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Evaluations of Ramanujan's Cubic Continued Fraction $G(q)$ by Modular Equations

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Abstract

In this paper, we establish certain modular equations related to Ramanujan's cubic continued fraction

$$G(q) := \frac{q^{1/3}}{1}\,_{+}\,\frac{q+q^2}{1}\,_{+}\,\frac{q^2+q^4}{1}\,_{+\cdots}, \qquad |q| < 1.$$

and obtain many explicit values of $G(e^{-\pi\sqrt{n}})$, for certain values of n.

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1 Introduction

Ramanujan's cubic continued fraction $G(q)$ is defined by

$$G(q) := \frac{q^{1/3}}{1}\,_{+}\,\frac{q+q^2}{1}\,_{+}\,\frac{q^2+q^4}{1}\,_{+\cdots}, \qquad |q| < 1. \tag{1.1}$$

Ramanujan recorded this continued fraction (1.1) on page 227 of his third notebook [10] and on page 44 of his lost notebook [11]. For the wonderful introduction to Ramanujan's continued fraction see [3], [5], [9] and for some beautiful subsequent work on Ramanujan's cubic contnued fraction [1], [2],[4], [12], [13].

In this paper, we establish certain general formulae for evaluating $G(q)$. In section 2 of this paper, we setup some preliminaries which are required to prove the general formulae. In section 3, we establish certain modular equations related to $G(q)$ and in the final section, we deduce the above stated general formulae and obtain many explicit values of $G(q)$.

We conclude this introduction by recalling an identity for $G(q)$ stated by Ramanujan.

$$8G^3(-q) = 1 - \frac{\varphi^4(q)}{\varphi^4(q^3)}, \tag{1.2}$$

where

$$\varphi(q) := \sum_{n=\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}. \tag{1.3}$$

The proof of (1.2) follows from Entry 1 (ii) and (iii) of Chapter 20 [5, p. 345].

2 Some Preliminary Results

As usual, for any complex number a,

$$(a;q)_0 := 1$$

and

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n, \qquad |x|<1,$$

with

$$(a)_n := a(a+1)(a+2)...(a+n-1).$$

Then, we say that β is of n^{th} degree over α and call the ratio

$$m := \frac{z_1}{z_n},$$

the multiplier, where $z_1 =_2 F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n =_2 F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

Theorem 2.1 Let $G(q)$ be as defined as in (1.1), then

$$G(q) + G(-q) + 2G^2(-q)G^2(q) = 0 \tag{2.1}$$

and

$$G^2(q) + 2G^2(q^2)G(q) - G(q^2) = 0. \tag{2.2}$$

For a proof, see [9].

Theorem 2.2 Let β and γ be of the third and ninth degrees, respectively, with respect to α. Let $m = z_1/z_3$ and $m' = z_3/z_9$. Then,

$$(i) \qquad \left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4} - \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = \frac{-3m}{m'} \tag{2.3}$$

and

$$(ii) \qquad \left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} = \frac{m'}{m}. \tag{2.4}$$

For a proof, see [5, Entry 3 (xii) and (xiii), pp. 352-353].

Theorem 2.3 Let α, β, γ and δ be of the first, third, fifth and fifteenth degrees respectively. Let m denote the multiplier connecting α and β and let m' be the multiplier relating γ and δ. Then,

$$(i) \qquad \left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \sqrt{\frac{m'}{m}} \tag{2.5}$$

and

$$(ii) \qquad \left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}}. \tag{2.6}$$

For a proof, see [5, Entry 11 (viii) and (ix), p. 383].

Theorem 2.4 If β, γ and δ are of degrees 3, 7 and 21 respectively, $m = z_1/z_3$ and $m' = z_7/z_{21}$, then

$$\begin{aligned}(i) \qquad &\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4} + \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} \\ -2&\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8}\left\{1 + \left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8} + \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8}\right\} = mm'\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}(ii) \qquad &\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/4} + \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/4} \\ -2&\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/8}\left\{1 + \left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/8}\right\} = \frac{9}{mm'}.\end{aligned} \tag{2.8}$$

For a proof, see [5, Entry 13 (v) and (vi), pp. 400-401].

3 Modular Equations

Theorem 3.1 Let

$$R := \frac{\varphi(q^3)\varphi(q^2)}{\varphi(q)\varphi(q^6)} \qquad \text{and} \qquad S := \frac{\varphi(q^6)\varphi(q^4)}{\varphi(q^2)\varphi(q^{12})}.$$

Then,

$$(RS)^2 + \frac{1}{(RS)^2} - 8\left(RS + \frac{1}{RS}\right) + 6\left(\sqrt{RS} - \frac{1}{\sqrt{RS}}\right)\left[\sqrt{\frac{R}{S}} + \sqrt{\frac{S}{R}}\right]$$

$$-2\left((RS)^{3/2}-\frac{1}{(RS)^{3/2}}\right)\left[\sqrt{\frac{R}{S}}+\sqrt{\frac{S}{R}}\right]+\left(RS+\frac{1}{RS}\right)\left(\frac{R}{S}+\frac{S}{R}\right)+10=0. \tag{3.1}$$

Proof. From (1.2) and the definition of R and S, it can be seen that

$$R^4(1-8A^3)+8C^3-1=0 \tag{3.2}$$

and

$$S^4(1-8C^3)+8D^3-1=0, \tag{3.3}$$

where $A=G(-q), B=G(q^2), C=G(-q^2)$ and $D=G(-q^4)$.
On changing q to q^2 in (2.1), we obtain

$$G(q^2)+G(-q^2)+2G^2(-q^2)G^2(q^2)=0 \tag{3.4}$$

and also change q to $-q$ in (2.2), then

$$G^2(-q)+2G^2(q^2)G(-q)-G(q^2)=0. \tag{3.5}$$

Eliminating $G(q^2)$ between (3.4) and (3.5) using Maple, we have

$$2(AC)^4-4(AC)^3+3(AC)^2+AC+A^3+C^3=0. \tag{3.6}$$

Now, on eliminating A between (3.2) and (3.6) using Maple, we obtain

$$L(R,C)M(R,C)N(R,C)=0,$$

where

$$L(R,C)=R-R^2-R^3-C^3+R^4+8R^3C^3+R^4C^3-8RC^3+8R^2C^3+8C^6,$$
$$M(R,C)=R+R^2-R^3+C^3-R^4+8R^3C^3-R^4C^3-8RC^3-8R^2C^3-8C^6$$

and

$$N(R,C)=64C^{12}-16C^9+16R^4C^9-128C^9R^2+206R^4C^6+C^6R^8+C^6+96C^6R^2$$
$$+48C^6R^6-18R^2C^3-30R^6C^3-50R^4C^3+2R^8C^3+R^2+3R^6+3R^4+R^8.$$

