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**MULTIVARIATE FUZZY APPROXIMATION BY NEURAL  
NETWORK OPERATORS ACTIVATED BY A GENERAL  
SIGMOID FUNCTION**

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**Abstract:** Here is studied in detail the multivariate fuzzy approximation to the multivariate unit by multivariate fuzzy neural network operators activated by a general sigmoid function. These operators are multivariate fuzzy analogs of earlier studied multivariate Banach space valued ones. The derived results generalize earlier Banach space valued ones into the fuzzy level. Here the high order multivariate fuzzy pointwise and uniform convergences with rates to the multivariate fuzzy unit operator are given through multivariate fuzzy Jackson type inequalities involving the multivariate fuzzy moduli of continuity of the  $m$ th order ( $m \geq 0$ )  $H$ -fuzzy partial derivatives, of the involved multivariate fuzzy number valued function. The treated operators are of averaged, quasi-interpolation, Kantorovich and quadrature types at the multivariate fuzzy setting.

**Keywords and Phrases:** General sigmoid activation function, multivariate fuzzy real analysis, multivariate fuzzy: quasi-interpolation, Kantorovich and Quadrature neural network operators, multivariate fuzzy modulus of continuity and multivariate Jackson type inequalities.

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## 1. Introduction

The author in [1], [2] and [3], see chapters 2-5, was the first to derive neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types [25], by employing the modulus of continuity of the engaged function or its high order derivative, and deriving very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" function are assumed to be of compact support.

The author motivated by [26], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [12] - [20], by treating both the univariate and multivariate cases.

Continuation of the author's works ([4] - [11], [22] and especially of [21], Ch. 21) is this article, where the multivariate fuzzy neural network approximation is based on a general sigmoid activation function, which among others, may result into higher rates of approximation. We involve the fuzzy partial derivatives of the multivariate fuzzy function under approximation or itself, and we establish tight multivariate fuzzy Jackson type inequalities. An extensive background is given on fuzzy multivariate analysis and real neural network approximation, all needed to present our results.

Our fuzzy multivariate feed-forward neural networks (FFNNs) are with one hidden layer. For neural networks in general you may read [29], [32], [33]. For the fractional aspect see [34].

## 2. Fuzzy Real Analysis background

See also [21], Ch. 21, pp. 466-473.

We need the following background

**Definition 2.1.** (see [36]) *Let  $\mu : \mathbb{R} \rightarrow [0, 1]$  with the following properties*

- (i) *is normal, i.e.,  $\exists x_0 \in \mathbb{R}; \mu(x_0) = 1$ .*
- (ii)  *$\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in \mathbb{R}, \forall \lambda \in [0, 1]$  ( $\mu$  is called a convex fuzzy subset).*
- (iii)  *$\mu$  is upper semicontinuous on  $\mathbb{R}$ , i.e.  $\forall x_0 \in \mathbb{R}$  and  $\forall \varepsilon > 0, \exists$  neighborhood  $V(x_0) : \mu(x) \leq \underline{\mu}(x_0) + \varepsilon, \forall x \in V(x_0)$ .*
- (iv) *The set  $\text{sup } p(\mu)$  is compact in  $\mathbb{R}$ , (where  $\text{sup } p(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$ ).*

*We call  $\mu$  a fuzzy real number. Denote the set of all  $\mu$  with  $\mathbb{R}_{\mathcal{F}}$ .*

*E.g.  $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$ , for any  $x_0 \in \mathbb{R}$ , where  $\chi_{\{x_0\}}$  is the characteristic function at  $x_0$ .*

For  $0 < r \leq 1$  and  $\mu \in \mathbb{R}_{\mathcal{F}}$  define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\} \quad (2.1)$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each  $r \in [0, 1]$ ,  $[\mu]^r$  is a closed and bounded interval on  $\mathbb{R}$  ([28]).

