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Construction of a q-deformed Hilbert Space to analyze some deformed states in Quantum Optics

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abstract: In this paper we construct a q-deformed Hilbert space and define annihilation and creation operators to generate deformed states. We show that these states are useful to throw some insight in the theory of Quantum Optics.

1 Introduction:

Coherent states associated with various dynamical symmetry groups are important in many problems of quantum physics. Glauber's coherent states[17] of simple harmonic oscillator and coherent states of various Lie algebras[18], due to Perelomov, are useful in the study of quantum optics. There are three basic ways one can generate coherent states which refer to vectors in a finite or infinite dimensional Hilbert space. In the first approach, Glauber defined coherent states as the right-hand eigenstates of the non-Hermitian boson annihilation operator of the radiation field. In the second approach, the coherent states are generated from vacuum by the action of the so called unitary displacement operator, that is, they are displacement operator states. This is also known as the group theoretic approach to generate coherent states. In the third approach,the coherent states can be defined as states that minimize Heisenberg uncertainty relation, or, simply as minimum uncertainty states.

We adopt the first approach to generate coherent vectors of a backwardshift acting on a deformed Hilbert space.This gives a generalisation of coherent states, as an eigenstate of photon annihilation operator, which are studied in various contexts of quantum optics.

To deal with the fluctuating fields we introduce a distribution for the complex field amplitude in classical coherence theory. By integrating over the strength of the field we then obtain the phase distribution. The description of the phase in quantum mechanical terms has been influenced by the difficulty of ascribing an operator to it in the quantum sense. To define a Hermitian phase operator in the quantum mechanical description of phase goes back to the work of Dirac [1], who attempted a defination of a phase operator with the help of polar decomposition of the annihilation operator in radiation field. But a polar decomposition of the one-mode field complex amplitude operator does not give a unitary operator exponential of the phase. Thereafter, Susskind and Glogower[2], Carruthers and Nieto[3], Pegg and Barnett[4], Shapiro and co-workers[5] have studied further in this topic. Susskind and Glogower modified Dirac's phase operator though it is one-sided unitary operator. Nevertheless, their phase operator has been extensively used in quantum optics. Shapiro and co-workers introduced phase measurement statistics through quantum estimation theory[6]. Pegg and Barnett carried out a polar decomposition of the annihilation operator in a truncated Hilbert space of dimension $s+1$, and defined a Hermitian phase operator in this finite-dimensional space. Now, given a state in the finite-dimensional Hilbert space one first computes the expectation value with the restricted state to the $s+1$-dimensional space. It is natural now to take the limit s to infinity and recover an Hermitian phase operator on the full Hilbert space. However, in this limit the PB phase operator does not converge to an Hermitian phase operator, but the distribution does converge to the SG phase distribution. Thus it appears to be computationally advantageous to describe the quantum-mechanical phase via a phase distribution rather than through a phase operator. This view was manifested in the work of Shapiro and co-workers. Agarwal and co-workers[7] adopted this point of view in investigating the quantum-mechanical phase properties of the nonlinear optical phenomena.

2 Preliminaries and Notations

We consider the set

$$H_q = \{f : f(z) = \sum a_n z^n \text{where} \sum [n]! \, |a_n|^2 < \infty\},$$

where $[n] = \dfrac{1-q^n}{1-q}$, $0 < q < 1$.

For $f, g \in H_q$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ we define addition and scalar multiplication as follows:

$$(f+g)(z) = f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n \tag{1}$$

and

$$(\lambda \, o \, f)(z) = \lambda \, o \, f(z) = \sum_{n=0}^{\infty} \lambda \, a_n z^n. \tag{2}$$

It is easily seen that H_q forms a vector space with respect to usual pointwise scalar multiplication and pointwise addition by (1) and (2). We observe that $e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!}$ belongs to H_q.

Now we define the inner product of two functions $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ belonging to H_q as

$$(f\,,\,g) = \sum [n]!\,\bar{a_n}\,b_n. \tag{3}$$

Corresponding norm is given by

$$\|f\|^2 = (f\,,\,f) = \sum [n]!\,|a_n|^2 < \infty\,.$$

With this norm derived from the inner product it can be shown that H_q is a complete normed space.Hence H_q forms a Hilbert space.