It is easy to see that R and C have the following series expansion:

$$R=1+2q+2q^2+6q^3+6q^4+4q^5+14q^6+8q^7+6q^8+\cdots$$

and

$$C=1+2q+2q^2+6q^3+6q^4+4q^5+14q^6+8q^7+6q^8+\cdots.$$

Using these in $L(R,C), M(R,C)$ and $N(R,C)$, we obtain

$$L(R,C)=32q^3+96q^4+384q^5+1120q^6+2688q^7+6176q^8+12896q^9+23552q^{10}+\cdots$$

$$M(R,C)=8q+32q^2+120q^3+400q^4+1008q^5+2528q^6+5440q^7+11056q^8+\cdots$$

and

$$N(R,C)=8+80q+544q^2+2800q^3+12064q^4+45408q^5+153056q^6+\cdots.$$

One can see that $q^{-1}M(R,C)$, $q^{-1}N(R,C)$ does not tends to 0 as $q\to 0$, whereas $q^{-1}L(R,C)$ tend to 0 as $q\to 0$. Hence $q^{-1}L(R,C)=0$ in some neighborhood $q=0$. By analytic continuation $q^{-1}L(R,C)=0$ in $|q|<1$. Thus we have

$$L(R,C)=0. \tag{3.7}$$

Changing q to q^4 in (3.6), we see that

$$2(CD)^4 - 4(CD)^3 + 3(CD)^2 + CD + C^3 + D^3 = 0. \tag{3.8}$$

Eliminating D between (3.3) and (3.8) using Maple, we obtain

$$L'(C,S)N'(C,S) = 0,$$

where

$$L'(C,S) = 1 + 2C^3 + C^6 - 3S^2 + 64C^12S^8 - 16C^9S^8 + 128C^9S^6 + 16C^9S^4 + C^6S^8$$
$$-96C^6S^6 + 206C^6S^4 - 48C^6S^2 + 18C^3S^6 - 50C^3S^4 + 30C^3S^2 - S^6 + 3S^4 = 0$$

and

$$N'(C,S) = 64C^12S^8 - 16C^9S^8 - 128C^9S^6 + C^6S^8 + 16C^9S^4 + 96C^6S^6 + 206C^6S^4$$
$$-18C^3S^6 + 48C^6S^2 - 50C^3S^4 + C^6 + S^6 - 30C^3S^2 + 3S^4 + 2C^3 + 3S^2 + 1.$$

Using the series expansion of C and S, where

$$S = 1 + 2q^2 + 2q^4 + 6q^6 + 6q^8 + 4q^{10} + 14q^{12} + 8q^{14} + 6q^{16} + \cdots$$

in the above, we find that

$$L'(C,S) = 128q^4 + 768q^6 - 1280q^8 - 42752q^{10} \cdots$$

and

$$N'(C,S) = 8 + 144q^2 + 1568q^4 + 11568q^6 + 65056q^8 + 299872q^{10} + 1196960q^{12} + \cdots$$

Again, $N'(C,S)$ does not tend to 0 as $q \to 0$, whereas both $L'(C,S) \to 0$ as $q \to 0$. Thus, $L'(C,S) = 0$ in some neighborhood $q = 0$. By analytic continuation, we have

$$L'(C,S) = 0, \qquad\qquad |q| < 1. \tag{3.9}$$

Finally, on eliminating C between (3.7) and (3.9) using Maple, we obtain

$$X_1(R,S)X_2(R,S)X_3(R,S)X_4(R,S) = 0,$$

where

$$X_1(R,S) = R^2S^2 + R^2S + RS^2 + 2RS - R - S + 1,$$
$$X_2(R,S) = R^2S^2 - R^2S + RS^2 - 2RS - R + S + 1,$$
$$X_3(R,S) = R^4S^4 + 2R^4S^3 - 2R^3S^4 + R^4S^2 + 8R^3S^3 + R^2S^4 + 6R^3S^2 - 6R^2S^3$$
$$+10R^2S^2 + 6R^2S - 6RS^2 + R^2 + 8RS + S^2 + 2R - 2S + 1$$

and

$$X_4(R,S) = R^4S^4 - 2R^4S^3 - 2R^3S^4 + R^4S^2 - 8R^3S^3 + R^2S^4 + 6R^3S^2 + 6R^2S^3$$
$$+10R^2S^2 - 6R^2S - 6RS^2 + R^2 - 8RS + S^2 + 2R + 2S + 1.$$

On using the series expansion of R and S, in the above,

$$X_1(R,S) = 4 + 12q + 32q^2 + 92q^3 + 176q^4 + 352q^5 + 680q^6 + 1144q^7 + 1952q^8 + \cdots,$$

$$X_2(R,S) = 4q + 4q^3 - 16q^4 - 64q^5 - 136q^6 - 312q^7 - 656q^8 - \cdots,$$

$$X_3(R,S) = 32 + 160q + 544q^2 + 1888q^3 + 5376q^4 + 13824q^5 + 34592q^6 + 79360q^7 + 173504q^8 + \cdots$$

and

$$X_4(R,S) = 64q^3 + 320q^4 + 1408q^5 + 4544q^6 + 12736q^7 + 31680q^8 + 72640q^9 + \cdots.$$

As usual, $q^{-1}X_1(R,S)$, $q^{-1}X_2(R,S)$ and $q^{-1}X_3(R,S)$ does not tend to 0 as $q \to 0$ whereas $q^{-1}X_4(R,S) \to 0$ as $q \to 0$.Thus,$q^{-1}X_4(R,S) = 0$ in some neighborhood of $q = 0$. By analytic continuation $q^{-1}X_4(R,S) = 0$ in $|q| < 1$. Thus

$$X_4(R,S) = 0.$$

On dividing throughout by $(RS)^2$ and upon simplifying, we obtain the required result.

Theorem 3.2 If

$$R := \frac{\varphi^2(q^3)}{\varphi(q)\varphi(q^9)} \qquad \text{and} \qquad S := \frac{\varphi^2(q^6)}{\varphi(q^2)\varphi(q^{18})},$$

then

$$\left(\frac{R}{S}\right)^2 + \left(\frac{S}{R}\right)^2 - 2\left(RS - \frac{3}{RS}\right)\left(\frac{R}{S} + \frac{S}{R}\right) + (RS)^2 + \frac{9}{(RS)^2}$$
$$-16\left(RS + \frac{3}{RS}\right) - 44 = 0. \tag{3.10}$$

Proof. Let

$$P := \frac{\varphi^2(-q^3)}{\varphi(-q)\varphi(-q^9)} \qquad \text{and} \qquad Q := \frac{\varphi^2(-q^6)}{\varphi(-q^2)\varphi(-q^{18})}.$$

On using Entry 10(ii)and (iii) of Chapter 17 [5, p. 122] in P and Q, we deduce

$$\frac{P}{Q} = \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\delta)}\right)^{1/8} \qquad \text{and} \qquad \frac{P}{Q^2} = \left(\frac{z_1 z_9}{z_3^2}\right)^{1/2}$$

Using these in (2.3) and (2.4), it is easy to see that

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} = \frac{P^2(Q^2+3)}{Q^2(P^2-Q^2)} \qquad \text{and} \qquad \left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} = \frac{Q^2(Q^2-1)}{(P^2-Q^2)}.$$

Multiplying these two, we arrive at

$$P^4 - 4P^2Q^2 + Q^4 + 3P^2 - P^2Q^4 = 0. \tag{3.11}$$

Changing q to $-q$ in the above,

$$R^4 - 4R^2Q^2 + Q^4 + 3R^2 - R^2Q^4 = 0. \tag{3.12}$$

Eliminating Q between (3.11) and (3.12) using Maple, we obtain

$$(P^2R^2 - P^2 - 4PR - R^2 - 3)(P^2R^2 - P^2 + 4PR - R^2 - 3)(P-R)^2 = 0. \tag{3.13}$$

The series expansion of P and R are

$$P = 1 - 6q + 12q^2 - 8q^3 + 6q^4 - 24q^5 + 24q^6 + \cdots$$

and

$$R = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + 24q^6 + \cdots.$$

On using these in (3.13), we observe that $(P^2R^2 - P^2 - 4PR - R^2 - 3)$does not tends to 0 as $q \longrightarrow 0$ whereas $P^2R^2 - P^2 + 4PR - R^2 - 3$ tends to 0 as $q \longrightarrow 0$. Thus, $P^2R^2 - P^2 + 4PR - R^2 - 3 = 0$ in some neighborhood $q = 0$. By analytic continuation in $|q| < 1$,