For  $u, v \in \mathbb{R}_{\mathcal{F}}$  and  $\lambda \in \mathbb{R}$ , we define uniquely the sum  $u \oplus v$  and the product  $\lambda \odot u$  by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall r \in [0, 1],$$

where  $[u]^r + [v]^r$  means the usual addition of two intervals (as subsets of  $\mathbb{R}$ ) and  $\lambda [u]^r$  means the usual product between a scalar and a subset of  $\mathbb{R}$  (see, e.g. [36]).

Notice  $1 \odot u = u$  and it holds

$$u \oplus v = v \oplus u, \quad \lambda \odot u = u \odot \lambda.$$

If  $0 \leq r_1 \leq r_2 \leq 1$  then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually  $[u]^r = [u_-^{(r)}, u_+^{(r)}]$ , where  $u_-^{(r)} \leq u_+^{(r)}$ ,  $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$ ,  $\forall r \in [0, 1]$ .

For  $\lambda > 0$  one has  $\lambda u_{\pm}^{(r)} = (\lambda \odot u)_{\pm}^{(r)}$ , respectively.

Define  $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$  by

$$D(u, v) := \sup_{r \in [0, 1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\}, \quad (2.2)$$

where

$$[v]^r = [v_-^{(r)}, v_+^{(r)}]; \quad u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that  $D$  is a metric on  $\mathbb{R}_{\mathcal{F}}$ .

Then  $(\mathbb{R}_{\mathcal{F}}, D)$  is a complete metric space, see [36], [37].

Let  $f, g : W \subseteq \mathbb{R}^m \rightarrow \mathbb{R}_{\mathcal{F}}$ . We define the distance

$$D^*(f, g) = \sup_{x \in W} D(f(x), g(x)).$$

**Remark 2.2.** We determine and use

$$D^*(f, \tilde{\delta}) = \sup_{x \in W} D(f(x), \tilde{\delta}) =$$

$$\sup_{x \in W} \sup_{r \in [0,1]} \max \left\{ \left| f_-^{(r)}(x) \right|, \left| f_+^{(r)}(x) \right| \right\}.$$

By the principle of iterated suprema we find that

$$D^*(f, \tilde{o}) = \sup_{r \in [0,1]} \max \left\{ \left\| f_-^{(r)} \right\|_\infty, \left\| f_+^{(r)} \right\|_\infty \right\}, \quad (2.3)$$

under the assumption  $D^*(f, \tilde{o}) < \infty$ , that is  $f$  is a fuzzy bounded function.

Above  $\|\cdot\|_\infty$  is the supremum norm of the function over  $W \subseteq \mathbb{R}^m$ .

Here  $\Sigma^*$  stands for fuzzy summation and  $\tilde{0} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$  is the neutral element with respect to  $\oplus$ , i.e.,

$$u \oplus \tilde{0} = \tilde{0} \oplus u = u, \quad \forall u \in \mathbb{R}_{\mathcal{F}}.$$

We need

**Remark 2.3.** ([5]) Here  $r \in [0, 1]$ ,  $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$ ,  $i = 1, \dots, m \in \mathbb{N}$ . Suppose that

$$\sup_{r \in [0,1]} \max \left( x_i^{(r)}, y_i^{(r)} \right) \in \mathbb{R}, \text{ for } i = 1, \dots, m.$$

Then one sees easily that

$$\sup_{r \in [0,1]} \max \left( \sum_{i=1}^m x_i^{(r)}, \sum_{i=1}^m y_i^{(r)} \right) \leq \sum_{i=1}^m \sup_{r \in [0,1]} \max \left( x_i^{(r)}, y_i^{(r)} \right). \quad (2.4)$$

**Definition 2.4.** Let  $f \in C(W)$ ,  $W \subseteq \mathbb{R}^m$ ,  $m \in \mathbb{N}$ , which is bounded or uniformly continuous, we define ( $h > 0$ )

$$\omega_1(f, h) := \sup_{x, y \in W, \|x-y\|_\infty \leq h} |f(x) - f(y)|, \quad (2.5)$$

where  $x = (x_1, \dots, x_m)$ ,  $y = (y_1, \dots, y_m)$ .

