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Spherical Gravitational Collapse of a Radiating Star

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Dedicated to Prof. Hari M. Srivastava on his 75^{th} ***birth anniversary***

Abstract: In the present paper we have obtained a new model by using the Tewari [1] algorithm for a collapsing radiating star and the matching conditions required for the description of physically meaningful fluid. The interior matter fluid is shear-free spherically symmetric isotropic and undergoing radial heat flow. The interior metric obeyed all the relevant physical and thermodynamic conditions and matched with Vaidya exterior metric over the boundary. Initially the interior solutions represent a static configuration of perfect fluid which then gradually starts evolving into radiating collapse. The apparent luminosity as observed by the distant observer at rest at infinity and the effective surface temperature are zero in remote past at the instant when the collapse begins and at the stage when collapsing configuration reaches the horizon of the black hole.

Keywords: Exact solutions, Collapsing radiating star, Black hole.

1. Introduction

When a body does not have substantially strong pressure gradient force, it may continue collapsing because of its own gravity, this phenomena is called gravitational collapse. It is one of the important issues in relativistic astrophysics whether the end state of gravitational collapse is a black hole or a naked singularity (Joshi and Malafarina [2] and references therein). In relativistic astrophysics, a detailed description of gravitational collapse of massive stars and the modeling of the structure of compact objects under various conditions is the most interesting phenomena.The study of the gravitational collapse was started by Oppenheimer and Snyder [3], in which they assumed a spherically symmetric distribution of state in the form of dust with Schwarzschild exterior. Later on taking into account the outgoing radiation from collapsing spherical fluid Vaidya [4] initiated the problem and the modified equations proposed by Misner [5] for an adiabatic distribution of matter.

It is an established fact that gravitational collapse is a high energy dissipating process (Herrera and Santos [6]; Herrera et al. [7]; Mitra [8] and references therein) which plays a dominant role in the formation and evolution of stars. However, the dissipation of energy from collapsing fluid distribution is described in two limiting cases. The first case describes the free streaming and a number of models of radiating stars in this case discussed by Tewari ([9]-[12]). While second one is diffusion approximation and in this case the dissipation is modeled by heat flow type vector and in this the model proposed by Glass [13] has been extensively studied by Santos [14] for the junction conditions of collapsing spherically symmetric shear-free non-adiabatic fluid with radial heat flow. On a similar ground a number of stellar models [de Oliveira et al. [15]; Bonnor et al. [16]; Banerjee et al. [17]; Herrera et al. [18]; Tewari [19]; Sharif and Abbas [20]; Tewari and Charan ([21]-[23]) and also references therein have been reported with the impact of various dissipative processes on the evolution.

Keeping in view of generality of solution due to Tewari [1] we present a special solution and its detailed study, in order to construct a realistic model of collapsing radiating star. The interior space-time metric is matched with Vaidya exterior metric [4] over the boundary, and the final fate of our model is formation of a black hole. The paper is organised as follows: In sec. 2 the field equations and the junction conditions which match the interior metric of the collapsing fluid with the exterior metric are given. In section 3 a new class of exact solutions of the field equations are presented. In section 4 a detailed study of a class of solutions for a collapsing radiating star is given and finally in section 5 some concluding remarks have been made.

2. The field equations and junction conditions

The metric in the interior of a shear-free spherically symmetric fluid distribution is given by

$$ds_-^2 = -A^2(r,t)dt^2 + B^2(r,t)\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\} \tag{1}$$

The energy-momentum tensor for the matter distribution with isotropy in pressure is

$$T_{\mu\nu} = (\epsilon + p)w_\mu w_\nu + p g_{\mu\nu} + q_\mu w_\nu + q_\nu w_\mu \tag{2}$$

where ϵ is the energy density of the fluid, p the isotropic pressure, w_μ is the four velocity and q_μ the radial heat flux vector. Assuming comoving coordinates, we have $w^\mu = \delta_0^\mu$. The heat flow vector q^μ is orthogonal to the velocity vector so that $q^\mu w_\mu = 0$ and takes the form $q^\mu = q\delta_1^\mu$.