$$P^2R^2 - P^2 + 4PR - R^2 - 3 = 0. \tag{3.14}$$

On changing q to q^2 in the above, we have

$$Q^2S^2 - Q^2 + 4QS - S^2 - 3 = 0. \tag{3.15}$$

Eliminating Q between (3.14) and (3.15) using Maple, we get

$$X_1(R,S)X_2(R,S)X_3(R,S) = 0,$$

where

$$X_1(R,S) = R^4 + 3R^2 - 4R^2S^2 + S^4 - R^2S^4,$$

$$X_2(R,S) = R^4S^4 - 2R^4S^2 + 16R^3S^3 - 2R^2S^4 + R^4 + 44R^2S^2 + S^4 + 6R^2 + 48RS + 6S^2 + 9$$

and

$$X_3(R,S) = R^4S^4 - 2R^4S^2 - 16R^3S^3 - 2R^2S^4 + R^4 + 44R^2S^2 + S^4 + 6R^2 - 48RS + 6S^2 + 9.$$

Using the series expansion of R and S, in the above

$$X_1(R,S) = 96q^2 + 576q^3 + 2880q^4 + 7680q^5 + \cdots,$$

$$X_2(R,S) = 128 + 1152q + 6912q^2 + 31488q^3 + 124416q^4 + \cdots$$

and

$$X_3(R,S) = 4608q^3 + 3686q^4 + 190464q^5 + 737280q^6 + \cdots.$$

We find that $q^{-2}X_1(R,S)$ and $q^{-2}X_2(R,S)$ does not tend to zero as $q \to 0$ whereas $q^{-2}X_3(R,S)$ tends to zero as $q \to 0$. Hence $q^{-2}X_3(R,S) = 0$ in some neighborhood $q = 0$. By analytic continuation $q^{-2}X_3(R,S) = 0$ in $|q| < 1$. Thus, we have

$$X_3(R,S) = 0.$$

On dividing throughout by $(RS)^2$, we obtain the required result.

Theorem 3.3 If

$$R := \frac{\varphi(q^3)\varphi(q^5)}{\varphi(q)\varphi(q^{15})} \qquad \text{and} \qquad S := \frac{\varphi(q^6)\varphi(q^{10})}{\varphi(q^2)\varphi(q^{30})},$$

then

$$\begin{aligned}(RS)^2 + \frac{1}{(RS)^2} - 24\left(RS + \frac{1}{RS}\right) + \left(\frac{R}{S}\right)^2 + \left(\frac{S}{R}\right)^2 \\ + \left[6\left(RS + \frac{1}{RS}\right) - 8\right]\left(\frac{R}{S} + \frac{S}{R}\right) - 4\left(\sqrt{RS} - \frac{1}{\sqrt{RS}}\right)\left[\left(\frac{R}{S}\right)^{3/2} + \left(\frac{S}{R}\right)^{3/2}\right] \\ + \left[16\left(\sqrt{RS} - \frac{1}{\sqrt{RS}}\right) - 4\left((RS)^{3/2} - \frac{1}{(RS)^{3/2}}\right)\right]\left[\left(\frac{R}{S}\right)^{1/2} + \left(\frac{S}{R}\right)^{1/2}\right] + 36 = 0.\end{aligned} \tag{3.16}$$

Proof. Let
$$P := \frac{\varphi(-q^3)\varphi(-q^5)}{\varphi(-q)\varphi(q^{-15})} \qquad \text{and} \qquad Q := \frac{\varphi(-q^6)\varphi(-q^{10})}{\varphi(-q^2)\varphi(-q^{30})}.$$
On employing Entry 10(ii) and (iii) of Chapter 17 [5, p. 122] in P and Q, we deduce
$$\frac{P}{Q} = \frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)} \qquad \text{and} \qquad \frac{P}{Q^2} = \left(\frac{z_1 z_{15}}{z_3 z_5}\right)^{1/2}$$
Using these in (2.5) and (2.6), it is easy to see that
$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} = \frac{Q(Q-1)}{P-Q} \qquad \text{and} \qquad \left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} = \frac{P(Q+1)}{Q(P-Q)}.$$
Multiplying these, we have
$$P^2 + Q^2 - 2PQ - PQ^2 + P = 0. \tag{3.17}$$
Changing q to $-q$ in (3.17), we have
$$R^2 + Q^2 - 2RQ - RQ^2 + R = 0. \tag{3.18}$$
Eliminating Q between (3.17) and (3.18) using Maple, we find that
$$P^2R^2 + P^2 - 2P^2R - 4PR - 2R^2P + 2P + R^2 + 2R + 1 = 0. \tag{3.19}$$
Changing q to q^2 in (3.19), we obtain
$$Q^2S^2 + Q^2 - 2Q^2S - 4QS - 2S^2Q + 2Q + S^2 + 2S + 1 = 0. \tag{3.20}$$
On eliminating Q between (3.18) and (3.20) using Maple, we find that
$$R^4S^4 - 4R^4S^3 - 4R^3S^4 + 6R^4S^2 - 24R^3S^3 + 6R^2S^4 - 4R^4S + 16R^3S^2 + 16R^2S^3 - 4RS^4 + R^4$$
$$-8R^3S + 36R^2S^2 - 8RS^3 + S^4 + 4R^3 - 16R^2S - 16RS^2 + 4S^3 + 6R^2 - 24RS + 6S^2 + 4R + 4S + 1 = 0.$$
Now, by dividing the above by $(RS)^2$ and then simplifying, we obtain the required result.

Theorem 3.4 If
$$R := \frac{\varphi(q^3)\varphi(q^{21})}{\varphi(q)\varphi(q^7)} \qquad \text{and} \qquad S := \frac{\varphi(q^6)\varphi(q^{42})}{\varphi(q^2)\varphi(q^{14})}$$
then
$6561x_8 - 104976x_7 + 2846016x_6 - 36869904x_5 + 259144272x_4 - 1106664336x_3 + 3047088960x_2 - 5546113680x_1 + y_8 - 8(122+3x_1)y_7 + (114368 - 54384x_1 + 252x_2)y_6 - (1094864 - 968736x_1 + 409104x_2 + 1512x_3)y_5 + (1670096 - 1094448x_1 + 616896x_2 - 397872x_3 + 5670x_4)y_4 + (64673968 - 53731488x_1 + 29372400x_2 - 9192528x_3 + 1069200x_4 - 13608x_5)y_3 - (338325312 - 267181584x_1 + 130664664x_2 - 38120112x_3 + 5925312x_4 - 283824x_5 - 20412x_6)y_2 - (822886992 - 728114640x_1 + 483574320x_2 - 218720736x_3 + 59749488x_4 - 8514720x_5 + 408240x_6 + 17496x_7)y_1 + 6762080356 = 0,$
where
$$x_n = (3RS)^n + \frac{1}{(3RS)^n} \qquad \text{and} \qquad y_n = \left(\frac{R}{S}\right)^n + \left(\frac{S}{R}\right)^n$$

Proof. Let
$$P := \frac{\varphi(-q^3)\varphi(-q^{21})}{\varphi(-q)\varphi(-q^7)} \qquad \text{and} \qquad Q := \frac{\varphi(-q^6)\varphi(-q^{42})}{\varphi(-q^2)\varphi(-q^{14})}$$

Using Entry 10(ii) and (iii) of Chapter 17 [5, p. 122] in P and Q, we can deduce

$$\frac{P}{Q}=\left(\frac{(1-\beta)(1-\delta)}{(1-\delta)(1-\gamma)}\right)^{1/8} \qquad \text{and} \qquad \frac{P}{Q^2}=\left(\frac{z_1 z_7}{z_3 z_{21}}\right)^{1/2}$$

Using these in (2.7) and (2.8) respectively, it is easy to see that

$$(P-Q)^2Q^2x^2-2PQ^2(P+Q)x+P^2(Q^2-1)=0 \tag{3.21}$$

and

$$\frac{(P-Q)^2}{x^2}-2\frac{Q(P+Q)}{x}+Q^2(1-9Q^2)=0, \tag{3.22}$$

where $x=\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4}$. Eliminating x between (3.22) and (3.23)using Maple, we obtain

$$\begin{aligned}&81P^4Q^8-18P^6Q^4-36P^5Q^5+72P^4Q^6-36P^3Q^7-18P^2Q^8+P^8-12P^7Q\\&+24P^6Q^2-12P^5Q^3-44P^4Q^4-12P^3Q^5+24P^2Q^6-12PQ^7+Q^8-2P^6-4P^5Q\\&+8P^4Q^2-4P^3Q^3-2P^2Q^4+P^4=0.\end{aligned} \tag{3.23}$$

