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SOME FINITE SUMS INVOLVING MULTIPLE GAUSSIAN HYPERGEOMETRIC FUNCTIONS OF EXTON AND SRIVASTAVA

M.I. Qureshi[1], K.A. Quraishi[2], Mohd. Sadiq Khan[3] and R. Singh[4]

[1]Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi-25 (India)
E-mail:- miqureshi_delhi@yahoo.co.in

[2]Mathematics Section, Mewat Engineering Collage (Waqf),
Palla, Nuh, Mewat, Haryana-122107 (India)
E-mail:- kaleemspn@yahoo.co.in

[3]Department of Mathematics, Shibli National College,
Azamgarh, Uttar Pradesh-276001 (India)
E-mail:- msadiqkhan.snc@gmail.com

[4]Department of Applied Sciences and Humanities,
Accurate Institute of Engineering and Technology,
Knowledge Park-III, Greater Noida,
Gautam Buddha Nagar, U.P.-201306(India)

Abstract: In this paper, we obtain some interesting finite sums (not recorded earlier) of general triple hypergeometric series $F^{(3)}$ of Srivastava in terms of general double hypergeometric series G of Exton, by series rearrangement technique.

Keywords and Phrases: Pochhammer's symbol; Double hypergeometric functions of Exton and Srivastava; Combinatorial identity; Series iteration technique.

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1. Introduction

In 1967, Srivastava[17,p.428] defined the general triple hypergeometric function $F^{(3)}$ in the following form:

$$F^{(3)}\left[\begin{array}{c} (a_A) :: (b_B);\ (d_D);\ (e_E) : (g_G);\ (h_H);\ (\ell_L); \\ (q_Q) :: (r_R);\ (s_S);\ (t_T) : (u_U);\ (v_V);\ (w_W); \end{array} x,\ y,\ z\right]$$

$$= \sum_{i,j,k=0}^{\infty} \frac{[(a_A)]_{i+j+k}\,[(b_B)]_{i+j}\,[(d_D)]_{j+k}\,[(e_E)]_{k+i}\,[(g_G)]_i\,[(h_H)]_j\,[(\ell_L)]_k\ x^i\ y^j\ z^k}{[(q_Q)]_{i+j+k}\,[(r_R)]_{i+j}\,[(s_S)]_{j+k}\,[(t_T)]_{k+i}\,[(u_U)]_i\,[(v_V)]_j\,[(w_W)]_k\ i!\ j!\ k!} \quad (1.1)$$

which is the unification and generalization of Lauricella's complete fourteen triple hypergeometric functions of second order $F_1,\ F_2,\ F_3,\ \cdots,\ F_{14}$ [6, pp. 113-114] including Saran's ten triple hypergeometric functions $F_E,\ F_F,\ F_G,\ F_K,\ F_M,\ F_N,\ F_P,\ F_R,\ F_S,\ F_T$ [7;8], extended triple hypergeometric function F_K of Sharma [9, p. 613(2)] and three additional triple hypergeometric functions $H_A,\ H_B,\ H_C$ of Srivastava [16, pp. 99-100; see also 13; 14; 15; 18].

In 1984, Exton [3,p.113(1.2); see also 1, p. 12 (1.7.1)] defined the double hypergeometric function G in the following form:

$$G^{A:B;D}_{E:H;M}\left[\begin{array}{l} (a_A) : (b_B);\ (d_D); \\ \\ (e_E) : (h_H);\ (m_M); \end{array} x; y\right]$$

$$= F^{A:B;D}_{E:H;M}\left(\begin{array}{l} [(a_A):1,-1] : [(b_B):1];\ [(d_D):1]; \\ \\ [(e_E):1,-1] : [(h_H):1];\ [(m_M):1]; \end{array} x; y\right)$$

$$= \sum_{i,j=0}^{\infty} \frac{[(a_A)]_{i-j}[(b_B)]_i[(d_D)]_j x^i y^i}{[(e_E)]_{i-j}[(h_H)]_i[(m_M)]_j i! j!} \quad (1.2)$$

It is the generalization and unification of Horn's non confluent double hypergeometric functions G_2 [2, p. 224(11)], H_2 [2, p. 225 (14)] and Horn's confluent double hypergeometric functions $\Gamma_1, \Gamma_2, H_2, H_3, H_4, H_5, H_{11}$ [2, pp. 226-227(27, 28, 30, 31, 32, 33, 39); see also 4; 5].