3 Orthonormal Set

Proposition-1. The set $\{ \frac{z^n}{\sqrt{[n]!}}\,, n = 0,1,2,3\ldots\}$ forms a complete orthonormal set.

Proof. If $f_n = \frac{z^n}{\sqrt{[n]!}}\,, n = 0,1,2,3\ldots$, then,

$$\|\,f_n\,\| = (f_n\,,\,f_n)^{1/2} = 1.$$

and $(f_n\,,\,f_m) = 0$. Hence $\{\,f_n\,\}$ forms an orthonormal set.
Also it is complete,for if $f(z) = \sum a_n z^n \in H_q$, then

$$(f_n\,,\,f) = [n]!\,a_n.\frac{1}{\sqrt{[n]!}} = \sqrt{[n]!}\,a_n.$$

Hence

$$\sum |\,(f_n\,,f)\,|^2 = \sum [n]!\,|\,a_n\,|^2 = \|\,f\,\|^2.$$

By Parseval's theorem, $\{\,f_n\,\}$ is complete.

4 Reproducing Kernel

H_q being a functional Hilbert space, the linear functional $f \to f(z)$ on H_q is bounded for every $z \in \mathbb{C}$. Consequently, there exists, for each $z \in \mathbb{C}$, an element K_z of H_q such that $f(z) = (\,K_z\,,\,f\,)$ for all $f \in H_q$. The function $K(w,z) = K_z(w)$ is called the kernel function or the reproducing kernel of H_q.

Consider the Fourier expansion of K_z with respect to the orthonormal basis $\{f_n\}$:

$$K_z = \sum_n (f_n, K_z) f_n = \sum \bar{f}_n(z) f_n.$$

Hence

$$\begin{aligned} K(w,z) &= K_z(w) = (K_w, K_z) = \sum f_n(w) \bar{f}_n(z) \\ &= \sum \frac{w^n}{\sqrt{[n]!}} \cdot \frac{\bar{z}^n}{\sqrt{[n]!}} = \sum \frac{(\bar{z}w)^n}{[n]!} = e_q(\bar{z}w) \end{aligned}$$

Thus $K(w,z) = e_q(\bar{z}w)$ is the reproducing kernel for H_q.

5 Eigenvectors

We consider the following actions on H_q:

$$\begin{aligned} T f_n &= \sqrt{[n]}\, f_{n-1} \\ T^* f_n &= \sqrt{[n+1]}\, f_{n+1} \end{aligned} \tag{4}$$

T is the backward shift and its adjoint T^* is the forward shift operator on H_q.

5.1 Backwardshift

Now we shall find the solution of the following eigenvalue equation:

$$T f_\alpha = \alpha f_\alpha. \tag{5}$$

$$f_\alpha(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n(z). \tag{6}$$

or

$$f_\alpha = \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n.$$

$$\begin{aligned} T f_\alpha &= \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, T f_n = \sum_{n=1}^{\infty} a_n \sqrt{[n]!} \sqrt{[n]}\, f_{n-1} \\ &= \sum_{n=0}^{\infty} a_{n+1} \sqrt{[n+1]!} \sqrt{[n+1]}\, f_n. \end{aligned} \tag{7}$$

$$\alpha f_\alpha(z) = \alpha \sum_{n=0}^{\infty} a_n z^n = \alpha \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n(z). \tag{8}$$

or

$$\alpha\, f_\alpha \;=\; \alpha \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n.$$

From (5), (6), (7) and (8) we observe that a_n satisfies the following difference equation:

$$a_{n+1} \sqrt{[n+1]}\, \sqrt{[n+1]} \;=\; \alpha\, a_n. \tag{9}$$

That is,

$$a_{n+1} \;=\; \frac{\alpha\, a_n}{[n+1]}. \tag{10}$$

Hence,

$$a_1 \;=\; \frac{\alpha\, a_0}{[1]}\,,\; a_2 \;=\; \frac{\alpha\, a_1}{[2]} \;=\; \frac{\alpha^2\, a_0}{[2]!}\,,\; a_3 \;=\; \frac{\alpha\, a_2}{[3]} \;=\; \frac{\alpha^3\, a_0}{[3]!}\,,\; \ldots.$$

Thus,

$$a_n \;=\; \frac{\alpha^n\, a_0}{[n]!}.$$

Hence,

$$f_\alpha \;=\; \sum a_n \sqrt{[n]!}\, f_n \;=\; a_0 \sum \frac{\alpha^n}{\sqrt{[n]!}}\, f_n.$$