Changing q to $-q$ in the above, we obtain

$$\begin{aligned}&81Q^8R^4-18Q^8R^2-36Q^7R^3+72Q^6R^4-36Q^5R^5-18Q^4R^6+Q^8-12Q^7R\\&+24Q^6R^2-12Q^5R^3-44Q^4R^4-12Q^3R^5+24Q^2R^6-12QR^7+R^8-2R^6-4Q^3R^3\\&+8Q^2R^4-4QR^5-2Q^4R^2+R^4=0.\end{aligned} \tag{3.24}$$

Now on eliminating Q between (3.24) and (3.25), using Maple and then proceeding as in proof of Theorem 3.2, we obtain

$$\begin{aligned}&6561P^8R^8-2916P^8R^6+11664P^7R^7-2916P^6R^8+486P^8R^4-3888P^6R^6\\&+486P^4R^8-36P^8R^2-1584P^7R^3+2952P^6R^4-8496P^5R^5+2952P^4R^6-1584P^3R^7\\&-36P^2R^8+P^8-96P^7R+272P^6R^2-3360P^5R^3+3396P^4R^4-3360P^3R^5+272P^2R^6\\&-96PR^7+R^8-4P^6-176P^5R+328P^4R^2-944P^3R^3+328P^2R^4-176PR^5-4R^6+6P^4\\&-48P^2R^2+6R^4-4P^2+16PR-4R^2+1=0.\end{aligned} \tag{3.25}$$

Changing q to q^2 in (3. 26), we find that

$$\begin{aligned}&6561Q^8S^8-2916Q^8S^6+11664Q^7S^7-2916Q^6S^8+486Q^8S^4-3888Q^6S^6\\&+486Q^4S^8-36Q^8S^2-1584Q^7S^3+2952Q^6S^4-8496Q^5S^5+2952Q^4S^6-1584Q^3S^7\\&-36Q^2S^8+Q^8-96Q^7S+272Q^6S^2-3360Q^5S^3+3396Q^4S^4-3360Q^3S^5+272Q^2S^6\\&-96QS^7+S^8-4Q^6-176Q^5S+328Q^4S^2-944Q^3S^3+328Q^2S^4-176QS^5-4S^6+6Q^4\\&-48Q^2S^2+6S^4-4Q^2+16QS-4S^2+1=0.\end{aligned} \tag{3.26}$$

Eliminating Q between (3.25) and (3.27)using Maple and then proceeding as in proof of Theorem 3.2, we obtain

$$\begin{aligned}&43046721R^{16}S^{16}-38263752R^{16}S^{14}-229582512R^{15}S^{15}-38263752R^{14}S^{16}\\&+14880348R^{16}S^{12}-297606960R^{15}S^{13}+2074745664R^{14}S^{14}-297606960R^{13}S^{15}\\&+14880348R^{12}S^{16}-3306744R^{16}S^{10}+68969232R^{15}S^{11}+2069076960R^{14}S^{12}\\&-8959386672R^{13}S^{13}+2069076960R^{12}S^{14}+68969232R^{11}S^{15}-3306744R^{10}S^{16}\end{aligned}$$

$$
\begin{aligned}
&+459270R^{16}S^{8}+86605200R^{15}S^{9}-479950272R^{14}S^{10}-4839708528R^{13}S^{11}\\
&+20990686032R^{12}S^{12}-4839708528R^{11}S^{13}-479950272R^{10}S^{14}+86605200R^{9}S^{15}\\
&+459270R^{8}S^{16}-40824R^{16}S^{6}-10742544R^{15}S^{7}-248198256R^{14}S^{8}+1029243024R^{13}S^{9}\\
&+5905459872R^{12}S^{10}-29879937072R^{11}S^{11}+5905459872R^{10}S^{12}+1029243024R^{9}S^{13}\\
&-248198256R^{8}S^{14}-10742544R^{7}S^{15}-40824R^{6}S^{16}+2268R^{16}S^{4}-3681936R^{15}S^{5}\\
&+5552064R^{14}S^{6}+264351600R^{13}S^{7}-1175981976R^{12}S^{8}-4352168880R^{11}S^{9}\\
&+27423800640R^{10}S^{10}-4352168880R^{9}S^{11}-1175981976R^{8}S^{12}+264351600R^{7}S^{13}\\
&+5552064R^{6}S^{14}-3681936R^{5}S^{15}+2268R^{4}S^{16}-72R^{16}S^{2}-163152R^{15}S^{3}\\
&+2906208R^{14}S^{4}-3283344R^{13}S^{5}-161194464R^{12}S^{6}+801544752R^{11}S^{7}+2184343920R^{10}S^{8}\\
&-16638341040R^{9}S^{9}+2184343920R^{8}S^{10}+801544752R^{7}S^{11}-161194464R^{6}S^{12}\\
&-3283344R^{5}S^{13}+2906208R^{4}S^{14}-163152R^{3}S^{15}-72R^{2}S^{16}+R^{16}-976R^{15}S+114368R^{14}S^{2}\\
&-1094864R^{13}S^{3}+1670096R^{12}S^{4}+64673968R^{11}S^{5}-338325312R^{10}S^{6}-822886992R^{9}S^{7}\\
&+6762080356R^{8}S^{8}-822886992R^{7}S^{9}-338325312R^{6}S^{10}+64673968R^{5}S^{11}+1670096R^{4}S^{12}\\
&-1094864R^{3}S^{13}+114368R^{2}S^{14}-976RS^{15}+S^{16}-8R^{14}-18128R^{13}S+322912R^{12}S^{2}\\
&-364816R^{11}S^{3}-17910496R^{10}S^{4}+89060528R^{9}S^{5}+242704880R^{8}S^{6}-1848704560R^{7}S^{7}\\
&+242704880R^{6}S^{8}+89060528R^{5}S^{9}-17910496R^{4}S^{10}-364816R^{3}S^{11}+322912R^{2}S^{12}\\
&-18128RS^{13}-8S^{14}+28R^{12}-45456R^{11}S+68544R^{10}S^{2}+3263600R^{9}S^{3}-14518296R^{8}S^{4}\\
&-53730480R^{7}S^{5}+338565440R^{6}S^{6}-53730480R^{5}S^{7}-14518296R^{4}S^{8}+3263600R^{3}S^{9}\\
&+68544R^{2}S^{10}-45456RS^{1}1+28S^{12}-56R^{1}0-14736R^{9}S-340464R^{8}S^{2}+1411856R^{7}S^{3}\\
&+8100768R^{6}S^{4}-40987568R^{5}S^{5}+8100768R^{4}S^{6}+1411856R^{3}S^{7}-340464R^{2}S^{8}-14736RS^{9}\\
&-56S^{1}0+70R^{8}+13200R^{7}S-73152R^{6}S^{2}-737648R^{5}S^{3}+3199312R^{4}S^{4}-737648R^{3}S^{5}\\
&-73152R^{2}S^{6}+13200RS^{7}+70S^{8}-56R^{6}+1168R^{5}S+35040R^{4}S^{2}-151728R^{3}S^{3}+35040R^{2}S^{4}\\
&+1168RS^{5}-56S^{6}+28R^{4}-560R^{3}S+3904R^{2}S^{2}-560RS^{3}+28S^{4}-8R^{2}-48RS-8S^{2}+1=0,
\end{aligned}
$$

Dividing this throughout by $(RS)^8$ and then upon rearranging, we obtain (3.21).