**Definition 2.5.** Let  $f : W \rightarrow \mathbb{R}_{\mathcal{F}}$ ,  $W \subseteq \mathbb{R}^m$ , we define the fuzzy modulus of continuity of  $f$  by

$$\omega_1^{(\mathcal{F})}(f, h) = \sup_{x, y \in W, \|x-y\|_\infty \leq h} D(f(x), f(y)), \quad h > 0. \quad (2.6)$$

where  $x = (x_1, \dots, x_m)$ ,  $y = (y_1, \dots, y_m)$ .

For  $f : W \rightarrow \mathbb{R}_{\mathcal{F}}$ ,  $W \subseteq \mathbb{R}^m$ , we use

$$[f]^r = \left[ f_-^{(r)}, f_+^{(r)} \right], \quad (2.7)$$

where  $f_{\pm}^{(r)} : W \rightarrow \mathbb{R}, \forall r \in [0, 1]$ .

We need

**Proposition 2.6.** ([5]) *Let  $f : W \rightarrow \mathbb{R}_{\mathcal{F}}$ . Assume that  $\omega_1^{\mathcal{F}}(f, \delta), \omega_1(f_{-}^{(r)}, \delta), \omega_1(f_{+}^{(r)}, \delta)$  are finite for any  $\delta > 0, r \in [0, 1]$ .*

Then

$$\omega_1^{(\mathcal{F})}(f, \delta) = \sup_{r \in [0, 1]} \max \left\{ \omega_1(f_{-}^{(r)}, \delta), \omega_1(f_{+}^{(r)}, \delta) \right\}. \quad (2.8)$$

We denote by  $C_{\mathcal{F}}^U(W)$  the space of fuzzy uniformly continuous functions from  $W \rightarrow \mathbb{R}_{\mathcal{F}}$ , also  $C_{\mathcal{F}}(W)$  is the space of fuzzy continuous functions on  $W \subseteq \mathbb{R}^m$ , and  $C_B(W, \mathbb{R}_{\mathcal{F}})$  is the fuzzy continuous and bounded functions.

We mention

**Proposition 2.7.** ([7]) *Let  $f \in C_{\mathcal{F}}^U(W)$ , where  $W \subseteq \mathbb{R}^m$  is convex. Then  $\omega_1^{(\mathcal{F})}(f, \delta) < \infty$ , for any  $\delta > 0$ .*

**Proposition 2.8.** ([7]) *It holds*

$$\lim_{\delta \rightarrow 0} \omega_1^{(\mathcal{F})}(f, \delta) = \omega_1^{(\mathcal{F})}(f, 0) = 0, \quad (2.9)$$

iff  $f \in C_{\mathcal{F}}^U(W)$ ,  $W \subseteq \mathbb{R}^m$ , where  $W$  is convex and compact.

**Proposition 2.9.** ([7]) *Let  $f \in C_{\mathcal{F}}(W)$ ,  $W \subseteq \mathbb{R}^m$  open or compact. Then  $f_{\pm}^{(r)}$  are equicontinuous with respect to  $r \in [0, 1]$  over  $W$ , respectively in  $\pm$ .*

**Note 2.10.** *It is clear by Propositions 6, 8, that if  $f \in C_{\mathcal{F}}^U(W)$ , then  $f_{\pm}^{(r)} \in C_U(W)$  (uniformly continuous on  $W$ ). Also if  $f \in C_B(W, \mathbb{R}_{\mathcal{F}})$ , it implies by (2.3) that  $f_{\pm}^{(r)} \in C_B(W)$  (continuous and bounded functions on  $W$ ).*

We need

**Definition 2.11.** *Let  $x, y \in \mathbb{R}_{\mathcal{F}}$ . If there exists  $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$ , then we call  $z$  the  $H$ -difference on  $x$  and  $y$ , denoted  $x - y$ .*

**Definition 2.12.** ([36]) *Let  $T := [x_0, x_0 + \beta] \subset \mathbb{R}$ , with  $\beta > 0$ . A function  $f : T \rightarrow \mathbb{R}_{\mathcal{F}}$  is  $H$ -difference at  $x \in T$  if there exists an  $f'(x) \in \mathbb{R}_{\mathcal{F}}$  such that the limits (with respect to  $D$ )*

$$\lim_{h \rightarrow 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \rightarrow 0+} \frac{f(x) - f(x-h)}{h} \quad (2.10)$$

exist and are equal to  $f'(x)$ .