We choose a_0 so that f_α is normalized:

$$\begin{aligned} 1 \;=\; (\, f_\alpha\,,\, f_\alpha\,) \;&=\; \sum [n]!\, |\, a_n\,|^2 \;=\; \sum [n]!\, \frac{|\, a_0\,|^2\, |\, \alpha\,|^{2n}}{([n]!)^2} \\ &=\; |\, a_0\,|^2 . \sum \frac{(|\,\alpha\,|^2)^n}{[n]!} \;=\; |\, a_0\,|^2\, e_q\,(\,|\,\alpha\,|^2\,). \end{aligned}$$

Thus, aside from a trivial phase

$$a_n \;=\; e_q(|\alpha|^2)^{-\dfrac{1}{2}} . \frac{\alpha^n}{[n]!}.$$

So, the eigenvector of T is

$$f_\alpha \;= e_q(|\alpha|^2)^{-\dfrac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}}\, f_n. \tag{11}$$

We shall call f_α a **coherent vector** in H_q.

5.2 Square of Backwardshift

Here we shall find the solution of the following eigenvalue equation:

$$T^2 f_\alpha = \alpha^2 f_\alpha. \tag{12}$$

$$\begin{aligned} T^2 f_\alpha &= T \sum_{n=0}^{\infty} a_{n+1} \sqrt{[n+1]!}\sqrt{[n+1]}\, f_n \\ &= \sum_{n=1}^{\infty} a_{n+1} \sqrt{[n+1]!}\sqrt{[n+1]}\sqrt{[n]}\, f_{n-1} \\ &= \sum_{n=0}^{\infty} a_{n+2} \sqrt{[n+2]!}\sqrt{[n+2]}\sqrt{[n+1]}\, f_n. \end{aligned} \tag{13}$$

$$\alpha^2 f_\alpha = \sum_{n=0}^{\infty} \alpha^2 a_n \sqrt{[n]!}\, f_n. \tag{14}$$

From (12), (13) and (14) we see that a_n satisfies the following difference equation:

$$a_{n+2} \sqrt{[n+2]!}\sqrt{[n+2]}\sqrt{[n+1]} = \alpha^2 a_n \sqrt{[n]!}.$$

Thus,

$$a_{n+2} = \frac{\alpha^2 a_n}{[n+2][n+1]}. \tag{15}$$

Hence,

$$a_2 = \frac{\alpha^2 a_0}{[2]!}, a_4 = \frac{\alpha^2 a_2}{[4][3]} = \frac{\alpha^4 a_0}{[4]!}, a_6 = \frac{\alpha^2 a_4}{[6][5]} = \frac{\alpha^6 a_0}{[6]!}, \ldots.$$

and

$$a_3 = \frac{\alpha^2 a_1}{[3][2]} = \frac{\alpha^2 a_1}{[3]!}, a_5 = \frac{\alpha^2 a_3}{[5][4]} = \frac{\alpha^4 a_1}{[5]!} \ldots.$$

Thus,

$$
\begin{aligned}
f_\alpha &= a_0\, f_0 \,+\, a_1\, f_1 \,+\, a_2.\sqrt{[2]!}\, f_2 \,+\, a_3.\sqrt{[3]!}\, f_3 \,+\, \ldots \\
&= (\, a_0\, f_0 \,+\, a_2 \sqrt{[2]!}\, f_2 \,+\, a_4 \sqrt{[4]!}\, f_4 \,+\, \ldots) \\
&+ (\, a_1\, f_1 \,+\, a_3 \sqrt{[3]!}\, f_3 \,+\, a_5 \sqrt{[5]!}\, f_5 \,+\, \ldots) \\
&= a_0 [\, f_0 \,+\, \frac{\alpha^2}{\sqrt{[2]!}}\, f_2 \,+\, \frac{\alpha^4}{\sqrt{[4]!}}\, f_4 \,+\, \ldots] \\
&+ a_1 [\, f_1 \,+\, \frac{\alpha^2}{\sqrt{[3]!}}\, f_3 \,+\, \frac{\alpha^4}{\sqrt{[5]!}}\, f_5 \,+\, \ldots] \\
&= a_0 [\, \frac{g_\alpha \,+\, g_{-\alpha}}{2N}\,] \,+\, \frac{a_1}{\alpha} [\, \frac{g_\alpha \,-\, g_{-\alpha}}{2N}\,] \\
&= (\, \frac{a_0}{2N} \,+\, \frac{a_1}{2\,\alpha N}\,)\, g_\alpha \,+\, (\, \frac{a_0}{2N} \,-\, \frac{a_1}{2\,\alpha N}\,)\, g_{-\alpha} \\
&= K\, g_\alpha \,+\, K'\, g_{-\alpha}
\end{aligned}
$$