Theorem 3.5 If

$$P=\frac{\varphi(-q)}{\varphi(-q^3)} \qquad \text{and} \qquad Q=\frac{\varphi(-q^7)}{\varphi(-q^{21})},$$

then

$$
\begin{aligned}
&\left(\frac{P}{Q}\right)^4-\left(\frac{Q}{P}\right)^4+14\left(\left(\frac{P}{Q}\right)^2-\left(\frac{Q}{P}\right)^2\right)-7\left(\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2-1\right)\left(PQ+\frac{3}{PQ}\right)\\
&\qquad +(PQ)^3+\frac{27}{(PQ)^3}=0. \qquad (3.27)
\end{aligned}
$$

Proof. Let

$$M_n=\frac{f(-q^n)}{q^{n/2}f(-q^{3n})}.$$

It is easy to see that

$$P=\frac{M_1^2}{M_2} \qquad \text{and} \qquad Q=\frac{M_7^2}{M_{14}},$$

which implies

$$M_2 = \frac{M_1^2}{P}, \qquad M_{14} = \frac{M_7^2}{Q}. \tag{3.28}$$

From Entry 51 of Chapter 25 [6, p. 204],

$$(M_1M_2)^2 + \frac{9}{(M_1M_2)^2} = \left(\frac{M_2}{M_1}\right)^6 + \left(\frac{M_1}{M_7}\right). \tag{3.29}$$

Using (3.29) in (3.30), we deduce that

$$M_1^{12} = \frac{P^8(P^4-9)}{P^4-1}. \tag{3.30}$$

Changing q to q^7 in (3.31), we have

$$M_7^{12} = \frac{Q^8(Q^4-9)}{Q^4-1}.$$

Thus from the above and (3.31), we have

$$\left(\frac{M_1}{M_7}\right)^{12} = \frac{P^8(P^4-9)(Q^4-1)}{Q^8(P^4-1)(Q^4-1)}. \tag{3.31}$$

From Theorem 3.2 of [7], we have

$$\begin{aligned}(M_1M_2)^4(M_7M_{14})^2 + (M_1M_2)^2(M_7M_{14})^4 &= (M_1M_{14})^6 + (M_2M_7)^6 \\ &+4[(M_1M_{14})^4(M_2M_7)^2 + (M_1M_{14})^2(M_2M_7)^4].\end{aligned} \tag{3.32}$$

From (3.29), we have

$$M_2M_{14} = \frac{M_1^2M_7^2}{PQ}.$$

Using the above in (3.33) and then on dividing throughout by $(M_1M_7)^3$, we obtain

$$Q^4(P^2-Q^2)\left(\frac{M_1}{M_7}\right)^3 + P^4(Q^2-P^2)\left(\frac{M_7}{M_1}\right)^3 = 4P^2Q^4\left(\frac{M_1}{M_7}\right) + 4P^4Q^2\left(\frac{M_7}{M_1}\right). \tag{3.33}$$

Eliminating (M_1/M_7) between (3.32) and (3.34), we have

$$C(P,Q)D(P,Q) = 0,$$

where

$$\begin{aligned}C(P,Q) = P^7Q^7 - 7P^7Q^3 + 7P^5Q^5 - 7P^3Q^7 - P^8 - 14P^6Q^2 + 14P^2Q^6 + Q^8 \\ -21P^5Q + 21P^3Q^3 - 21PQ^5 + 27PQ\end{aligned}$$

and

$$\begin{aligned}D(P,Q) = P^7Q^7 - 7P^7Q^3 + 7P^5Q^5 - 7P^3Q^7 + P^8 + 14P^6Q^2 - 14P^2Q^6 - Q^8 \\ -21P^5Q + 21P^3Q^3 - 21PQ^5 + 27PQ.\end{aligned}$$

We have the following series expansion of P and Q:

$$P = 1 - 2q - 2q^3 + 6q^4 - 4q^7 - 2q^9 + \cdots$$

$$Q = 1 - 2q^7 - 2q^{21} + 6q^{28} - 4q^{49} - 2q^{63} + \cdots.$$

Using these in $C(P,Q)$ and $D(P,Q)$, we obtain

$$C(P,Q) = -256q + 1792q^2 - 5632q^3 + 13312q^4 - 35840q^5 91392q^6 - \cdots$$

and

$$D(P,Q) = 3584q^9 - 10752q^{10} + 14336q^{11} - 53760q^{12} + 129024q^{13} + \cdots.$$

It is easy to see that, $q^{-1}C(P,Q)$ does not tend to zero as $q \to 0$, whereas $q^{-1}D(P,Q)$ tend to zero as $q \to 0$. Hence $q^{-1}D(P,Q) = 0$ in some neighborhood $q = 0$. By analytic continuation $q^{-1}D(P,Q) = 0$ in $|q| < 1$. Thus we have

$$D(P,Q) = 0.$$

Finally, dividing this throughout by $(PQ)^4$ and rearranging, we obtain the required result.

4 Evaluations of Ramanujan's cubic continued fraction

Lemma 4.1 For $q = e^{-\pi\sqrt{n/3}}$, let

$$J_n := \frac{1}{\sqrt[4]{3}} \frac{\varphi(q)}{\varphi(q^3)}$$

Then

$$(i) \qquad J_n J_{1/n} = 1, \tag{4.1}$$

$$(ii) \qquad J_1 = 1, \tag{4.2}$$

$$(iii) \qquad D_n = \sqrt{\frac{\sqrt{3} - J_n^2}{1 + \sqrt{3}J_n^2}} \tag{4.3}$$

and

$$(iv) \qquad G(q) = \frac{1}{2}\sqrt[3]{1 - 3D_n^4}. \tag{4.4}$$

For a proof see [8].

Lemma 4.2

$$3J_n^2 J_{9n}^2 + \frac{3}{J_n^2 J_{9n}^2} = 3 + 6\frac{J_{9n}^2}{J_n^2} + \frac{J_{9n}^2}{J_n^4},$$

For a proof, see [8].

Lemma 4.3

$$3(J_n J_{25n})^2 + \frac{3}{(J_n J_{25n})^2} = \left(\frac{J_{25n}}{J_n}\right)^3 - \left(\frac{J_n}{J_{25n}}\right)^3$$
$$+5\left(\frac{J_{25n}}{J_n}\right)^2 + 5\left(\frac{J_n}{J_{25n}}\right)^2 + 5\left(\frac{J_{25n}}{J_n}\right) - 5\left(\frac{J_n}{J_{25n}}\right),$$

For a proof, see [8].

Theorem 4.1 If J_n is as defined as in Lemma 4.1, then

$$\left(\frac{J_{16n}}{J_n}\right)^2+\left(\frac{J_n}{J_{16n}}\right)^2-8\left(\frac{J_{16n}}{J_n}+\frac{J_n}{J_{16n}}\right)+6\left(\sqrt{\frac{J_{16n}}{J_n}}-\sqrt{\frac{J_n}{J_{16n}}}\right)$$

$$\times\left(\sqrt{\frac{J_{4n}^2}{J_nJ_{16n}}}+\sqrt{\frac{J_nJ_{16n}}{J_{4n}^2}}\right)-2\left(\left(\frac{J_{16n}}{J_n}\right)^{3/2}-\left(\frac{J_n}{J_{16n}}\right)^{3/2}\right)\left(\sqrt{\frac{J_{4n}^2}{J_nJ_{16n}}}+\sqrt{\frac{J_nJ_{16n}}{J_{4n}^2}}\right)$$

$$+\left(\frac{J_{16n}}{J_n}+\frac{J_n}{J_{16n}}\right)\left(\frac{J_{4n}^2}{J_nJ_{16n}}+\frac{J_nJ_{16n}}{J_{4n}^2}\right)+10=0. \tag{4.5}$$

Proof. It is enough to realize that (4.3) is same as the result of Theorem 3.1 with $R(q)=J_{4n}/J_n$ and $S(q)=J_{16n}/J_{4n}$.

