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OPERATIONS ON SPECTRAL FUZZY GRAPHS

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Abstract: The operations in spectral fuzzy graphs such as Cartesian product, strong product, bipartite double, Kronecker product and Lexicographic product (composition) are analysed to elucidate the impact of eigenvalue spectra. Also, the properties are derived which delves the effect on spectral fuzzy graphs.

Keywords and Phrases: Fuzzy graphs, Spectral analysis, Graph operations, Cartesian product, Strong product, Bipartite double, Kronecker product, Fuzzy eigenvalues.

2020 Mathematics Subject Classification: 05C50, 05C72, 05C76.

1. Introduction

Spectral graph theory investigates the spectral properties of graphs, focusing on their structural and spectral attributes [9]. This field emerged from the application of algebraic techniques to graph theory [5], particularly examining the eigenvalues and eigenvectors of matrices associated with a graph, such as the adjacency matrix (λ), and the Laplacian matrix (ϑ). Essam El Seidy computed the spectra of fundamental graphs resulting from various graph operations [8]. Comprehending the impact of these operations on the spectral properties of graphs is fundamental in domains of network analysis.

The concept of fuzzy graphs [17], introduced by L.A. Zadeh in 1965, addresses uncertainty in graph theory [10]. Kauffmann initially introduced fuzzy graphs [4], and Rosenfeld [14] subsequently incorporated fuzzy relations into their framework.

Building upon the seminal work of Yeh and Bang [16] in 1975, Samanta Pal, Rashmanlou, and other researchers established numerous concepts within the realm of fuzzy graphs. Nagoor Gani discussed the properties of Cartesian products and vertex degrees in composition [12]. Nirmal et al. elaborated on vertex degrees in fuzzy graphs, including tensor and normal products [12]. Moderson demonstrated that the cartesian product and union of two fuzzy subgraphs satisfy necessary and sufficient conditions [11]. D. Venugopalam extensively covered various operations on fuzzy graphs [15]. Shovan Dogra identified different product categories on fuzzy graphs and analyzed vertex degrees [7]. These operations extend from crisp graphs [3] to fuzzy graphs (FGs), utilizing adjacency matrices (A) and Laplacian matrices (L) of FGs.

The examination of spectral characteristics in fuzzy graphs frequently incorporates methods from graph theory, matrix theory, and fuzzy set theory. Compared to classical graphs, fuzzy graphs produced by different graph operations and products may have more complicated spectral characteristics. Degrees of membership (μ_i, μ_j) are introduced in fuzzy graphs, enabling a more flexible depiction of interactions between vertices. Insights into the spectral characteristics of fuzzy graphs, influenced by the structure of the original graphs, can be acquired through an examination of the product procedures.

The interplay between spectral graph theory and fuzzy graph theory opens new avenues for analyzing complex networks, especially in situations where uncertainty and imprecision are inherent. The spectral properties of fuzzy graphs can reveal underlying patterns and connections that are not immediately apparent in classical graph representations.

In recent years, there has been a growing interest in exploring the spectral properties of fuzzy graphs resulting from different graph operations, such as Cartesian products, tensor products, and normal products. Researchers have developed various techniques to analyze these properties, providing valuable insights into the behaviour of complex systems modelled by fuzzy graphs.

This paper is organized as follows: The authors provide the foundational concepts and background information in Section 2. Section 3 derives the main results relating eigenvalues of operated fuzzy graphs to the eigenvalues of original fuzzy graphs. Additionally, the spectral features on fuzzy graphs in terms of their eigenvalues (λ_i) and (η_j) are illustrated with examples. The spectrum of a fuzzy graph is demonstrated to be made up of the eigenvalues and eigenvectors of these matrices. Section 4 summarizes the spectral properties for different fuzzy graph operations and concludes the paper.