The notation $F^{A:B;D}_{E:H;M}(\cdot)$ in (1.2) is due to Srivastava-Daoust [21, pp. 64-65 (1.7.18, 1.7.19, 1.7.20)]. The symbol (a_A) denotes the array of A parameters in Slater's contracted notation [11, p. 54; see also 12, p. 41] given by $a_1, a_2, ..., a_A$ with similar interpretations for others.

The Pochhammer's symbol $[(b_B)]$ is defined by:

$$[(b_B)]_u = (b_1)_u(b_2)_u...(b_B)_u = \prod_{m=1}^{B}\{(b_m)_u\}$$

$$= \prod_{m=1}^{B}\left\{\frac{\Gamma(b_m+u)}{\Gamma(b_m)}\right\}, \quad \text{if} \quad b_m \neq 0, -1, -2, -3, ...$$

$$= \prod_{m=1}^{B} \{(b_m)(b_m+1)(b_m+2)...(b_m+u-1)\}, \quad \text{if} \quad u = 1, 2, 3, ...$$

and

$$(b_m)_0 = 1 \tag{1.3}$$

with similar interpretations for others. The notation Γ is used for Gamma function. The denominator parameters in (1.1) and (1.2) are neither zero nor negative integers, numerator parameters may be zero or negative integers.

In our investigation, we shall use the following results:

$$\sum_{n=0}^{m} \sum_{r=0}^{n} \sum_{s=0}^{m-n} \Phi(m, n, r, s) = \sum_{s=0}^{m} \sum_{r=0}^{m-s} \sum_{n=0}^{m-s-r} \Phi(m, n+r, r, s) \tag{1.4}$$

$$\sum_{n=0}^{k} (-1)^n \begin{pmatrix} k \\ n \end{pmatrix} = \sum_{n=0}^{k} \frac{(-k)_n}{n!} = \begin{cases} 1; & \text{if} \quad k = 0 \\ 0; & \text{if} \quad k = 1, 2, 3, \cdots \end{cases} \tag{1.5}$$

where $\begin{pmatrix} k \\ n \end{pmatrix}$ is binomial coefficient.

$$\sum_{s=0}^{m} \sum_{r=0}^{m-s} \frac{\Psi(r+s) x^s y^r}{s! r!} = \sum_{p=0}^{m} \frac{\Psi(p)(x+y)^p}{p!} \tag{1.6}$$

The finite triple series identity (1.4) and combinatorial identity (1.5) are due to Srivastava [20, pp. 95-96 (7.13, 7.14)]. The finite double series identity (1.6) is due to Srivastava [19, p. 4 (12, 15); see also 10, p. 41 (1)].

The following Pochhammer's symbols identities used in the derivations of (2.1) and (2.2), can be proved in view of the definition (1.3) of Pochhammer's symbol:

$$[1-(b_B)-n-r]_r = \frac{(-1)^{Br}[(b_B)]_{n+r}}{[(b_B)]_n} \tag{1.7}$$

$$[1-(e_E)-n-r]_{i+r} = \frac{[(e_E)]_{n+r}[1-(e_E)]_{i-n}}{(-1)^{E(n+r)}} \tag{1.8}$$

$$(-n-r)_r = \frac{(-1)^r (n+r)!}{n!} \tag{1.9}$$

$$[(k_K)+m]_{-s} = \frac{(-1)^{Ks}}{[1-(k_K)-m]_s} \tag{1.10}$$

$$[(a_A)]_{m+n+p} = [(a_A)]_m [(a_A)+m]_n [(a_A)+m+n]_p \tag{1.11}$$

2. Main Finite Summation Formulae

Since Pochhammer symbols are associated with Gamma functions therefore numerator, denominator parameters and arguments are adjusted in such a way that each side is completely meaningful and defined, then without any loss of convergence, we have:

$$\sum_{n=0}^{m}\frac{(-m)_n}{n!}\ F^{(3)}\left[\begin{array}{l}(a_A):: (d_D); -; (w_W): -n, (k_K); -m+n, (p_P); (t_T);\\ (b_B) :: (e_E); -;\ (h_H) : (\ell_L);\ (q_Q);\ (u_U);\end{array} x, y, z\right]$$

$$= \frac{[(a_A)]_m[(d_D)]_m[(w_W)]_m[(k_K)]_m x^m}{[(b_B)]_m[(e_E)]_m[(h_H)]_m[(\ell_L)]_m} G^{W:A+T;1+L+P}_{H:B+U;K+Q}\left[\begin{array}{l}(w_W)+m:(a_A)+m, (t_T);\\ (h_H)+m :(b_B)+m, (u_U);\end{array}\right.$$