Theorem 4.2 We have

$$J_4=\frac{(\sqrt{3}+1)(\sqrt{2}-1)}{\sqrt{2}}$$

and

$$J_{1/4}=\frac{(\sqrt{3}-1)(\sqrt{2}+1)}{\sqrt{2}}.$$

Proof. Put $n=1/4$ in (4.3) and using (4.1), we obtain

$$J_4^4+J_4^{-4}-4(J_4^3-J_4^{-3})-6(J_4^2+J_4^{-2})+12(J_4-J_4^{-1})+10=0.$$

Setting $J_4-J_4^{-1}=t$ in the above equation, we have

$$t^2(t^2-4t-2)=0.$$

Since J_n is positive, and decreasing, we choose

$$J_4-J_4^{-1}=2-\sqrt{6},$$

and

$$J_4=\frac{(\sqrt{3}+1)(\sqrt{2}-1)}{\sqrt{2}}.$$

Which implies

$$J_{1/4}=\frac{(\sqrt{3}-1)(\sqrt{2}+1)}{\sqrt{2}}.$$

Corollary 4.1 We have

$$(i) \qquad D_4 = \frac{\sqrt{\sqrt{3}-1}}{\sqrt[4]{2}}$$

and

$$(ii) \qquad D_{1/4} = \sqrt{6-4\sqrt{2}-3\sqrt{3}+2\sqrt{6}}.$$

Proof. By applying the results of Theorem 4.2 to (4.3) we at once have the required result.

Corollary 4.2 We have

$$G(e^{-2\pi/\sqrt{3}}) = \frac{1}{2}\sqrt[3]{3\sqrt{3}-5}$$

and

$$G(e^{-\pi/2\sqrt{3}}) = \frac{1}{2}\sqrt[3]{252\sqrt{2}+204\sqrt{3}-144\sqrt{6}-356}.$$

Proof. For proving part (i), we apply part (i) of Corollary 4.1 to equation 4.4. Similarly part (ii) is obtained from part (ii) of Corollary 4.1 and equation 4.4.

Theorem 4.3 If J_n is defined as in Lemma 4.1, then

$$\left(\frac{J_{9n}J_{4n}}{J_nJ_{36n}}\right)^2 + \left(\frac{J_nJ_{36n}}{J_{9n}J_{4n}}\right)^2 - 2\left(\frac{J_{9n}J_{36n}}{J_nJ_{4n}} - 3\frac{J_nJ_{4n}}{J_{9n}J_{36n}}\right)\left(\frac{J_{9n}J_{4n}}{J_nJ_{36n}} + \frac{J_nJ_{36n}}{J_{9n}J_{4n}}\right)$$
$$-16\left(\frac{J_{9n}J_{36n}}{J_nJ_{4n}} + 3\frac{J_nJ_{4n}}{J_{9n}J_{36n}}\right) + \left(\frac{J_{9n}J_{36n}}{J_nJ_{4n}}\right)^2 + 9\left(\frac{J_nJ_{4n}}{J_{9n}J_{36n}}\right)^2 + 44 = 0. \tag{4.6}$$

Proof. Proof is similar to Theorem 4.1. Indeed (4.4) is same as (3.10) with $R = \frac{J_{9n}}{J_n}$ and $S = \frac{J_{36n}}{J_{4n}}$.

Theorem 4.4 We have

$$J_6 = \sqrt[4]{603-426\sqrt{2}-348\sqrt{3}+246\sqrt{6}} = J_{1/6}^{-1}$$

and

$$J_{2/3} = \sqrt[4]{\frac{9-2\sqrt{2}-4\sqrt{3}+2\sqrt{6}}{3}} = J_{3/2}^{-1}.$$

Proof. On setting $n = 1/6$ in Theorem 4.3, we have

$$3\left(\frac{x^2}{3}+\frac{3}{x^2}\right) - 16\sqrt{3}\left(\frac{x}{\sqrt{3}}+\frac{\sqrt{3}}{x}\right) - 4\left(\frac{x}{\sqrt{3}}-\frac{\sqrt{3}}{x}\right) + 46 = 0,$$

where $x = (J_6/J_{2/3})^2$.
On setting $\frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{x} = t$, then $\frac{x^2}{3} + \frac{3}{x^2} = t^2 - 2$ and $\frac{x}{\sqrt{3}} - \frac{\sqrt{4}}{x} = \sqrt{t^2-4}$.
Using these in the above, we obtain

$$9t^4 - 96\sqrt{3}t^3 + 960t^2 - 1280\sqrt{3}t + 1792 = 0.$$

Solving this and since J_n is decreasing, we obtain,

$$t = \frac{4}{\sqrt{3}}(3 - 2\sqrt{2}).$$

Again on solving and since $x < 1$,

$$\frac{J_6}{J_{2/3}} = \sqrt{9 - 6\sqrt{2}}. \tag{4.7}$$

Put $n = 2/3$, in Lemma 4.3, we have

$$x + \frac{1}{x} = 70 - 48\sqrt{2}$$

where $x = (J_6 J_{2/3})^2$. On solving we choose

$$J_6 J_{2/3} = \sqrt{35 - 24\sqrt{2} - 20\sqrt{3} + 14\sqrt{6}},$$

From (4.5) and the above, we obtain the required result.

Corollary 4.3 We have

$$(i) \qquad D_6 = \sqrt{3 - \sqrt{6}},$$

$$(ii) \qquad D_{1/6} = \frac{\sqrt{\sqrt{6} - \sqrt{2} - 1}}{\sqrt[4]{3}},$$

$$(iii) \qquad D_{2/3} = \frac{\sqrt{\sqrt{3} - \sqrt{2}}}{\sqrt[4]{3}}$$

and

$$(iv) \qquad D_{3/2} = \sqrt{12 + 9\sqrt{2} - 7\sqrt{3} - 5\sqrt{6}}.$$

Proof. For proving part (i) and (iii), we need only apply Theorem 4.4 to equation (4.3). Part (ii) and (iv) follows similarly.

Corollary 4.4 We have

$$(i) \qquad G(e^{-\pi\sqrt{2}}) = \frac{1}{2}\sqrt[3]{18\sqrt{6} - 44},$$

$$(ii) \qquad G(e^{-\pi/3\sqrt{2}}) = \frac{1}{2}\sqrt[3]{2\sqrt{6} - 2\sqrt{2} + 4\sqrt{3} - 8},$$

$$(iii) \qquad G(e^{-\pi\sqrt{2}/3}) = \frac{1}{2}\sqrt[3]{2\sqrt{6} - 4}$$

$$(iv) \qquad G(e^{-\pi/\sqrt{2}}) = \frac{1}{2}\sqrt[3]{738\sqrt{6} - 1278\sqrt{2} + 1044\sqrt{3} - 1808}.$$

Proof. For proving part (i), we need only apply part (i) of Corollary 4.3 to (4.4). Similarly, part (ii), (iii) and (iv) is obtained.