2. Preliminary and Definitions

Definition 2.1. [10] *A fuzzy graph $G = (V, \sigma, \mu)$ is a triple consisting of a non-empty set V together with a pair of functions $\sigma : V \longrightarrow [0, 1]$ is a fuzzy vertex set and $\mu : V \times V \longrightarrow [0, 1]$ is a fuzzy edge set such that $\mu(ij) \leq \sigma(i) \wedge \sigma(j)$ for all $i, j \in V$.*

Definition 2.2. [1] *For a square matrix M, the multiset of eigenvalues of M is called the spectrum of M and is denoted by $\Gamma(G) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_p^{(m_p)}\}$ where each λ_i is a distinct eigenvalue of M with multiplicity m_i, for all $i = 1, 2, \ldots, p$.*

Definition 2.3. [2] *The adjacency matrix A of a fuzzy graph $G = (V, \sigma, \mu)$ is an $n \times n$ matrix defined as $A = [a_{ij}]$ where $a_{ij} = \mu(v_i, v_j)$. The eigenvalues are denoted by $\lambda_i : \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$ of A.*

Definition 2.4. [8] *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then Cartesian product of $G_1 \Box G_2$ is a pair of functions $(\sigma_1 \Box \sigma_2, \mu_1 \Box \mu_2)$ with underlying vertex set $V_1 \Box V_2 = \{(u_1, v_1) : u_1 \in V_1$ and $v_1 \in V_2\}$ and underlying edge set $E_1 \Box E_2 = \{((u_1, v_1)(u_2, v_2)) : u_1 = u_2, v_1v_2 \in E_2$ or $u_1u_2 \in E_1, v_1 = v_2\}$ with*

$$
\begin{aligned}
(\sigma_1 \Box \sigma_2)(u_1, v_1) &= \sigma_1(u_1) \wedge \sigma_2(u_2), \text{ where } u_1 \in V_1 \text{ and } v_1 \in V_2 \\
(\mu_1 \Box \mu_2)((u_1, v_1)(u_2, v_2)) &= \sigma_1(u_1) \wedge \mu_2(v_1v_2), \text{ if } u_1 = u_2 \text{ and } v_1v_2 \in E_2 \\
&= \mu_1(u_1u_2) \wedge \sigma_2(v_1) \text{ if } u_1u_2 \in E_1 \text{ and } v_1 = v_2
\end{aligned}
$$

Definition 2.5. [8] *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then Strong product of $G_1 \boxtimes G_2$ is a pair of functions $(\sigma_1 \boxtimes \sigma_2, \mu_1 \boxtimes \mu_2)$ with underlying vertex set $V_1 \boxtimes V_2 = \{(u_1, v_1) : u_1 \in V_1$ and $v_1 \in V_2\}$ and underlying edge set $E_1 \boxtimes E_2 = \{((u_1, v_1)(u_2, v_2)) : u_1 = u_2, v_1v_2 \in E_2$ or $u_1u_2 \in E_1, v_1 = v_2\}$ with*

$$
\begin{aligned}
(\sigma_1 \boxtimes \sigma_2)(u_1, v_1) &= \sigma_1(u_1) \wedge \sigma_2(u_2), \text{ where } u_1 \in V_1 \text{ and } v_1 \in V_2 \\
(\mu_1 \boxtimes \mu_2)((u_1, v_1)(u_2, v_2)) &= \sigma_1(u_1) \wedge \mu_2(v_1v_2), \text{ if } u_1 = u_2 \text{ and } v_1v_2 \in E_2 \\
&= \mu_1(u_1u_2) \wedge \sigma_2(v_1) \text{ if } u_1u_2 \in E_1 \text{ and } v_1 = v_2 \\
(\mu_1 \boxtimes \mu_2)((u_1, v_1)(u_2, v_2)) &= \mu_1(u_1u_2) \wedge \mu_2(v_1v_2), \text{ if } u_1u_2 \in E_1 \text{ and } v_1v_2 \in E_2
\end{aligned}
$$

Definition 2.6. [6] *Let $G_1 = (\sigma_1, \mu_1)$ and K_2 be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. If G_1 is bipartite, its double is just the union of two disjoint copies. If G_1 is connected and not bipartite, then its double is connected and bipartite. If G_1 has spectrum Φ, then $G_1 \otimes K_2$ has spectrum $\Phi \cup -\Phi$.*