We call  $f'$  the  $H$ -derivative or fuzzy derivative of  $f$  at  $x$ .

Above is assumed that the  $H$ -differences  $f(x+h) - f(x)$ ,  $f(x) - f(x-h)$  exists in  $\mathbb{R}_{\mathcal{F}}$  in a neighborhood of  $x$ .

**Definition 2.13.** We denote by  $C_{\mathcal{F}}^{N^*}(W)$ ,  $N^* \in \mathbb{N}$ , the space of all  $N^*$ -times fuzzy continuously differentiable functions from  $W$  into  $\mathbb{R}_{\mathcal{F}}$ ,  $W \subseteq \mathbb{R}^m$  open or compact which is convex.

Here fuzzy partial derivatives are defined via Definition 2.12 in the obvious way as in the ordinary real case.

We mention

**Theorem 2.14.** ([30]) Let  $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}_{\mathcal{F}}$  be  $H$ -fuzzy differentiable. Let  $t \in [a, b]$ ,  $0 \leq r \leq 1$ . Clearly

$$[f(t)]^r = \left[ f(t)_-^{(r)}, f(t)_+^{(r)} \right] \subseteq \mathbb{R}.$$

Then  $(f(t))_{\pm}^{(r)}$  are differentiable and

$$[f'(t)]^r = \left[ \left( f(t)_-^{(r)} \right)', \left( f(t)_+^{(r)} \right)' \right].$$

I.e.

$$(f')_{\pm}^{(r)} = \left( f_{\pm}^{(r)} \right)', \quad \forall r \in [0, 1]. \quad (2.11)$$

**Remark 2.15.** (see also [6]) Let  $f \in C^{N^*}([a, b], \mathbb{R}_{\mathcal{F}})$ ,  $N^* \geq 1$ . Then by Theorem ?? we obtain  $f_{\pm}^{(r)} \in C^{N^*}([a, b])$  and

$$[f^{(i)}(t)]^r = \left[ \left( f(t)_-^{(r)} \right)^{(i)}, \left( f(t)_+^{(r)} \right)^{(i)} \right],$$

for  $i = 0, 1, 2, \dots, N^*$ , and in particular we have

$$(f^{(i)})_{\pm}^{(r)} = \left( f_{\pm}^{(r)} \right)^{(i)}, \quad (2.12)$$

for any  $r \in [0, 1]$ .

Let  $f \in C_{\mathcal{F}}^{N^*}(W)$ ,  $W \subseteq \mathbb{R}^m$ , open or compact, which is convex, denote  $f_{\tilde{\alpha}} := \frac{\partial^{\tilde{\alpha}} f}{\partial x^{\tilde{\alpha}}}$ , where  $\tilde{\alpha} := (\tilde{\alpha}_1, \dots, \tilde{\alpha}_m)$ ,  $\tilde{\alpha}_i \in \mathbb{Z}^+$ ,  $i = 1, \dots, m$  and

$$0 < |\tilde{\alpha}| := \sum_{i=1}^m \tilde{\alpha}_i \leq N^*, \quad N^* > 1.$$

Then by Theorem 2.14 we get that

$$\left(f_{\pm}^{(r)}\right)_{\tilde{\alpha}} = (f_{\tilde{\alpha}})_{\pm}^{(r)}, \quad \forall r \in [0, 1], \quad (2.13)$$

and any  $\tilde{\alpha} : |\tilde{\alpha}| \leq N^*$ . Here  $f_{\pm}^{(r)} \in C^{N^*}(W)$ .