The line element (1) corresponds to shear- free spherically symmetric fluid (Glass [24]), as the shear tensor vanishes identically. The fluid collapse rate $\Theta = w^{\mu}_{;\mu}$ of the fluid distribution (1) is given by

$$\Theta = \frac{3\dot{B}}{AB} \tag{3}$$

Non-trivial Einsteins field equations in view of (1) and (2) are given by following system of equations

$$\kappa\epsilon = -\frac{1}{B^2}\Big(\frac{2B''}{B} - \frac{B'^2}{B^2} + \frac{4B'}{rB}\Big) + \frac{3\dot{B}^2}{A^2B^2} \tag{4}$$

$$\kappa p = \frac{1}{B^2}\Big(\frac{B'^2}{B^2} + \frac{2A'B'}{AB} + \frac{2A'}{AB} + \frac{2A'}{rA} + \frac{2B'}{rB}\Big) + \frac{1}{A^2}\Big(-\frac{2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{2\dot{A}\dot{B}}{AB}\Big) \tag{5}$$

$$\kappa p = \frac{1}{B^2}\Big(\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{B'}{rB} + \frac{A''}{A} + \frac{A'}{rA}\Big) + \frac{1}{A^2}\Big(-\frac{2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{2\dot{A}\dot{B}}{AB}\Big) \tag{6}$$

$$\kappa q = \frac{-2}{AB^2}\Big(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'\dot{B}}{AB}\Big) \tag{7}$$

here and hereafter the primes and dots stand for differentiation with respect to r and t respectively. The coupling constant in geometrized units is $\kappa = 8\pi (i.e. G = c = 1)$.

The exterior space-time is described by Vaidyas exterior metric [4] which represents an outgoing radial flow of radiation

$$ds^2_+ = -\Big(1 - \frac{2M(v)}{R}\Big)dv^2 - 2dRdv + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{8}$$

where v is the retarded time and M(v) is the exterior Vaidya mass.

The junction conditions for matching two line elements (1) and (8) continuously across a spherically symmetric time-like hyper surface Σ are well known and obtained by Santos [14]

$$(rB)_\Sigma = R_\Sigma(v) = \mathcal{R}(\tau) \tag{9}$$

$$(p_r)_\Sigma = (qB)_\Sigma \tag{10}$$

$$m_\Sigma(r,t) = M(v) = \left\{ \frac{r^3B\dot{B}^2}{2A^2} - r^2B' - \frac{r^3B'^2}{2B} \right\}_\Sigma \tag{11}$$

where m_Σ is the mass function calculated in the interior at $r = r_\Sigma$ (Cahill et al. [25], Misner and Sharp [26]).

Some other characteristics of the model such as the surface luminosity and the boundary redshift z_Σ observed on Σ are

$$L_\Sigma = \tfrac{\kappa}{2}\{r^2 B^3 q\}_\Sigma \tag{12}$$

$$z_\Sigma = \left[1 + \tfrac{rB'}{B} + \tfrac{r\dot{B}}{A}\right]_\Sigma^{-1} - 1 \tag{13}$$

The total luminosity for an observer at rest at infinity is

$$L_\infty = -\frac{dM}{dv} = \frac{L_\Sigma}{(1 + z_\Sigma)^2} \tag{14}$$

3. Solution of the field equations

In order to solve the field equations we choose a particular form of the metric coefficients given in (1) into functions of r and t coordinates as $A(r,t) = A_0(r)g(t)$ and $B(r,t) = B_0(r)f(t)$.

In view of the above metric coordinates the Einstein's field equations (6)-(9) lead to the following system of equations

$$\kappa\epsilon = \tfrac{\epsilon_0}{f^2} + \tfrac{3\dot{f}^2}{A_0^2 g^2 f^2} \tag{15}$$

$$\kappa p = \tfrac{p_0}{f^2} + \tfrac{1}{A_0^2 g^2}\left(- \tfrac{2\ddot{f}}{f} - \tfrac{\dot{f}^2}{f^2}\right) \tag{16}$$

where

$$\epsilon_0 = -\tfrac{1}{B_0^2}\left(\tfrac{2B_0''}{B_0} - \tfrac{B_0'^2}{B_0^2} + \tfrac{4B_0'}{rB_0}\right) \tag{17}$$

$$p_0 = \tfrac{1}{B_0^2}\left(\tfrac{B_0'^2}{B_0^2} + \tfrac{2B_0'}{rB_0} + \tfrac{2A_0'B_0'}{A_0 B_0} + \tfrac{2A_0'}{rA_0}\right) \tag{18}$$

here the quantities with the suffix 0 corresponds to the static star model with metric components $A_0(r),\ B_0(r)$.