where g_α and $g_{-\alpha}$ are normalized coherent vectors. Also we have taken $K = \frac{a_0}{2N} + \frac{a_1}{2\alpha N}$ and $K' = \frac{a_0}{2N} - \frac{a_1}{2\alpha N}$ with $N = e_q(|\alpha|^2)^{-\frac{1}{2}}$.

We choose a_0 and a_1 so that f_α is normalized:

$$1 = (f_\alpha, f_\alpha) = |K|^2 + |K'|^2 + e_q(-|\alpha|^2)e_q(|\alpha|^2)^{-1}[2ReK\bar{K}']$$

where we have used the facts

$$
\begin{aligned}
(g_\alpha, g_\alpha) &= 1 \\
(g_{-\alpha}, g_{-\alpha}) &= 1 \\
(g_\alpha, g_{-\alpha}) &= e_q(-|\alpha|^2)e_q(|\alpha|^2)^{-1}
\end{aligned}
$$

6 Hilbert Space Properties of Coherent Vectors

Coherent vectors are not orthogonal, for

$$
\begin{aligned}
(f_\alpha, f_{\alpha'}) &= e_q(|\alpha|^2)^{-\frac{1}{2}}.e_q(|\alpha'|^2)^{-\frac{1}{2}}.\sum_{n=0}^{\infty}[n]!\frac{\bar{\alpha}^n}{[n]!}\frac{\alpha'^n}{[n]!} \\
&= e_q(|\alpha|^2)^{-\frac{1}{2}}.e_q(|\alpha'|^2)^{-\frac{1}{2}}.e_q(\bar{\alpha}\alpha').
\end{aligned} \tag{16}
$$

Nevertheless, the coherent vectors are complete, in fact, overcomplete -they form a resolution of the identity [21]

$$I = \frac{1}{2\pi}\int_{\alpha\in\mathbb{C}} d\mu(\alpha)|f_\alpha >< f_\alpha|. \tag{17}$$

where

$$d\mu(\alpha) = e_q(|\alpha|^2)e_q(-|\alpha|^2)d_q|\alpha|^2 d\theta \tag{18}$$

where $\alpha = re^{i\theta}$.

To prove this we define the operator

$$|f_\alpha >< f_\alpha| : H_q \to H_q \tag{19}$$

by

$$|f_\alpha >< f_\alpha| f = (f_\alpha, f) f_\alpha \tag{20}$$

with $f(z) = \sum_0^\infty b_n z^n$. Now,

$$(f_\alpha, f) = e_q(|\alpha|^2)^{-\frac{1}{2}} \sum_{n=0}^\infty [n]! \frac{\bar{\alpha}^n}{[n]!} b_n.$$

Then,

$$(f_\alpha, f) f_\alpha = e_q(|\alpha|^2)^{-1} \sum_{m,n=0}^\infty \frac{\alpha^m}{\sqrt{[m]!}} \bar{\alpha}^n b_n f_m.$$

Hence,

$$\begin{aligned} \frac{1}{2\pi}\int_{\alpha\in\mathbb{C}} d\mu(\alpha)|f_\alpha >< f_\alpha|f &= \sum_{m,n=0}^\infty \frac{f_m}{\sqrt{[m]!}} b_n \frac{1}{2\pi}\int_0^\infty d_q r^2 . e_q^{-r^2} r^{m+n} \\ &\times \int_0^{2\pi} d\theta . e^{i(m-n)\theta} \\ &= \sum_{n=0}^\infty \frac{f_n}{\sqrt{[n]!}} b_n \int_0^\infty d_q r^2 . e_q^{-r^2} . r^{2n} \\ &= \sum_{n=0}^\infty \frac{f_n}{\sqrt{[n]!}} b_n \int_0^\infty d_q x . e_q^{-x} . x^n \\ &= \sum_{n=0}^\infty \sqrt{[n]!} b_n f_n \\ &= f. \end{aligned} \tag{21}$$

Where we have taken $x = r^2$ and utilized the fact $\int_0^\infty d_q x . e_q^{-x} . x^n = [n]!$ [21].