Definition 2.7. [8] *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then Kronecker product of $G_1 \otimes G_2$ is a pair of functions $(\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2)$ with underlying vertex set $V_1 \otimes V_2 = \{(u_1, v_1) : u_1 \in V_1$ and $v_1 \in V_2\}$ and underlying edge set $E_1 \otimes E_2 = \{((u_1, v_1)(u_2, v_2)) : u_1u_2 \in E_1, v_1v_2 \in E_2\}$ with*

$$(\sigma_1 \otimes \sigma_2)(u_1, v_1) = \sigma_1(u_1) \wedge \sigma_2(u_2), \text{ where } u_1 \in V_1 \text{ and } v_1 \in V_2$$
$$(\mu_1 \otimes \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2), \text{ if } u_1u_2 \in E_1 \text{ and } v_1v_2 \in E_2$$

Definition 2.8. [13] *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then Lexicographic product of $G_1[G_2]$ is a pair of functions defined by*

$$\begin{aligned}(\sigma_1 \circ \sigma_2)(u_1, v_1) &= \sigma_1(u_1) \wedge \sigma_2(u_2), \text{ where } u_1 \in V_1 \text{ and } v_1 \in V_2\\ (\mu_1 \circ \mu_2)((u_1, v_1)(u_2, v_2)) &= \sigma_1(u_1) \wedge \mu_2(v_1, v_2), \text{ if } u_1 = u_2, (v_1, v_2) \in E_2\\ &= \sigma_2(v_1) \wedge \mu_1(u_1, u_2), \text{ if } v_1 = v_2, (u_1, u_2) \in E_1\\ &= \mu_1(u_1u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2), \text{ if } v_1 \neq v_2 \text{ and } (u_1, u_2) \in E_1\end{aligned}$$

3. Main Theorems

Theorem 3.1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of G_1 and $\eta_1, \eta_2, \ldots, \eta_m$ be the eigenvalues of G_2. Then, the eigenvalues of $G_1 \Box G_2$ are $\lambda_i + \eta_j$, for all $i \in [1, n]$ and $j \in [1, m]$.*

Proof. Let A_1 and A_2 denote the fuzzy adjacency matrices of G_1 and G_2 respectively. Then for every eigenvalue λ and every eigen vector x of A_1 and for every eigenvalue η and every eigen vector y of A_2, $A_1x = \lambda x$ and $A_2y = \eta y$.
From the definition of Cartesian product, it follows that

$$\begin{aligned}(A_1 \otimes I_m + I_n \otimes A_2)(x \otimes y) &= (A_1 \otimes I_m)(x \otimes y) + (I_n \otimes A_2)(x \otimes y)\\ &= A_1x \otimes I_my + I_nx \otimes A_2y\\ &= \lambda x \otimes y + x \otimes \eta y\\ &= \lambda(x \otimes y) + \eta(x \otimes y)\\ &= (\lambda + \eta)(x \otimes y)\end{aligned}$$

Thus, $\lambda_i + \eta_j$ is the eigenvalue of $G_1 \Box G_2$.

Example 3.2. Consider G_1 and G_2 as shown in Figure 3.1 having $\lambda_i = \{-0.5, -0.17, 0.6\}$ and $\eta_j = \{-0.3, 0.3\}$ respectively.

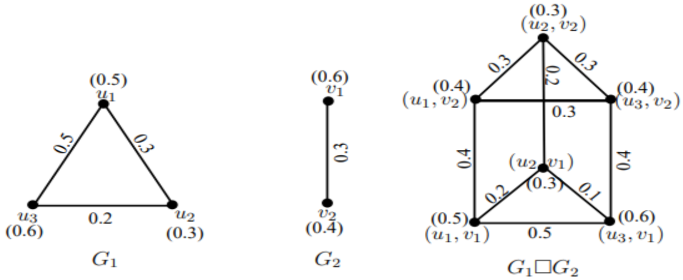

Figure 3.1. A fuzzy graph $G_1 \square G_2$

Then eigenvalue $G_1 \square G_2 = \{-0.8, -0.46, -0.01, 0.07, 0.2, 0.9\}$ satisfies $\lambda_i + \eta_j$.