In the absence of dissipative forces the equation (10), $(p)_\Sigma = (qB)_\Sigma$, reduces to the condition $[p_0]_\Sigma = 0$ and yields at $r = r_\Sigma = R_\Sigma$

$$\frac{2\ddot{f}}{f} + \frac{\dot{f}^2}{f^2} - \frac{2\dot{g}\dot{f}}{gf} = \frac{2\alpha g\dot{f}}{f^2} \tag{19}$$

where

$$\alpha = \left(\frac{A_0'}{B_0}\right)_\Sigma \tag{20}$$

To solve the equation (19), by assuming $g(t) = f(t)$ (Tewari [1]) obtained the following solution

$$\dot{f} = -2\alpha\sqrt{f}(1-\sqrt{f}) \tag{21}$$

$$t = \tfrac{1}{\alpha} ln(1-\sqrt{f}) \tag{22}$$

We observed that the function $f(t)$ decreases monotonically from the value $f(t) = 1$ at $t = -\infty$ to $f(t) = 0$ at $t = 0$.

The isotropy of pressure would give the equation

$$\frac{A_0''}{A_0} + \frac{B_0''}{B_0} = \left(\frac{2B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right) \tag{23}$$

where the quantities with the suffix 0 corresponds to the static star model metric components $A_0(r), B_0(r)$.

The new parametric class of solutions of equation (23) obtained by Tewari [1] is

$$A_0 = D_2(1+C_1r^2)^{\frac{n}{l+1}} + D_1(1+C_1r^2)^{\frac{2-n}{l+1}+1} \tag{24}$$

$$B_0 = C_2(1+C_1r^2)^{\frac{1}{l+1}} \tag{25}$$

$$n = \frac{1}{2}\left\{(l+3) \pm (l^2+10l+17)^{\frac{1}{2}}\right\} \tag{26}$$

where n, l, C_1, C_2, D_1 and D_2 are constants and n is real if $l \geq -5+2\sqrt{2}$ or $l \leq -5-2\sqrt{2}$.

One can arrive at a number of solutions for different values of n from above class of solutions. When $n = 0$, we rediscover the Schwarzschild interior solution and dissipative collapsing model in this case has been studied by de Oliveira et al. [15] and Bonner et al. [16]. When $n = -1$,it reduces into Banerjee et al. [17] solution, for $n = -2, -3/2$, it reduces to Tewari ([19], [1]), for $n = -1-\sqrt{2}$, horizon-free case studied by Tewari and Charan [21] and one more case for $n = -5/3$ discussed in detail by Tewari and Charan [22] in which they present a Supernovae model.

4. Physical analysis of the model

In order to construct the new realistic model we assume $n = -6/5$ and, from (24) and (25) we obtain,

$$A_0 = D_2(1+C_1r^2)^{\frac{6}{71}} + D_1(1+C_1r^2)^{\frac{55}{71}} \tag{27}$$

$$B_0 = C_2(1 + C_1 r^2)^{\frac{-5}{71}} \tag{28}$$

In view of (27) and (28) the equations (17) and (18) reduces in following expressions

$$\epsilon_0 = \frac{20C_1}{5041C_2^2(1 + C_1 r^2)^{\frac{132}{71}}}(213 + 66C_1 r^2) \tag{29}$$

$$p_0 = \frac{4C_1}{5041C_2^2(1 + C_1 r^2)^{\frac{132}{71}}}[(71 + 36C_1 r^2) +$$

$$\frac{49D_1(1 + C_1 r^2)^{\frac{49}{71}}}{\{D_2 + D_1(1 + C_1 r^2)^{\frac{49}{71}}\}}(71 + 61C_1 r^2)] \tag{30}$$