Theorem 4.5 If J_n is as defined as in Lemma 4.1, then

$$\left(\frac{J_{25n}J_{100n}}{J_nJ_{4n}}\right)^2 + \left(\frac{J_nJ_{4n}}{J_{25n}J_{100n}}\right)^2 - 24\left(\frac{J_{25n}J_{100n}}{J_nJ_{4n}} + \frac{J_nJ_{4n}}{J_{25n}J_{100n}}\right) + \left(\frac{J_{25n}J_{4n}}{J_nJ_{100n}}\right)^2 + \left(\frac{J_nJ_{100n}}{J_{25n}J_{4n}}\right)^2$$

$$\left[6\left(\frac{J_{25n}J_{100n}}{J_nJ_{4n}}+\frac{J_nJ_{4n}}{J_{25n}J_{100n}}\right)-8\right]\left\{\frac{J_{25n}J_{4n}}{J_nJ_{100n}}+\frac{J_nJ_{100n}}{J_{25n}J_{4n}}\right\}-4\left[\left(\frac{J_{25n}J_{100n}}{J_nJ_{4n}}\right)^{1/2}-\left(\frac{J_nJ_{4n}}{J_{25n}J_{100n}}\right)^{1/2}\right]$$
$$\times\left\{\left(\frac{J_{25n}J_{4n}}{J_nJ_{100n}}\right)^{3/2}+\left(\frac{J_nJ_{100n}}{J_{25n}J_{4n}}\right)^{3/2}\right\}+\left[16\left\{\left(\frac{J_{25n}J_{100n}}{J_nJ_{4n}}\right)^{1/2}-\left(\frac{J_nJ_{4n}}{J_{25n}J_{100n}}\right)^{1/2}\right\}\right.$$
$$\left.-4\left\{\left(\frac{J_{25n}J_{100n}}{J_nJ_{4n}}\right)^{3/2}-\left(\frac{J_nJ_{4n}}{J_{25n}J_{100n}}\right)^{3/2}\right\}\right]\left[\left(\frac{J_{25n}J_{4n}}{J_nJ_{100n}}\right)^{1/2}+\left(\frac{J_nJ_{100n}}{J_{25n}J_{4n}}\right)^{1/2}\right]+36=0. \tag{4.8}$$

Proof. It is easy to see that (4.6) is same as the result of Theorem 3.3 with $R=\frac{J_{25n}}{J_n}$ and $S=\frac{J_{100n}}{J_{4n}}$.

Theorem 4.6 We have

$$J_{10}=\sqrt{2-\sqrt{6}+2\sqrt{2}-\sqrt{3}}\sqrt[4]{99-40\sqrt{6}-18\sqrt{30}+44\sqrt{5}}=J_{1/10}^{-1}$$

and

$$J_{2/5}=\frac{\sqrt[4]{99-40\sqrt{6}-18\sqrt{30}+44\sqrt{5}}}{\sqrt{2-\sqrt{6}+2\sqrt{2}-\sqrt{3}}}=J_{5/2}^{-1}.$$

Proof. Setting $n=1/10$ in (4.6) and using (4.1), we obtain

$$\left(\frac{J_{10}}{J_{2/5}}\right)^4+\left(\frac{J_{2/5}}{J_{10}}\right)^4-8\left\{\left(\frac{J_{10}}{J_{2/5}}\right)^3-\left(\frac{J_{2/5}}{J_{10}}\right)^3\right\}-12\left\{\left(\frac{J_{10}}{J_{2/5}}\right)^2+\left(\frac{J_{2/5}}{J_{10}}\right)^2\right\}$$
$$+24\left\{\frac{J_{10}}{J_{2/5}}-\frac{J_{2/5}}{J_{10}}\right\}+22=0. \tag{4.9}$$

Put $x=J_{10}/J_{2/5}$, then,

$$x^4+x^{-4}-8(x^3-x^{-3})-12(x^2+x^{-2})+24(x-x^{-1})+22=0.$$

Setting $x-x^{-1}=t$ in the above equation, we have

$$t^2(t^2-8t-8)=0.$$

Since J_n is positive, and decreasing in n, we have

$$x-x^{-1}=4-2\sqrt{6}. \tag{4.10}$$

On solving this,

$$\frac{J_{10}}{J_{2/5}}=2-\sqrt{6}+2\sqrt{2}-\sqrt{3}. \tag{4.11}$$

Put $n=2/5$ in Lemma 4.3 and on using (4.7), we obtain

$$(J_{2/5}J_{10})^2+\frac{1}{(J_{2/5}J_{10})^2}=198-80\sqrt{6}.$$

Since J_n is positive and decreasing in n, on solving this, we have

$$J_{2/5}J_{10}=\sqrt{99-40\sqrt{6}-18\sqrt{30}+44\sqrt{5}}. \tag{4.12}$$

From (4.9) and (4.10), we obtain J_{10} and $J_{2/5}$.

Corollary 4.5 We have

$$(i) \qquad D_{10} = \sqrt{\sqrt{10} + \sqrt{5} - 2\sqrt{2} - 2},$$

$$(ii) \qquad D_{1/10} = \sqrt{6\sqrt{30} + 7\sqrt{10} + 9\sqrt{6} - 10\sqrt{5} - 13\sqrt{3} - 16\sqrt{2} - 4\sqrt{30} + 22},$$

$$(iii) \qquad D_{2/5} = \sqrt{\sqrt{10} - \sqrt{5} - 2\sqrt{2} + 2}$$

and

$$(iv) \qquad D_{5/2} = \sqrt{4\sqrt{30} + 6\sqrt{15} + 7\sqrt{10} - 9\sqrt{6} + 10\sqrt{5} - 13\sqrt{3} - 16\sqrt{2} - 22}.$$

Proof. For part (i), we apply part (i) of Theorem 4.6 to (4.3). Similarly, part (ii), (iii) and (iv) are obtained from Theorem 4.6 and (4.3).

Corollary 4.6 We have

$$(i) \qquad G(e^{-\pi\sqrt{10/3}}) = \frac{1}{2}\sqrt[3]{24\sqrt{10} + 36\sqrt{5} - 54\sqrt{2} - 80},$$

$$(ii) \qquad G(e^{-\pi/\sqrt{30}}) = \frac{1}{2}$$

$$\times \sqrt[3]{822\sqrt{30} - 1152\sqrt{15} - 1416\sqrt{10} - 1836\sqrt{6} + 2016\sqrt{5} + 2604\sqrt{3} + 6318\sqrt{2} - 9296},$$

$$(iii) \qquad G(e^{-\pi\sqrt{2/15}}) = \frac{1}{2}\sqrt[3]{54\sqrt{2} + 36\sqrt{5} - 24\sqrt{10} - 80}$$

and

$$(iv) \qquad G(e^{-\pi\sqrt{5/6}}) = \frac{1}{2}$$

$$\times \sqrt{2190\sqrt{30} + 3096\sqrt{15} + 3792\sqrt{10} - 4896\sqrt{6} + 5364\sqrt{5} - 6924\sqrt{3} - 8478\sqrt{2} - 11996}.$$

Proof. For proving part (i), we need only to apply part (i) of Corollary 4.5 to (4.4). Similarly, for part (ii), (iii) and (iv), we apply part (ii), (iii) and (iv) of Corollary 4.5 to (4.4).

Theorem 4.7 If J_n is as defined as in Lemma 4.1, then
$6561x_8 - 104976x_7 + 2846016x_6 - 36869904x_5 + 259144272x_4 - 1106664336x_3 + 3047088960x_2 - 5546113680x_1 + y_8 - 8(122 + 3x_1)y_7 + (114368 - 54384x_1 + 252x_2)y_6 - (1094864 - 968736x_1 + 409104x_2 + 1512x_3)y_5 + (1670096 - 1094448x_1 + 616896x_2 - 397872x_3 + 5670x_4)y_4 + (64673968 - 53731488x_1 + 29372400x_2 - 9192528x_3 + 1069200x_4 - 13608x_5)y_3 - (338325312 - 267181584x_1 + 130664664x_2 - 38120112x_3 + 5925312x_4 - 283824x_5 - 20412x_6)y_2 - (822886992 - 728114640x_1 + 483574320x_2 - 218720736x_3 + 59749488x_4 - 8514720x_5 + 408240x_6 + 17496x_7)y_1 + 6762080356 = 0,$
where

$$x_n = (J_n J_{4n} J_{49n} J_{196n})^n + \frac{1}{(J_n J_{4n} J_{49n} J_{196n})^n} \qquad \text{and} \qquad y_n = \left(\frac{J_n J_{49n}}{J_{4n} J_{196n}}\right)^n + \left(\frac{J_{4n} J_{196n}}{J_n J_{49n}}\right)^n$$

Proof. It is enough to realize that (4.12) is same as the result of Theorem 3.4 with $R(q) = 1/J_n J_{49n}$ and $S(q) = 1/J_{4n} J_{196n}$.