**Notation 2.16.** We denote

$$\left(\sum_{i=1}^2 D\left(\frac{\partial}{\partial x_i}, \tilde{0}\right)\right)^2 f(\vec{x}) := \quad (2.14)$$

$$D\left(\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2}, \tilde{0}\right) + D\left(\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2}, \tilde{0}\right) + 2D\left(\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}, \tilde{0}\right).$$

In general we denote ( $j = 1, \dots, N^*$ )

$$\left(\sum_{i=1}^m D\left(\frac{\partial}{\partial x_i}, \tilde{0}\right)\right)^j f(\vec{x}) := \quad (2.15)$$

$$\sum_{(j_1, \dots, j_m) \in \mathbb{Z}_+^m : \sum_{i=1}^m j_i = j} \frac{j!}{j_1! j_2! \dots j_m!} D\left(\frac{\partial^j f(x_1, \dots, x_m)}{\partial x_1^{j_1} \partial x_2^{j_2} \dots \partial x_m^{j_m}}, \tilde{0}\right).$$

We mention also a particular case of the Fuzzy Henstock integral ( $\delta(x) = \frac{\delta}{2}$ ), see [36].

**Definition 2.17.** ([27], p. 644) Let  $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ . We say that  $f$  is Fuzzy-Riemann integrable to  $I \in \mathbb{R}_{\mathcal{F}}$  if for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any division  $P = \{[u, v]; \xi\}$  of  $[a, b]$  with the norms  $\Delta(P) < \delta$ , we have

$$D\left(\sum_P^* (v - u) \odot f(\xi), I\right) < \varepsilon.$$

We write

$$I := (FR) \int_a^b f(x) dx. \quad (2.16)$$

We mention

**Theorem 2.18.** ([28]) Let  $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$  be fuzzy continuous. Then

$$(FR) \int_a^b f(x) dx$$

exists and belongs to  $\mathbb{R}_{\mathcal{F}}$ , furthermore it holds

$$\left[ (FR) \int_a^b f(x) dx \right]^r = \left[ \int_a^b (f)_-^{(r)}(x) dx, \int_a^b (f)_+^{(r)}(x) dx \right],$$

$\forall r \in [0, 1]$ .

For the definition of general fuzzy integral we follow [31] next.

**Definition 2.19.** Let  $(\Omega, \Sigma, \mu)$  be a complete  $\sigma$ -finite measure space. We call  $F : \Omega \rightarrow \mathbb{R}_{\mathcal{F}}$  measurable iff  $\forall$  closed  $B \subseteq \mathbb{R}$  the function  $F^{-1}(B) : \Omega \rightarrow [0, 1]$  defined by

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \text{ all } w \in \Omega$$

is measurable, see [31].

**Theorem 2.20.** ([31]) For  $F : \Omega \rightarrow \mathbb{R}_{\mathcal{F}}$ ,

$$F(w) = \left\{ \left( F_-^{(r)}(w), F_+^{(r)}(w) \right) \mid 0 \leq r \leq 1 \right\},$$

the following are equivalent

- (1)  $F$  is measurable,
- (2)  $\forall r \in [0, 1]$ ,  $F_-^{(r)}$ ,  $F_+^{(r)}$  are measurable.

Following [31], given that for each  $r \in [0, 1]$ ,  $F_-^{(r)}$ ,  $F_+^{(r)}$  are integrable we have that the parametrized representation

$$\left\{ \left( \int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) \mid 0 \leq r \leq 1 \right\} \quad (2.17)$$

is a fuzzy real number for each  $A \in \Sigma$ .

The last fact leads to

**Definition 2.21.** ([31]) A measurable function  $F : \Omega \rightarrow \mathbb{R}_{\mathcal{F}}$ ,

$$F(w) = \left\{ \left( F_-^{(r)}(w), F_+^{(r)}(w) \right) \mid 0 \leq r \leq 1 \right\}$$

is integrable if for each  $r \in [0, 1]$ ,  $F_{\pm}^{(r)}$  are integrable, or equivalently, if  $F_{\pm}^{(0)}$  are integrable.