Theorem 3.3. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of G_1 and $\eta_1, \eta_2, \ldots, \eta_m$ be the eigenvalues of G_2. Then the eigenvalues of $G_1 \boxtimes G_2$ is approximately equal to $(\lambda_i + 1)(\eta_j + 1) - 1$ or $\lambda_i \eta_j + \lambda_i + \eta_j$ for all $i \in [1, n]$ and $j \in [1, m]$.*

Proof. Let A_1 and A_2 denote the fuzzy adjacency matrices of fuzzy graph G_1 and G_2 respectively. Then for every eigenvalue λ and every eigen vector x of A_1 and for every eigenvalue η and every eigen vector y of A_2, $A_1 x = \lambda x$ and $A_2 y = \eta y$.
It follows that,

$$
\begin{aligned}
(((A_1 + I) \boxtimes (A_2 + I)) - I)(x \boxtimes y) &= ((A_1 + I) \boxtimes (A_2 + I))(x \boxtimes y) - (x \boxtimes y) \\
&= (A_1 + I)x \boxtimes (A_2 + I)y - (x \boxtimes y) \\
&= (A_1 x + x) \boxtimes (A_2 y + y) - (x \boxtimes y) \\
&= (\lambda x + x) \boxtimes (\eta y + y) - (x \boxtimes y) \\
&\simeq (\lambda + 1)x \boxtimes (\eta + 1)y - (x \boxtimes y) \\
&\simeq ((\lambda + 1)(\eta + 1) - 1)(x \boxtimes y)
\end{aligned}
$$

Thus, $\lambda_i \eta_j + \lambda_i + \eta_j$ is the eigenvalue of $G_1 \boxtimes G_2$.

Example 3.4. Consider G_1 and G_2 as shown in Figure 3.2 having $\lambda_i = \{-0.5, 0.5\}$ and $\eta_j = \{-0.3606, 0, 0.3606\}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

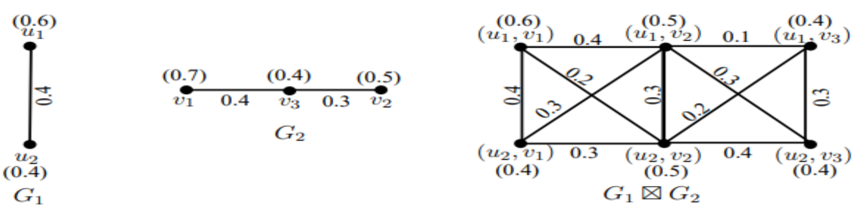

Figure 3.2. A fuzzy graph $G_1 \boxtimes G_2$

The eigenvalue of $G_1 \boxtimes G_2 = \{-0.53, -0.5, -0.34, -0.25, 0.50, 1.1\}$ which approximately equals $((\lambda+1)(\eta+1)-1)(x \boxtimes y)$.

Theorem 3.5. *For any λ_n eigenvalues of G_1 and η_m eigenvalues of G_2, the eigenvalues of $G_1 \otimes G_2$ results in $2(\lambda_i \eta_j)$, for all $i \in [1, n]$ and $j \in [1, m]$.*

Proof. Let A_1 and A_2 be any fuzzy adjacency matrices of G_1 and G_2 respectively. For every eigenvalue λ and every eigen vector x of A_1 and eigenvalue η, eigen vector y of A_2, $A_1 x = \lambda x$ and $A_2 y = \eta y$. It follows that,

$$\begin{aligned}(A_1 \otimes A_2)(x \otimes y) &= A_1 x \otimes A_2 y\\ &= \lambda x \otimes \eta y\\ &= \lambda\eta(x \otimes y)\\ 2(A_1 \otimes A_2)(x \otimes y) &= 2(\lambda\eta(x \otimes y))\end{aligned}$$

Therefore, $G_1 \otimes G_2$ has the eigenvalue $2(\lambda_i \eta_j)$.