Theorem 4.9 We have

$$J_{14} = \sqrt{\sqrt{2}-1}\sqrt[4]{5\sqrt{7}-5\sqrt{6}+2\sqrt{42}-12} = J_{1/14}^{-1}$$

and

$$J_{2/7} = \frac{\sqrt{\sqrt{2}-1}}{\sqrt[4]{5\sqrt{7}-5\sqrt{6}+2\sqrt{42}-12}} = J_{7/2}^{-1}.$$

Proof. Put $n = 1/14$ in (4.11), we obtain

$$x^8 + x^{-8} - 1024(x^7 + x^{-7}) + 6104(x^6 + x^{-6}) + 21376(x^5 + x^{-5}) - 69412(x^4 + x^{-4})$$
$$-318080(x^3 + x^{-3}) - 293400(x^2 + x^{-2}) + 314112(x + x^{-1}) + 746182 = 0,$$

where $x = (J_{2/7}J_{14})^2$. Setting $x + x^{-1} = t$, and then simplifying, we obtain

$$t^8 - 1024t^7 + 6096t^6 + 28544t^5 - 106016t^4 - 439296t^3 + 39168t^2 + 1382400t + 1181952 = 0.$$

On solving, and since J_n is positive and decreasing function, we choose

$$x + x^{-1} = 6$$

and we arrive at

$$J_{2/7}J_{14} = \sqrt{3-2\sqrt{2}} = \sqrt{2}-1. \tag{4.13}$$

Set $n = 2/7$ in (4.12) and on using (4.13)

$$x^2 - \frac{1}{x^2} - 14\left(x - \frac{1}{x}\right) + 14\sqrt{6}\left(x + \frac{1}{x}\right) - 44\sqrt{6} = 0,$$

where $x = (J_{14}/J_{2/7})^2$. Again on setting $x + x^{-1} = v$ in this, it is easy to see that $x - x^{-1} = \sqrt{v^2-4}$. On using these in the above we obtain

$$(v^2-4)(v+14)^2 = 6(-14v+44)^2$$

On solving, and since J_n is decreasing, choose

$$\frac{J_{14}}{J_{2/7}} = \sqrt{5\sqrt{7}+2\sqrt{42}-5\sqrt{6}-12} \tag{4.14}$$

From (4.13) and (4.14) we have J_{14} and $J_{2/7}$.

Corollary 4.7 We have

$$(i) \qquad D_{14} = \sqrt{\frac{\sqrt{3}-a}{1+\sqrt{3}a}},$$

$$(ii) \qquad D_{1/14} = \sqrt{\frac{\sqrt{3}a-1}{a+\sqrt{3}}},$$

$$(iii) \qquad D_{2/7} = \sqrt{\frac{\sqrt{3}b-\sqrt{2}+1}{b+\sqrt{6}-\sqrt{3}}}$$

and

$$(iv) \qquad D_{7/2} = \sqrt{\frac{\sqrt{6}-\sqrt{3}-b}{\sqrt{2}-1+b}}.$$

Where

$$a = (\sqrt{2}-1)b \qquad \text{and} \qquad b = \sqrt{5\sqrt{7}-5\sqrt{6}+2\sqrt{42}-12}.$$

Proof. By applying the results of Theorem 4.9 to (4.3), we have the required result.

Corollary 4.8 We have

$$(i) \qquad G(e^{-\pi\sqrt{14/3}}) = \frac{1}{2}\sqrt[3]{1-3c},$$

$$(ii) \qquad G(e^{-\pi/\sqrt{42}}) = \frac{1}{2}\sqrt[3]{1-3d},$$

$$(iii) \qquad G(e^{-\pi\sqrt{2/21}}) = \frac{1}{2}\sqrt[3]{1-3e}$$

and

$$(iv) \qquad G(e^{-\pi\sqrt{7/6}}) = \frac{1}{2}\sqrt[3]{1-3f}.$$

where

$$c = \left(\frac{\sqrt{3}-a}{1+\sqrt{3}a}\right)^2 \qquad d = \left(\frac{\sqrt{3}a-1}{a+\sqrt{3}}\right)^2 \qquad e = \left(\frac{\sqrt{3}b-\sqrt{2}+1}{}\right)^2 \qquad f = \left(\frac{\sqrt{6}-\sqrt{3}-b}{\sqrt{2}-1+b}\right)^2.$$

with a and b are as in Corollary 4.7.

Proof. For proving part (i), we need only apply part (i) of Corollary 4.7 to (4.4). Similarly, part (ii), (iii) and (iv) is obtained.

Theorem 4.8 If J_n is as defined as in Lemma 4.1, then

$$\left(\frac{J_{49n}}{J_n}\right)^4 - \left(\frac{J_n}{J_{49n}}\right)^4 - 14\left[\left(\frac{J_{49n}}{J_n}\right)^2 - \left(\frac{J_n}{J_{49n}}\right)^2\right] - 7\sqrt{3}\left(J_nJ_{49n} + \frac{1}{J_nJ_{49n}}\right)$$
$$+7\sqrt{3}\left(J_nJ_{49n} + \frac{1}{J_nJ_{49n}}\right)\left[\left(\frac{J_{49n}}{J_n}\right)^2\left(\frac{J_n}{J_{49n}}\right)^2\right] - 3\sqrt{3}\left((J_nJ_{49n})^3 + \frac{1}{(J_nJ_{49n})^3}\right) = 0. \tag{4.15}$$

Proof. It is enough to realize that (4.12) is same as the result of Theorem 3.5 with $P(-q) = \sqrt[4]{3}J_n$ and $Q(-q) = \sqrt[4]{3}J_{49n}$.

Theorem 4.10 We have

$$J_7 = \sqrt[4]{4\sqrt{7}-6\sqrt{3}+2\sqrt{21}-9} = J_{1/7}^{-1}. \tag{4.16}$$

Proof. Put $n = 1/7$ in (4.12), we obtain

$$x^2 - x^{-2} + 14(x - x^{-1}) + 14\sqrt{3}(x + x^{-1}) - 20\sqrt{3} = 0,$$

where $x = J_7^4$. On solving this on the same lines, we obtain

$$v^4 + 28v^3 - 396v^2 + 1568v - 1984 = 0,$$

where $v = x + x^{-1}$. Since J_n is positive and decreasing

$$x + x^{-1} = -18 + 8\sqrt{7}.$$

Finally on solving this, we obtain the result.

Corollary 4.9 We have

$$(i) \qquad D_7 = \sqrt{\frac{\sqrt{3} - a}{1 + \sqrt{3}a}}$$

and

$$(ii) \qquad D_{1/7} = \sqrt{\frac{\sqrt{3}a - 1}{a + \sqrt{3}}}$$

where $a = \sqrt{2\sqrt{21} + 4\sqrt{7} - 6\sqrt{3} - 9}$.

Proof. For proving part (i), we apply Theorem 4.10 to (4.3). Similarly part (ii) also follows.

Corollary 4.10 We have

$$(i) \qquad G(e^{-\pi\sqrt{7/3}}) = \frac{1}{2}\sqrt[3]{1 - 3b}$$

and

$$(ii) \qquad G(e^{-\pi/\sqrt{21}}) = \frac{1}{2}\sqrt[3]{1 - 3c}.$$

where $b = \left(\frac{\sqrt{3}-a}{1+\sqrt{3}a}\right)^2$ and $c = \left(\frac{\sqrt{3}a-1}{a+\sqrt{3}}\right)^2$ with a is as in Corollary 4.9.

Proof. For proving part (i), we apply part (i) of Corollary 4.9 to 4.4. Similarly, for part (ii), we apply part (ii) of Corollary 4.10 to (4.4).

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