The junction condition $[p_0]_\Sigma = 0$ gives

$$D_2 = -\frac{25D_1(1 + C_1 r_\Sigma^2)^{\frac{49}{71}}(142 + 121C_1 r_\Sigma^2)}{(71 + 36C_1 r_\Sigma^2)} \tag{31}$$

We observed that $\epsilon_0 > 0$, $p_0 > 0$, $\frac{p_0}{\epsilon_0} < 1$, $\epsilon_0' < 0$, $p_0' < 0$ at the centre are satisfied with suitable choice of constants $C_1 > 0$, $C_2 > 0$, $D_2 > 0$, $D_1 < 0$ and $D_2 > -50D_1$. The total energy inside Σ for the static system

$$m_0 = \frac{10C_1 C_2 r_\Sigma^3(71 + 66C_1 r_\Sigma^2)}{5041(1 + C_1 r^2)^{\frac{147}{71}}} \tag{32}$$

Now the explicit expressions for ϵ, p, q, and Θ become

$$\epsilon = \frac{\epsilon_0}{f^2} + \frac{12\alpha^2}{A_0^2}\frac{(1 - \sqrt{f})^2}{f^3} \tag{33}$$

$$p = \frac{p_0}{f^2} + \frac{4\alpha^2}{A_0^2}\frac{(1 - \sqrt{f})}{f^{\frac{5}{2}}} \tag{34}$$

$$q = \frac{4C_1 r^2\{(\frac{6}{71})D_2 + \frac{55}{71}D_1(1 + C_1 r^2)^{\frac{49}{71}}\}}{C_2^2(1 + C_1 r^2)^{67/71}\{D_2 + D_1(1 + C_1 r^2)^{\frac{49}{71}}\}}\frac{2\alpha(1 - \sqrt{f})}{f^{\frac{7}{2}}} \tag{35}$$

$$\Theta = \frac{-6\alpha(1 - \sqrt{f})}{(1 + C_1 r^2)^{\frac{6}{71}}\{D_2 + D_1(1 + C_1 r^2)^{\frac{49}{71}}\}f^{\frac{3}{2}}} \tag{36}$$

where

$$\alpha = -\frac{490}{71} \frac{D_1 C_1 r_\Sigma}{C_2 (1 + C_1 r_\Sigma^2)^{\frac{11}{71}}} \frac{(71 + 66 C_1 r_\Sigma^2)}{(71 + 36 C_1 r_\Sigma^2)} \tag{37}$$

We can see the physical parameters ϵ, p, q are finite, positive monotonically decreasing at any instant with respect to radial coordinate for $0 \leq r \leq r_\Sigma$. Initially collapse is zero and it becomes infinite at final phase of the configuration.

The total energy entrapped inside Σ is given by

$$M(v) = \left[2 \left(\frac{10}{71} \right)^2 \frac{C_2 C_1^2 r_\Sigma^5 (71 + 66 C_1 r_\Sigma^2)^2}{(1 + C_1 r_\Sigma^2)^{\frac{147}{71}} (71 + 61 C_1 r_\Sigma^2)^2} (1 - \sqrt{f})^2 + m_0 f \right] \tag{38}$$

The luminosity and the red shift observed on Σ and luminosity observed by a distant observer are given by

$$L_\Sigma = 2 \left(\frac{10}{71} \right)^2 \frac{C_1^2 r_\Sigma^4 (71 + 66 C_1 r_\Sigma^2)^2}{(1 + C_1 r_\Sigma^2)^2 (71 + 61 C_1 r_\Sigma^2)^2} \frac{(1 - \sqrt{f})^2}{\sqrt{f}} \tag{39}$$

$$L_\infty = 2 \left(\frac{10}{71} \right)^2 \frac{C_1^2 r_\Sigma^4 (71 + 66 C_1 r_\Sigma^2)^2}{(1 + C_1 r_\Sigma^2)^2 (71 + 61 C_1 r_\Sigma^2)^2} \frac{(1 - \sqrt{f})^2}{\sqrt{f}} \frac{1}{(1 + z_\Sigma)^2} \tag{40}$$

$$z_\Sigma = \frac{\left\{ \frac{(71 + 66 C_1 r_\Sigma^2)^2 \sqrt{f} + 20 C_1 r_\Sigma^2 (71 + 66 C_1 r_\Sigma^2)(1 - \sqrt{f})}{71 \sqrt{f} (1 + C_1 r_\Sigma^2)(71 + 61 C_1 r^2 \Sigma)} \right\}_\Sigma}{(1 - \frac{2M}{r B_0 f})_\Sigma} - 1 \tag{41}$$