In this case, the fuzzy integral of  $F$  over  $A \in \Sigma$  is defined by

$$\int_A F d\mu := \left\{ \left( \int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) \mid 0 \leq r \leq 1 \right\}. \quad (2.18)$$



By [31],  $F$  is integrable iff  $w \rightarrow \|F(w)\|_{\mathcal{F}}$  is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}} := D(u, \tilde{0}), \quad \forall u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

**Theorem 2.22.** ([31]) *Let  $F, G : \Omega \rightarrow \mathbb{R}_{\mathcal{F}}$  be integrable. Then*

(1) *Let  $a, b \in \mathbb{R}$ , then  $aF + bG$  is integrable and for each  $A \in \Sigma$ ,*

$$\int_A (aF + bG) d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2)  *$D(F, G)$  is a real-valued integrable function and for each  $A \in \Sigma$ ,*

$$D\left(\int_A F d\mu, \int_A G d\mu\right) \leq \int_A D(F, G) d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}} d\mu.$$

Above  $\mu$  could be the multivariate Lebesgue measure, which we use in this article, with all the basic properties valid here too. Notice by [31], Fubini's theorem is valid for fuzzy integral (2.18).

Basically here we have that

$$\left[ \int_A F d\mu \right]^r = \left[ \int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right], \quad (2.19)$$

i.e.

$$\left( \int_A F d\mu \right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \quad \forall r \in [0, 1]. \quad (2.20)$$

### 3. About Real Neural Networks Background

Here we follow [24].

Let  $h : \mathbb{R} \rightarrow [-1, 1]$  be a general sigmoid function, such that it is strictly increasing,  $h(0) = 0$ ,  $h(-x) = -h(x)$ ,  $h(+\infty) = 1$ ,  $h(-\infty) = -1$ . Also  $h$  is strictly convex over  $(-\infty, 0]$  and strictly concave over  $[0, +\infty)$ , with  $h^{(2)} \in C(\mathbb{R})$ .

We consider the activation function

$$\psi(x) := \frac{1}{4} (h(x+1) - h(x-1)), \quad x \in \mathbb{R}, \quad (3.1)$$

As in [23], p. 88, we get that  $\psi(-x) = \psi(x)$ , thus  $\psi$  is an even function. Since  $x+1 > x-1$ , then  $h(x+1) > h(x-1)$ , and  $\psi(x) > 0$ , all  $x \in \mathbb{R}$ .

We see that

$$\psi(0) = \frac{h(1)}{2}. \quad (3.2)$$

Let  $x > 1$ , we have that

$$\psi'(x) = \frac{1}{4}(h'(x+1) - h'(x-1)) < 0,$$

by  $h'$  being strictly decreasing over  $[0, +\infty)$ .

Let now  $0 < x < 1$ , then  $1-x > 0$  and  $0 < 1-x < 1+x$ . It holds  $h'(x-1) = h'(1-x) > h'(x+1)$ , so that again  $\psi'(x) < 0$ . Consequently  $\psi$  is strictly decreasing on  $(0, +\infty)$ .

Clearly,  $\psi$  is strictly increasing on  $(-\infty, 0)$ , and  $\psi'(0) = 0$ .

See that

$$\lim_{x \rightarrow +\infty} \psi(x) = \frac{1}{4}(h(+\infty) - h(+\infty)) = 0, \quad (3.3)$$

and

$$\lim_{x \rightarrow -\infty} \psi(x) = \frac{1}{4}(h(-\infty) - h(-\infty)) = 0. \quad (3.4)$$

That is the  $x$ -axis is the horizontal asymptote on  $\psi$ .

Conclusion,  $\psi$  is a bell symmetric function with maximum

$$\psi(0) = \frac{h(1)}{2}.$$

We need

**Theorem 3.1.** ([24]) *We have that*

$$\sum_{i=-\infty}^{\infty} \psi(x-i) = 1, \quad \forall x \in \mathbb{R}. \quad (3.5)$$

**Theorem 3.2.** ([24]) *It holds*

$$\int_{-\infty}^{\infty} \psi(x) dx = 1. \quad (3.6)$$

Thus  $\psi(x)$  is a density function on  $\mathbb{R}$ .

