fixed point.
Proof: If we take $A = B = I$ an identity mapping in the theorem 2.1, and follow the similar proof as given in the theorem 2.1, we can establish this corollary.

Corollary 2.3: Let (X, d) be a complete d-metric space. Let $A, B : X \to X$ be two self mappings satisfying,
1. $d(x, y) \leq \alpha d(Ax, y) + \beta d(By, x) + \gamma d(Ax, By) + \eta d(By, y)$
for all $x, y \in X$ where $\alpha, \beta, \gamma, \eta \geq 0$ satisfying $\alpha + \beta + \gamma + \eta < \frac{1}{2}$
2. The range of one of the mapping A and B is a complete subspace of X. Then A and B have a unique common fixed point.
Proof: If we take $S = T = I$ an identity mapping in above theorem 2.1 and apply the similar proof as given in the theorem 2.1, we can establish this corollary 2.3.

Remark: Following is the procedure used in the proof of the theorem 2.1, we have the following next result in which we replace the condition $\alpha + \beta + \gamma + \eta < \frac{1}{2}$ by the condition $\alpha + \beta + \gamma + \eta \leq \frac{1}{2}$ for $\alpha, \beta, \gamma, \eta > 0$.

Theorem 2.2. Let A, B, T and $S : X \to X$ be four self-mappings of a complete d-metric space (X, d) such that
1. $TX \subset AX$ and $SX \subset BX$
2. The pairs (S, A) and (T,B) are weakly compatible;
3. $d(Sx, Ty) \leq \alpha d(Ax, Ty) + \beta d(By, Sx) + \gamma d(Ax, By) + \eta d(By, Ty)$ (2.7)
for all $x, y \in X$ where $\alpha, \beta, \gamma, \eta > 0$ satisfying $\alpha + \beta + \gamma + \eta \leq \frac{1}{2}$
4. The range of one of the mapping A, B, T or S is a complete subspace of X. Then A, B, T and S have a unique common fixed point in X.

Corollary 2.4. Let (X, d) be a complete d-metric space. Let A and S be two self mappings satisfying,
1. $SX \subset AX$
2. The pairs (S, A) is weakly compatible;
3. $d(Sx, Sy) \leq \alpha d(Ax, Sy) + \beta d(Ay, Sx) + \gamma d(Ax, Ay) + \eta d(Ay, Sy)$
for all $x, y \in X$ where $\alpha, \beta, \gamma, \eta > 0$ satisfying $\alpha + \beta + \gamma + \eta \leq \frac{1}{2}$
4. The range of one of the mapping A or S is a complete subspace of X. Then A and S have a unique common fixed point in X.
Proof: If we take $B = A$ and $S = T$ in theorem 2.2, and follow the similar proof as given in the theorem 2.1, we can establish this corollary.

Corollary 2.5. Let (X, d) be a complete d-metric space. Let S and $T : X \to X$

be two self mappings satisfying,
1. $d(Sx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y) + \eta d(y, Ty)$
for all $x, y \in X$ where $\alpha, \beta, \gamma, \eta > 0$ satisfying $\alpha + \beta + \gamma + \eta \leq \frac{1}{2}$
2. The range of one of the mapping S or T is a complete subspace of X. Then S and T have a unique common fixed point.
Proof: If we take $A = B = I$ an identity mapping in the theorem 2.2, and follow the similar proof as that in the theorem 2.1, we can establish this corollary.

Corollary 2.6. Let (X, d) be a complete d-metric space. Let $A, B : X \to X$ be two self mappings satisfying,
1. $d(x, y) \leq \alpha d(Ax, y) + \beta d(By, x) + \gamma d(Ax, By) + \eta d(By, y)$
for all $x, y \in X$ where $\alpha, \beta, \gamma, \eta > 0$ satisfying $\alpha + \beta + \gamma + \eta \leq \frac{1}{2}$
2. The range of one of the mapping A and B is a complete subspace of X. Then A and B have a unique common fixed point.
Proof: If we take $S = T = I$ an identity mapping in above Theorem 2.2 and apply the similar proof as given in the theorem 2.1, we can establish this corollary.

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