Example 3.6. Consider G_1 and G_2 as shown in Figure 3.3 having $\lambda_i = \{-0.3, 0.3\}$ and $\eta_j = \{-0.5, -0.28, 0.81\}$ for all $1 \le i \le n$ and $1 \le j \le m$.

The eigenvalue $G_1 \otimes G_2 = \{-0.5, -0.3, -0.18, 0.18, 0.3, 0.5\}$ clearly satisfies $2(\lambda_i \eta_j)$.

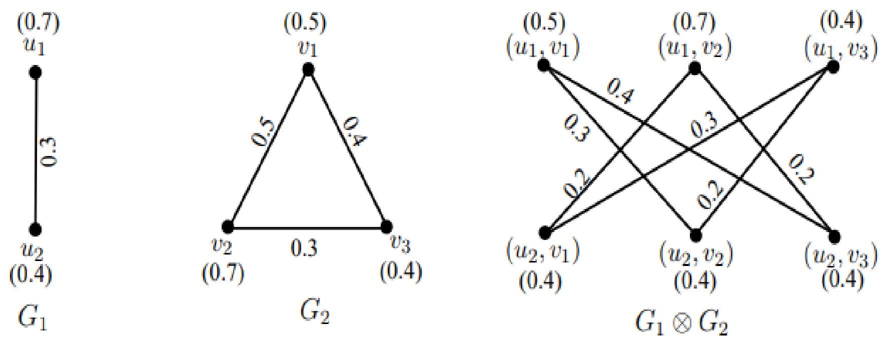

Figure 3.3. A fuzzy graph $G_1 \otimes G_2$

Theorem 3.7. *Let G hold eigenvalues λ_i and for any complete fuzzy graph K_2, its bipartite double is a fuzzy graph $G \otimes K_2$ resulting in the fuzzy spectrum with $\lambda_i \cup -\lambda_i$.*

Proof. Consider a fuzzy graph G, having V and E representing the vertex sets and edgesets respectively. Then for each fuzzy vertex, σ_i of G two vertices $\sigma_i^{'}$ and $\sigma_i^{''}$, and for each edge $\sigma_i \mu_j$ of G two edges $\sigma_i^{'} \mu_j^{''}$ and $\sigma_i^{''} \mu_j^{'}$ exists to form a bipartite double $G \otimes K_2$.

Case 1. Suppose a fuzzy graph G is bipartite, then the resulting bipartite double graph is just the representation of union of fuzzy graphs of two disjoint copies of G and K_2.

Case 2. Suppose G is not bipartite but a connected fuzzy graph, then the double is bipartite and also connected. If λ_i is the spectrum of G, then $G \otimes K_2$ has the spectrum $\lambda_i \cup -\lambda_i$.

Example 3.8. Consider G and K_2 having $\lambda_i = \{-0.616^{(2)}, 0.016^{(2)}\}$ and $K_2 = \{-0.5, 0.5\}$ for all $1 \leq i \leq n$ as shown in Figure 3.4 respectively.

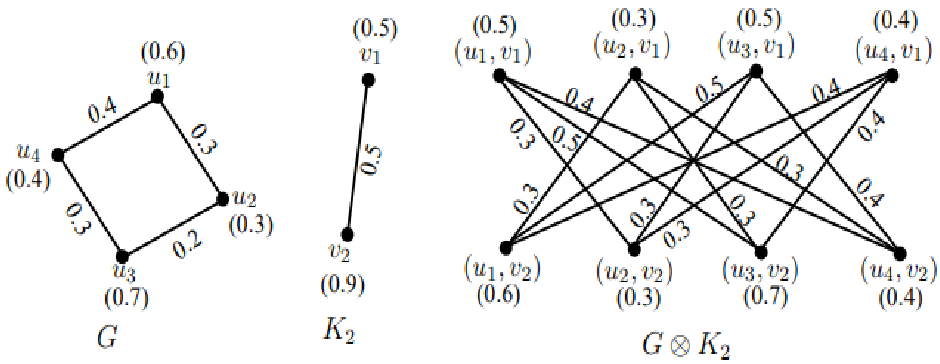

Figure 3.4. A fuzzy graph $G \otimes K_2$

The eigenvalue $G \otimes K_2 = \{-1.1, -0.5, -0.3, -0.24, 0.24, 0.37, 0.5, 1.11\}$ satisfies the above theorem.