The above expressions show that L_∞ vanishes in the beginning when$f(t) \rightarrow 1$ and at the stage when$z_\Sigma \rightarrow \infty$.

We obtain the black hole formation time as

$$\sqrt{f_{BH}} = \frac{20 C_1 r_\Sigma^2 (71 + 66 C_1 r_\Sigma^2)}{5041 (1 + C_1 r_\Sigma^2)^2} \tag{42}$$

and

$$t_{BH} = \frac{1}{\alpha} In \frac{(71 + 61 C_1 r_\Sigma^2)^2}{5041 (1 + C_1 r_\Sigma^2)^2} \tag{43}$$

The effective surface temperature observed by external observer can be calculate similar as Tewari [1]

$$T_\Sigma^4 = \frac{200}{5041\pi\delta C_2^2} \frac{C_1^2 r_\Sigma^2 (71+66C_1 r_\Sigma^2)^2}{(1+C_1 r_\Sigma^2)^{\frac{132}{71}} (71+61C_1 r_\Sigma^2)^2} \frac{(1-\sqrt{f})}{f^{\frac{5}{2}}} \frac{1}{(1+z_\Sigma)^2} \tag{44}$$

where the constant δ in Photon is given by

$$\delta = \frac{\pi^2 k^4}{15\hbar^3} \tag{45}$$

where k and $\hbar$ denoting respectively Boltzmann and Plank constants.
The temperature inside the star is given by Tewari [1]

$$T^4 = \left[\frac{T_0(t)}{(1+C_1r^2)^{\frac{24}{71}} \left\{ D_2 + D_1(1+C_1r^2)^{\frac{49}{71}} \right\}^4} \right] - \left[\frac{16\alpha(1-\sqrt{f})}{3\gamma f^{\frac{3}{2}} (1+C_1r^2)^{\frac{6}{71}} \left\{ D_2 + D_1(1+C_1r^2)^{\frac{49}{71}} \right\}} \right] \tag{46}$$

where

$$T_0(t) = \left\{ \frac{16\alpha}{3k\gamma} \frac{\{D_2 + D_1(1+C_1r^2)^{\frac{49}{71}}\}(1+C_1r^2)^{\frac{49}{71}}(1-\sqrt{f})}{f^{\frac{3}{2}}} \right\}_\Sigma + \left\{ \frac{2\alpha\{D_2 + D_1(1+C_1r^2)^{\frac{49}{71}}\}^2(1+C_1r^2)^{\frac{12}{71}}(1-\sqrt{f})}{\pi\delta r^2 f^{\frac{5}{2}}} \right\}_\Sigma \frac{1}{(1+z_\Sigma)^2} \tag{47}$$

It follows that the surface temperature of the collapsing star tends to zero at the beginning of the collapse [$f \to 1$] and the stage of formation of black hole [$z_\Sigma \to \infty$].

5. Conclusion

We have presented a new simple model corresponding to $n = -6/5$ of Tewari [1]. The model is physically and thermodynamically sound as it corresponds to well-behaved nature for the fluid density, isotropic pressure and radiation flux density throughout the fluid sphere. Initially the interior solutions represent a static configuration of perfect fluid which then gradually starts evolving into radiating collapse. The apparent luminosity as observed by the distant observer at rest at infinity is zero in remote past at the instance when collapse begins and at the stage when collapsing configuration reaches the horizon of the black hole. The surface temperature and the temperature inside the star of the collapsing body is zero at the beginning and become infinite at the final phase of the configuration. We have a number of applications of our work i.e. one can construct models of Quasar, Supernovae, Black holes and various high energy astronomical objects.

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