$$\left.\begin{array}{l}-m, 1-(\ell_L)-m, (p_P);\\ 1-(k_K)-m, (q_Q);\end{array} z, (-1)^{(K-L)}(\tfrac{y}{x})\right] \tag{2.1}$$

$$\sum_{n=0}^{m}\frac{(-1)^{n(B+D+E+T)}(-m)_n[(a_A)]_n[(d_D)]_n[(t_T)]_n(\frac{y}{x})^n}{n![(q_Q)]_n[(b_B)]_n[(e_E)]_n}\ F^{(3)}\left[\begin{array}{l}-::(g_G); -;\\ - ::(h_H); -;\end{array}\right.$$

$$\left.\begin{array}{l}1-(e_E)-n: -n, 1-(b_B)-n; -m+n, (a_A)+n; (\ell_L);\\ 1-(t_T)-n : 1-(d_D)-n;\ (q_Q)+n;\ (k_K);\end{array} x, y, z\right]$$

$$= G^{E:L;1+A+D}_{T:K;Q+B}\left[\begin{array}{l}1-(e_E):(\ell_L); -m, (a_A), (d_D);\\ 1-(t_T):(k_K); (q_Q), (b_B);\end{array} z, (-1)^{(B+D)}(\tfrac{y}{x})\right] \tag{2.2}$$

3. Derivations of (2.1) and (2.2)

Expressing Srivastava's triple hypergeometric function $F^{(3)}$ of left hand side of (2.1), into its power series form, we get;

$$V_1 = \sum_{n=0}^{m}\frac{(-m)_n}{n!}\sum_{r=0}^{n}\sum_{s=0}^{m-n}\sum_{i=0}^{\infty}\frac{[(a_A)]_{r+s+i}[(d_D)]_{r+s}[(w_W)]_{i+r}(-n)_r}{[(b_B)]_{r+s+i}[(e_E)]_{r+s}[(h_H)]_{i+r}}\times$$

$$\times\frac{[(k_K)]_r(-m+n)_s[(p_P)]_s[(t_T)]_i x^r y^s z^i}{[(\ell_L)]_r[(q_Q)]_s[(u_U)]_i r!s!i!}$$

$$= \sum_{i=0}^{\infty}\sum_{s=0}^{m}\sum_{r=0}^{m-s}\sum_{n=0}^{m-r-s}\frac{(-m)_{n+r+s}[(a_A)]_{r+s+i}[(d_D)]_{r+s}[(w_W)]_{i+r}}{[(b_B)]_{r+s+i}[(e_E)]_{r+s}[(h_H)]_{i+r}n!}\times$$

$$\times\frac{(-1)^r[(k_K)]_r[(p_P)]_s[(t_T)]_i x^r y^s z^i}{[(\ell_L)]_r[(q_Q)]_s[(u_U)]_i r!s!i!}$$

$$=\sum_{i=0}^{\infty}\sum_{s=0}^{m}\frac{(-m)_s[(a_A)]_{i+s}[(d_D)]_s[(w_W)]_i[(p_P)]_s[(t_T)]_i y^s z^i}{[(b_B)]_{i+s}[(e_E)]_s[(h_H)]_i[(q_Q)]_s[(u_U)]_i s!i!}\left[\sum_{r=0}^{m-s}\frac{(-1)^r(-m+s)_r}{[(b_B)+i+s]_r}\times\right.$$

$$\left.\times\frac{[(a_A)+i+s]_r[(d_D)+s]_r[(w_W)+i]_r[(k_K)]_r x^r}{[(e_E)+s]_r[(h_H)+i]_r[(\ell_L)]_r r!}\left\{\sum_{n=0}^{m-s-r}\frac{(-m+s+r)_n}{n!}\right\}\right] \quad (3.1)$$

When r varies from 0 to $m-s-1$, corresponding terms of surly bracket in (3.1) are zero in the light of combinatorial identity (1.5). When $r=m-s$ in (3.1), we get:

$$V_1=\sum_{i=0}^{\infty}\sum_{s=0}^{m}\frac{(-m)_m[(a_A)]_{m+i}[(d_D)]_m[(w_W)]_{i+m-s}(-1)^{m-s}}{[(b_B)]_{m+i}[(e_E)]_m[(h_H)]_{i+m-s}}\times$$

$$\times\frac{[(k_K)]_{m-s}[(p_P)]_s[(t_T)]_i x^{m-s} y^s z^i}{[(\ell_L)]_{m-s}[(q_Q)]_s[(u_U)]_i(m-s)!s!i!}$$