7 Phase Operator

Before going to define the phase operator we observe that

$$TT^* = [N+1], T^*T = [N] \tag{22}$$

where the operator N is such that

$$Nf_n = nf_n. \tag{23}$$

Also we can verify that

$$NT - TN = -T, NT^* - T^*N = T^* \tag{24}$$

and

$$TT^* - T^*T = q^N. \tag{25}$$

We can also show that q^N commutes with both T^*T and TT^*.

Now, analogous to the idea of Carruthers and Nieto [3], we initially proposed the phase operator to be

$$P = (q^N + T^*T)^{-1/2}T \tag{26}$$

where N is given by (23).

Now because of the relation

$$q^n + [n] = [n+1] \tag{27}$$

our phase operator(26) does not produce anything new but the phase distribution produced by Susskind-Glogower phase operator.

To circumvent this situation we propose our phase operator to be

$$P = (q^{N+1} + T^*T)^{-1/2}T \tag{28}$$

where N is given by (23).

The operator P (28) is not unitary but is one-sided unitary as we can easily verify

$$PP^* = I, P^*P \neq I. \tag{29}$$

8 Phase Distribution

In this section we describe the phase distribution in the deformed Hilbert space. To do this we introduce the phase vector and obtain its distributions in details.

8.1 Phase Vector

To obtain the phase vector we consider first the Susskind-Glogower type **phase operator** $P = (q^{N+1} + T^*T)^{-1/2}T$ as discussed above(28).

Now the phase vector is obtained by solving the eigenvalue equation

$$P f_\beta = \beta f_\beta \tag{30}$$

where $f_\beta(z) = \sum_{n=0}^{\infty} a_n\, z^n = \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n(z)$.That is,

$$f_\beta = \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n. \tag{31}$$

Then

$$\begin{aligned} P f_\beta &= \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\,(q^{N+1} + T^*T)^{-1/2} T f_n \\ &= \sum_{n=1}^{\infty} a_n \sqrt{[n]!}\,(q^{N+1} + T^*T)^{-1/2} \sqrt{[n]} f_{n-1} \\ &= \sum_{n=1}^{\infty} a_n \sqrt{[n]!}\sqrt{[n]}(q^n + [n-1])^{-1/2} f_{n-1} \\ &= \sum_{n=0}^{\infty} a_{n+1} \sqrt{[n+1]!}\sqrt{[n+1]}(q^{n+1} + [n])^{-1/2} f_n \end{aligned} \tag{32}$$

and

$$\beta f_\beta = \beta \sum_{n=0}^{\infty} a_n \sqrt{[n]!} f_n. \tag{33}$$

From (30), (31), (32) and (33) we observe that a_n satisfies the following difference equation:

$$a_{n+1}\sqrt{[n+1]!}\sqrt{[n+1]}(q^{n+1} + [n])^{-1/2} = \beta a_n \sqrt{[n]!}. \tag{34}$$

That is,

$$a_{n+1} = \frac{\beta a_n (q^{n+1} + [n])^{1/2}}{[n+1]}. \tag{35}$$

Hence,

$$\begin{aligned} a_1 &= \frac{\beta(q + [0])^{1/2} a_0}{[1]}. \\ a_2 &= \frac{\beta a_1 (q^2 + [1])^{1/2}}{[2]} = \frac{\beta^2 a_0 \sqrt{(q+[0])(q^2+[1])}}{[2]!}. \\ a_3 &= \frac{\beta a_2 (q^3 + [2])^{1/2}}{[3]} = \frac{\beta^3 a_0 \sqrt{(q+[0])(q^2+[1])(q^3+[2])}}{[3]!}. \end{aligned}$$

and so on. Thus,

$$a_n = \frac{\beta^n a_0 \sqrt{(q+[0])(q^2+[1])(q^3+[2])\dots(q^n+[n-1])}}{[n]!}.$$

Hence,

$$\begin{aligned} f_\beta &= \sum_{n=0}^{\infty} a_n \sqrt{[n]!} f_n \\ &= a_0 \sum_{n=0}^{\infty} \beta^n \sqrt{\frac{(q+[0])(q^2+[1])(q^3+[2])\dots(q^n+[n-1])}{[n]!}} f_n. \end{aligned}$$

where $\beta = |\beta| e^{i\theta}$ is a complex number. These vectors are normalizable in a strict sense only for $|\beta| < 1$.