We give

**Theorem 3.3.** ([24]) *Let  $0 < \alpha < 1$ , and  $n \in \mathbb{N}$  with  $n^{1-\alpha} > 2$ . It holds*

$$\sum_{k=-\infty}^{\infty} \psi(nx - k) < \frac{(1 - h(n^{1-\alpha} - 2))}{2}. \quad (3.7)$$

$$\left\{ \begin{array}{l} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{array} \right.$$

Notice that

$$\lim_{n \rightarrow +\infty} \frac{(1 - h(n^{1-\alpha} - 2))}{2} = 0.$$

Denote by  $[\cdot]$  the integral part of the number and by  $\lceil \cdot \rceil$  the ceiling of the number.

We further give

**Theorem 3.4.** ([24]) *Let  $x \in [a, b] \subset \mathbb{R}$  and  $n \in \mathbb{N}$  so that  $\lceil na \rceil \leq \lfloor nb \rfloor$ . It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k)} < \frac{1}{\psi(1)}, \quad \forall x \in [a, b]. \quad (3.8)$$

**Remark 3.5.** ([24]) *i) We have that*

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k) \neq 1, \quad (3.9)$$

for at least some  $x \in [a, b]$ .

*ii) For large enough  $n \in \mathbb{N}$  we always obtain  $\lceil na \rceil \leq \lfloor nb \rfloor$ . Also  $a \leq \frac{k}{n} \leq b$ , iff  $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$ .*

*In general, by Theorem 3.1, it holds*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k) \leq 1. \quad (3.10)$$

We introduce

$$Z(x_1, \dots, x_N) := Z(x) := \prod_{i=1}^N \psi(x_i), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N, \quad N \in \mathbb{N}. \quad (3.11)$$

It has the properties:

(i)  $Z(x) > 0, \quad \forall x \in \mathbb{R}^N,$

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x - k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} Z(x_1 - k_1, \dots, x_N - k_N) = 1, \quad (3.12)$$

where  $k := (k_1, \dots, k_n) \in \mathbb{Z}^N$ ,  $\forall x \in \mathbb{R}^N$ ,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx - k) = 1, \quad (3.13)$$

$\forall x \in \mathbb{R}^N$ ;  $n \in \mathbb{N}$ ,

and

(iv)

$$\int_{\mathbb{R}^N} Z(x) dx = 1, \quad (3.14)$$

that is  $Z$  is a multivariate density function.

Here denote  $\|x\|_{\infty} := \max\{|x_1|, \dots, |x_N|\}$ ,  $x \in \mathbb{R}^N$ , also set  $\infty := (\infty, \dots, \infty)$ ,  $-\infty := (-\infty, \dots, -\infty)$  upon the multivariate context, and

$$\lceil na \rceil := (\lceil na_1 \rceil, \dots, \lceil na_N \rceil), \quad (3.15)$$

$$\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, \dots, \lfloor nb_N \rfloor),$$

where  $a := (a_1, \dots, a_N)$ ,  $b := (b_1, \dots, b_N)$ .

We obviously see that

$$\begin{aligned} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left( \prod_{i=1}^N \psi(nx_i - k_i) \right) = \\ &= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left( \prod_{i=1}^N \psi(nx_i - k_i) \right) = \prod_{i=1}^N \left( \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i) \right). \end{aligned} \quad (3.16)$$

For  $0 < \beta < 1$  and  $n \in \mathbb{N}$ , a fixed  $x \in \mathbb{R}^N$ , we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) =$$

$$\sum_{\substack{[nb] \\ k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}} Z(nx - k) + \sum_{\substack{[nb] \\ k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}} Z(nx - k). \quad (3.17)$$

In the last two sums the counting is over disjoint vector sets of  $k$ 's, because the condition  $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$  implies that there exists at least one  $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$ , where  $r \in \{1, \dots, N\}$ .