$$=\frac{[(a_A)]_m[(d_D)]_m[(w_W)]_m[(k_K)]_m x^m}{[(b_B)]_m[(e_E)]_m[(h_H)]_m[(\ell_L)]_m}\sum_{i=0}^{\infty}\sum_{s=0}^{m}\frac{[(w_W)+m]_{i-s}[(a_A)+m]_i}{[(h_H)+m]_{i-s}[(b_B)+m]_i}\times$$

$$\times\frac{[(t_T)]_i(-m)_s[1-(\ell_L)-m]_s[(p_P)]_s z^i(-1)^{(K-L)s}}{[(u_U)]_i[1-(k_K)-m]_s[(q_Q)]_s i!s!}\left(\frac{y}{x}\right)^s$$

Now expressing above double power series into corresponding hypergeometric form, we get the right hand side of (2.1).

Similarly expressing $F^{(3)}$ of left hand side of (2.2) into its power series form, we get

$$V_2=\sum_{i=0}^{\infty}\sum_{n=0}^{m}\sum_{r=0}^{n}\sum_{s=0}^{m-n}\frac{(-m)_{n+s}[(a_A)]_{n+s}[(d_D)]_n[(t_T)]_n[(g_G)]_{r+s}[1-(e_E)-n]_{i+r}}{n![(q_Q)]_{n+s}[(b_B)]_n[(e_E)]_n[(h_H)]_{r+s}[1-(t_T)-n]_{i+r}}\times$$

$$\times\frac{(-n)_r[1-(b_B)-n]_r[(\ell_L)]_i x^{r-n} y^{s+n} z^i(-1)^{n(B+D+E+T)}}{[1-(d_D)-n]_r[(k_K)]_i r!s!i!}$$

$$=\sum_{i=0}^{\infty}\frac{[1-(e_E)]_i[(\ell_L)]_i z^i}{[1-(t_T)]_i[(k_K)]_i i!}\sum_{s=0}^{m}\sum_{r=0}^{m-s}\left\{\frac{(-m)_{s+r}[(a_A)]_{s+r}[(g_G)]_{s+r}y^s(-y)^r}{[(q_Q)]_{s+r}[(h_H)]_{s+r}s!r!}\times\right.$$

$$\left.\times\sum_{n=0}^{m-s-r}\frac{(-m+s+r)_n[(a_A)+s+r]_n}{n![(q_Q)+s+r]_n}\right\}\frac{[1-(e_E)+i]_{-n}[(d_D)]_n(-1)^{n(B+D)}(\frac{y}{x})^n}{[1-(t_T)+i]_{-n}[(b_B)]_n}$$

Now putting $r+s=p$ and applying identity (1.6), we get;

$$V_2=\sum_{i=0}^{\infty}\frac{[1-(e_E)]_i[\ell_L]_i z^i}{[1-(t_T)]_i[(k_K)]_i i!}\sum_{p=0}^{m}\frac{(-m)_p[(a_A)]_p[(g_G)]_p[(y)+(-y)]^p}{[(q_Q)]_p[(h_H)]_p p!}\times$$

$$\times\sum_{n=0}^{m-p}\frac{(-m+p)_n[(a_A)+p]_n[1-(e_E)+i]_{-n}[(d_D)]_n(-1)^{n(B+D)}(\frac{y}{x})^n}{n![(q_Q)+p]_n[1-(t_T)+i]_{-n}[(b_B)]_n}$$

Since argument corresponding to summation index p is zero, therefore we have

$$V_2=\sum_{i=0}^{\infty}\frac{[1-(e_E)]_i[\ell_L]_i z^i}{[1-(t_T)]_i[(k_K)]_i i!}\sum_{n=0}^{m}\frac{(-m)_n[(a_A)]_n[1-(e_E)+i]_{-n}[(d_D)]_n(-1)^{n(B+D)}(\frac{y}{x})^n}{n![(q_Q)]_n[1-(t_T)+i]_{-n}[(b_B)]_n}$$

$$=\sum_{i=0}^{\infty}\sum_{n=0}^{m}\frac{[1-(e_E)]_{i-n}[\ell_L]_i(-m)_n[(a_A)]_n[(d_D)]_n z^i(-1)^{n(B+D)}(\frac{y}{x})^n}{[1-(t_T)]_{i-n}[(k_K)]_i[(q_Q)]_n[(b_B)]_n i!n!}$$

Now expressing above double power series into its hypergeometric form, we get the right hand side of (2.2).

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