Now, if we take $a_0 = 1$ and $|\beta| = 1$ we have

$$f_\beta = \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q+[0])(q^2+[1])(q^3+[2])\dots(q^n+[n-1])}{[n]!}} f_n. \tag{36}$$

Henceforth, we shall denote this vector as

$$f_\theta = \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q+[0])(q^2+[1])(q^3+[2])\dots(q^n+[n-1])}{[n]!}} f_n, \tag{37}$$

$0 \le \theta \le 2\pi$ and call f_θ a **phase vector** in H_q.

8.2 Completeness of Phase Vectors

The phase vectors f_θ are neither normalizable nor orthogonal. The completeness relation

$$I = \frac{1}{2\pi} \int_X \int_0^{2\pi} d\nu(x,\theta) |f_\theta >< f_\theta| \tag{38}$$

where

$$d\nu(x,\theta) = d\mu(x) d\theta \tag{39}$$

may be proved as follows:

Here we consider the set X consisting of the points $x = 0, 1, 2, \dots$ and $\mu(x)$ is the measure on X which equals

$$\mu_n \equiv \frac{[n]!}{(q+[0])(q^2+[1])\dots(q^n+[n-1])}$$

at the point $x = n$ and θ is the Lebesgue measure on the circle.

Define the operator

$$|f_\theta >< f_\theta| : H_q \to H_q \tag{40}$$

by

$$|f_\theta >< f_\theta| f = (f_\theta, f) f_\theta \tag{41}$$

with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ Now,

$$\begin{aligned}
&(f_\theta, f) \\
= &\sum_{n=0}^{\infty} [n]! \frac{e^{-in\theta}}{\sqrt{[n]!}} \sqrt{\frac{(q+[0])(q^2+[1])(q^3+[2])\dots(q^n+[n-1])}{[n]!}} a_n \\
= &\sum_{n=0}^{\infty} e^{-in\theta} \sqrt{(q+[0])(q^2+[1])(q^3+[2])\dots(q^n+[n-1])} a_n.
\end{aligned} \tag{42}$$

Then,

$$\begin{aligned}
&(f_\theta, f) f_\theta \\
= &\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \, e^{i(m-n)\theta} \sqrt{\frac{(q+[0])(q^2+[1])\dots(q^m+[m-1])}{[m]!}} \\
\times &\sqrt{(q+[0])(q^2+[1])\dots(q^n+[n-1])} f_m.
\end{aligned} \tag{43}$$

Using

$$\int_0^{2\pi} d\theta e^{i(m-n)\theta} = 2\pi \delta_{mn} \tag{44}$$

we have

$$
\begin{aligned}
& \frac{1}{2\pi}\int_X\int_0^{2\pi} d\nu(x,\theta)|f_\theta><f_\theta|f \\
= & \int_X d\mu(x)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\, f_m\sqrt{\frac{(q+[0])(q^2+[1])\ldots(q^m+[m-1])}{[m]!}} \\
\times & \sqrt{(q+[0])(q^2+[1])\ldots(q^n+[n-1])}\frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta}d\theta \\
= & \sum_{n=0}^{\infty} a_n f_n \int_X \frac{(q+[0])(q^2+[1])\ldots(q^n+[n-1])}{\sqrt{[n]!}} d\mu(x) \\
= & \sum_{n=0}^{\infty} a_n f_n \frac{(q+[0])(q^2+[1])\ldots(q^n+[n-1])}{\sqrt{[n]!}} \\
\times & \frac{[n]!}{(q+[0])(q^2+[1])\ldots(q^n+[n-1])} \\
= & \sum_{n=0}^{\infty} \sqrt{[n]!} a_n f_n \\
= & f.
\end{aligned}
\tag{45}
$$

Thus,(38) follows.