Theorem 3.9. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of G_1 and $\eta_1, \eta_2, \ldots, \eta_m$ be the eigenvalues of G_2 such that G_2 is chosen to be a regular fuzzy graph. Then the eigenvalues of $G_1[G_2]$ are $2(\lambda_i \sigma_n + r_2)$ where r_2 is the degree of a regular fuzzy graph and σ_n denotes n^{th} vertex membership value of G_2.*

Proof. Let us denote the adjacency matrix of G_1 as A_1 and the adjacency matrix of G_2 as A_2. Since G_2 is chosen to be a regular fuzzy graph, A_2 will have a constant row sum which equals the degree r_2 of the regular fuzzy graph. Let us consider Lexicographic product $G_1[G_2]$. The adjacency matrix $A_{1[2]}$ of the product is given by, $A_{1[2]} = A_1 \circ I_m + I_n \circ A_2$ where I_n is the identity matrix of size $n \times n$ (corresponding to the vertices of G_1) and I_m is the identity matrix of size $m \times m$ (corresponding to the vertices of G_2).

The eigenvalues of $G_1[G_2]$ can be found by solving the characteristic equation:

$$det(A_{1[2]} - \lambda I_{nm}) = 0$$

where I_{nm} is the identity matrix of size $nm \times nm$. Given that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of G_1 and $\eta_1, \eta_2, \ldots, \eta_m$ are the eigenvalues of G_2,

$$\begin{aligned} det(A_{1[2]} - \lambda I_{nm}) &= det(A_1 \circ I_m + I_n \circ A_2 - \lambda I_{nm}) \\ &= det(A_1 \circ I_m - \lambda I_n \circ I_m + I_n \circ A_2 - \lambda I_{nm}) \\ &= det((A_1 - \lambda I_n) \circ I_m + I_n \circ (A_2 - \lambda I_m)) \\ &= det(A_1 - \lambda I_n)^m \cdot det(A_2 - \lambda I_m)^n \end{aligned}$$

Since G_2 is chosen to be a regular fuzzy graph, A_2 will have eigenvalues r_2 with multiplicity m where r_2 is the degree of G_2. Thus, the eigenvalues of A_2 are r_2 repeated m times.
Now, Let us consider λ_i as an eigenvalue of A_1 with multiplicity k_i. Then, using the above result, the eigenvalues of $G_1[G_2]$ will be $2(\lambda_i \sigma_n + r_2)$, each repeated $k_i \cdot m$ times, where σ_n denotes n^{th} the vertex membership value of G_2.

Example 3.10. Consider G_1 and G_2 as shown in Figure 3.5 having $\lambda_i = \{-0.3, 0.3\}$ and $\eta_j = \{-0.2, -0.2, 0.4\}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

