J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol. 1, No. 1 (2012), pp. 81-110

Integral and Series Representations of Riemann's Zeta function, Dirichelet's Eta Function and a Medley of Related Results

Michael S. Milgram[1]
Consulting Physicist, Geometrics Unlimited, Ltd.
Box 1484, Deep River, Ont. Canada. K0J 1P0

Abstract

Abstract: Contour integral representations for Riemann's Zeta function and Dirichelet's Eta (alternating Zeta) function are presented and investigated. These representations flow naturally from methods developed in the 1800's, but somehow they do not appear in the standard reference summaries. textbooks or literature. Using these representations as a basis, alternate derivations of known series and integral representations for the Zeta and Eta function are obtained on a unified basis that differs from the textbook approach, and results are developed that appear to be new.

1 Introduction

Riemann's Zeta function $\zeta(s)$, and its sibling, Dirichelet's (alternating zeta) function $\eta(s)$ play an important role in physics, complex analysis and number theory, and have been studied extensively for several centuries. In the same vein, the general importance of a contour integral representation for any function has been known for almost two centuries, so it is surprising that contour integral representations for both $\zeta(s)$ and $\eta(s)$ exist that cannot be found in any of the modern handbooks (NIST, [29], Section 25.5(iii); Abramowitz and Stegun, [1], Chapter 23). textbooks (Apostol, [5], Chapter 12; Olver, [30], Chapter 8.2; Titchmarsh, [37], Chapter 4; Whittaker and Watson, [39], Section (13.13)), summaries (Edwards, [12], Ivic, [19], Chapter 4: Patterson, [31]; Srivastava and Choi, [34] and [35]), compendia (Bateman, [14], Section 1.12 and Chapter 17), tables (Gradshteyn and Ryzhik, [17], 9.512; Prudnikov et. al, [32], Appendix II.7) and websites ([13], [29], [40], [41], [42], [43]) that summarize what is known about these functions. One can only conclude that such representations, generally discussed in the literature of the late 1800's, have been long-buried and their significance overlooked by modern scholars.

It is the purpose of this work to disinter these representations, revisit and explore some of the consequences. In particular, it is possible to obtain series and integral representations of $\zeta(s)$ and $\eta(s)$ that unify and generalize well-known results in various limits and combinations and reproduce results that are only now being discovered and investigated by alternate means. Additionally, it is possible to explore the properties of $\zeta(s)$ in the complex plane and on the critical line, and obtain results that appear to be new. Many of the results obtained are disparate and difficult to categorize in a unified manner, but share the common theme that they are all somehow obtained from a study of the revived integral representations. That is the unifying theme of this work. To maintain a semblance of brevity, most of the derivations are only sketched, with citations sufficient enough to allow the reader to reproduce any result for her/himself.

[1] mike@geometrics-unlimited.com