8.3 Distribution

We use the vectors f_θ to associate, to a given density operator ρ, a phase distribution as follows:

$$
\begin{aligned}
P(\theta) &= \tfrac{1}{2\pi}(f_\theta, \rho f_\theta) \\
&= \tfrac{1}{2\pi}\textstyle\sum_{m,n=0}^{\infty} \sqrt{\tfrac{(q+[0])\ldots(q^m+[m-1])}{[m]!}}.\sqrt{\tfrac{(q+[0])\ldots(q^n+[n-1])}{[n]!}}.e^{i(n-m)}.(f_m, \rho f_n)
\end{aligned}
\tag{46}
$$

The $P(\theta)$ as defined in (46) is positive, owing to the positivity of ρ, and is normalized

$$\int_X\int_0^{2\pi} P(\theta) d\nu(x,\theta) = 1 \tag{47}$$

where

$$d\nu(x,\theta) = d\mu(x)d\theta \tag{48}$$

for,

$$\begin{array}{rcl}\int_X \int_0^{2\pi} P(\theta) d\nu(x,\theta) & = & \int_X d\mu(x) \sum_{m,n=0}^{\infty} \sqrt{\frac{(q+[0])...(q^m+[m-1])}{[m]!}} . \sqrt{\frac{(q+[0])...(q^n+[n-1])}{[n]!}} . \\ & & \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta} d\theta . (f_m, \rho f_n) \\ & = & \int_X d\mu(x) \sum_{n=0}^{\infty} \frac{(q+[0])...(q^n+[n-1])}{[n]!} . (f_n, \rho f_n) \\ & = & \sum_{n=0}^{\infty} (f_n, \rho f_n) \\ & = & 1. \end{array} \tag{49}$$

In particular, the **phase distribution** over the window $0 \leq \theta \leq 2\pi$ for any vector f is then defined by

$$\begin{array}{rcl} P(\theta) & = & \frac{1}{2\pi}(f_\theta, |f >< f| f_\theta) \\ & = & \frac{1}{2\pi}|(f_\theta, f)|^2. \end{array} \tag{50}$$

8.4 Examples

We now consider some specific vectors in the Hilbert space H_q and compute their corresponding phase distributions.

8.4.1 Incoherent Vectors

For the incoherent vectors we take the density operator to be

$$\rho = \sum_{n=0}^{\infty} p_n |f_n >< f_n|, \tag{51}$$

with

$$p_n \geq 0 \text{ and } \sum_{n=0}^{\infty} p_n = 1.$$

Now we calculate the phase distribution $P(\theta)$ as

$$\begin{array}{rcl} P(\theta) & = & \frac{1}{2\pi}(f_\theta, \rho f_\theta) \\ & = & \frac{1}{2\pi}\sum_{n=0}^{\infty} p_n (f_\theta, |f_n >< f_n| f_\theta) \\ & = & \frac{1}{2\pi}\sum_{n=0}^{\infty} p_n |(f_\theta, f_n)|^2 \\ & = & \frac{1}{2\pi}\sum_{n=0}^{\infty} p_n . \frac{(q+[0])...(q^n+[n-1])}{[n]!} \end{array} \tag{52}$$

8.4.2 Coherent Vectors

For the coherent vectors $f_\alpha(11)$,

$$f_\alpha = e_q(|\alpha|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}} f_n. \tag{53}$$

we take the density operator to be

$$\rho = |f_\alpha >< f_\alpha|, \ \alpha = |\alpha| e^{i\theta_0} \tag{54}$$

and calculate the phase distribution $P(\theta)$ as

$$\begin{aligned} P(\theta) &= \frac{1}{2\pi}(f_\theta, \rho f_\theta) \\ &= \frac{1}{2\pi}(f_\theta, |f_\alpha >< f_\alpha| f_\theta) \\ &= \frac{1}{2\pi}|(f_\theta, f_\alpha)|^2 \\ &= \frac{1}{2\pi}|\sum_{n=0}^{\infty} e^{in(\theta_0-\theta)}.\frac{|\alpha|^n}{\sqrt{[n]!}}.e_q(|\alpha|^2)^{-\frac{1}{2}}.\sqrt{\frac{(q+[0])...(q^n+[n-1])}{[n]!}}|^2 \end{aligned} \quad (55)$$

9 Conclusion

The basic difference of this paper with the previous works is its functional analysis approach.Our observation that annihilation operator is a backwardshift has been reflected in our work. With the proposed phase operator we describe a phase distribution to calculate phase distribution for specific vectors in the deformed space.

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