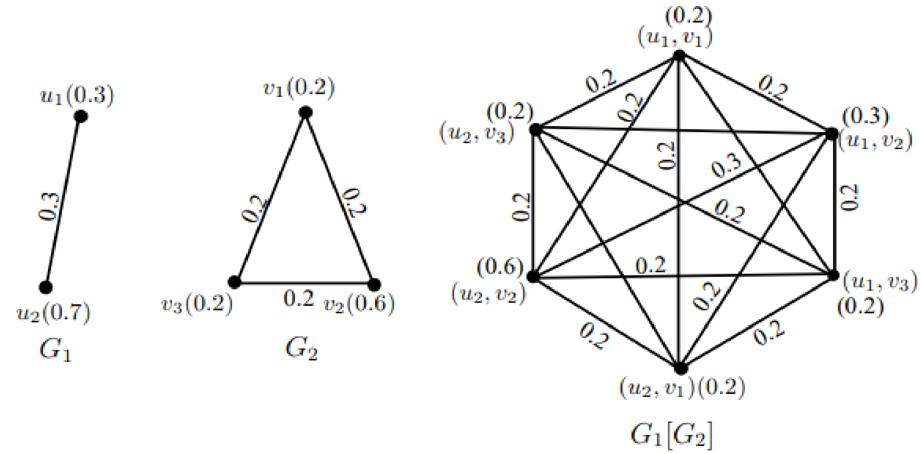

Figure 3.5. A fuzzy graph $G_1[G_2]$

The eigenvalue $G_1[G_2] = \{-0.3, -0.2^{(3)}, -0.1352, 1.0352\}$ satisfies $2(\lambda_i \sigma_n + r_2)$.

4. Conclusion

The authors analyzed the various operations on fuzzy graphs including Cartesian product, Strong product, bipartite double, Kronecker product, and Lexicographic product. These operations demonstrated to adhere the spectral fuzzy properties accompanied by pertinent examples. Furthermore, the utility of these properties in addressing the challenges encountered in protein structure networks will be explored in forthcoming papers.

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References

[1] Alhevaz, A., Baghipur, M., and Paul, S., Spectrum of graphs obtained by operations, Asian-European Journal of Mathematics, 13(02) (2020), 2050045.

[2] Anjali, N., and Mathew, S., Energy of a fuzzy graph, Annals of Fuzzy Mathematics and Informatics, 6(3) (2013), 455-465.

[3] Barik, S., Kalita, D., Pati, S., and Sahoo, G., Spectra of Graphs Resulting from Various Graph Operations and Products: A Survey, Special Matrices, 6(1) (2018), 323-342.

[4] Bhattacharya, P., Some Remarks on Fuzzy Graphs, Pattern Recognition Letters, 6(5) (1987), 297-302.

[5] Bondy, J. A., and Murty, U. S. R., Graph Theory with Applications, Elsevier, New York, 1976.

[6] Brouwer, A. E., and Haemers, W. H., Spectra of Graphs, Springer, New York, 2012.

[7] Dogra, S., Different Types of Products of Fuzzy Graphs, Progress in Nonlinear Dynamics and Chaos, 3(1) (2015), 41-56.

[8] El Seidy, E., Hussein, S. E., and AboElkher, A., Spectra on Some Operations on Graphs, Journal of Advances in Mathematics, 11(9) (2016), 5654-5660.

[9] Jiang, J., An Introduction to Spectral Graph Theory. Lecture Notes. https://math.uchicago.edu/ REU2012/REUPapers/JiangJ.pdf

[10] Mordeson, J. N., and Nair, P. S., Fuzzy Graphs and Fuzzy Hypergraphs, Physica Verlag, New York, 2000.

[11] Mordeson, J. N., and Peng, C. S., Operations on Fuzzy Graphs, Information Sciences, 79 (1994), 150-170.

[12] Mythili, V., Kaliyappan, M., and Hariharan, S., A Review of Fuzzy Graph Theory, International Journal of Pure and Applied Mathematics, 113(12) (2017), 187-195.

[13] Radha, K., and Arumugam, S., On Lexicographic Products of Two Fuzzy Graphs, International Journal of Fuzzy Mathematical Archive, 7(2) (2015), 169-176.

[14] Rosenfeld, A., Fuzzy Graphs in Fuzzy Sets and Their Applications (L.A. Zadeh, K. S. Fu, and M. Shimura, eds.), Academic Press, New York, (1975), 77-95.

[15] Venugopalam, D., Kumari, N. M., and Kumar, M. V., Operations on Fuzzy Graphs, South Asian Journal of Mathematics, 3(5) (2013), 333-338.

[16] Yeh, R. T., and Bang, S. Y., Fuzzy Relations, Fuzzy Graphs, and Their Applications to Clustering Analysis, in Fuzzy Sets and Their Applications (L. A. Zadeh, K. S. Fu, and M. Shimura, eds.), Academic Press, New York, (1975), 125-149.

[17] Zadeh, L. A., Fuzzy sets, Information and Control, 8(3) (1965), 338-353.