(v) As in [21], pp. 379-380, we derive that

$$\sum_{\substack{[nb] \\ k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}} Z(nx - k) \stackrel{??}{<} \frac{1 - h(n^{1-\beta} - 2)}{2}, \quad 0 < \beta < 1, \quad (3.18)$$

with  $n \in \mathbb{N} : n^{1-\beta} > 2$ ,  $x \in \prod_{i=1}^N [a_i, b_i]$ .

(vi) By Theorem 3.4 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{[nb]} Z(nx - k)} < \frac{1}{(\psi(1))^N} =: \gamma(N), \quad (3.19)$$

$\forall x \in \left( \prod_{i=1}^N [a_i, b_i] \right)$ ,  $n \in \mathbb{N}$ .

It is also clear that

(vii)

$$\sum_{\substack{\infty \\ k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}} Z(nx - k) < \frac{1 - h(n^{1-\beta} - 2)}{2} =: c(\beta, n), \quad (3.20)$$

$0 < \beta < 1$ ,  $n \in \mathbb{N} : n^{1-\beta} > 2$ ,  $x \in \mathbb{R}^N$ .

Furthermore it holds

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{[nb]} Z(nx - k) \neq 1, \quad (3.21)$$

for at least some  $x \in \left( \prod_{i=1}^N [a_i, b_i] \right)$ .

Let  $f \in C \left( \prod_{i=1}^N [a_i, b_i] \right)$ , and  $n \in \mathbb{N}$  such that  $\lceil na_i \rceil \leq [nb_i]$ ,  $i = 1, \dots, N$ .

We define the multivariate averaged positive linear neural network operators ( $x := (x_1, \dots, x_N) \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ ):

$$A_n(f, x_1, \dots, x_N) := A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} = \quad (3.22)$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^N \psi(nx_i - k_i)\right)}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i)\right)}.$$

For large enough  $n \in \mathbb{N}$  we always obtain  $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$ ,  $i = 1, \dots, N$ . Also  $a_i \leq \frac{k_i}{n} \leq b_i$ , iff  $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$ ,  $i = 1, \dots, N$ .

When  $f \in C_B(\mathbb{R}^N)$  we define

$$B_n(f, x) := B_n(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) := \quad (3.23)$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^N \psi(nx_i - k_i)\right),$$

$n \in \mathbb{N}$ ,  $\forall x \in \mathbb{R}^N$ ,  $N \in \mathbb{N}$ , the multivariate quasi-interpolation neural network operators.

Also for  $f \in C_B(\mathbb{R}^N)$  we define the multivariate Kantorovich type neural network operators

$$C_n(f, x) := C_n(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty} \left( n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt \right) Z(nx - k) := \quad (3.24)$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \left( n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, \dots, t_N) dt_1 \dots dt_N \right) \cdot \left( \prod_{i=1}^N \psi(nx_i - k_i) \right),$$

$n \in \mathbb{N}$ ,  $\forall x \in \mathbb{R}^N$ .

Again for  $f \in C_B(\mathbb{R}^N)$ ,  $N \in \mathbb{N}$ , we define the multivariate neural network operators of quadrature type  $D_n(f, x)$ ,  $n \in \mathbb{N}$ , as follows. Let  $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{N}^N$ ,  $\bar{r} =$

$(r_1, \dots, r_N) \in \mathbb{Z}_+^N$ ,  $w_{\bar{r}} = w_{r_1, r_2, \dots, r_N} \geq 0$ , such that  $\sum_{\bar{r}=0}^{\theta} w_{\bar{r}} = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \dots, r_N} = 1$ ;  $k \in \mathbb{Z}^N$  and

$$\begin{aligned}
 \delta_{nk}(f) &:= \delta_{n, k_1, k_2, \dots, k_N}(f) := \sum_{\bar{r}=0}^{\theta} w_{\bar{r}} f\left(\frac{k}{n} + \frac{\bar{r}}{n\theta}\right) := \\
 &\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \dots, r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, \dots, \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \quad (3.25)
 \end{aligned}$$

where  $\frac{\bar{r}}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, \dots, \frac{r_N}{\theta_N}\right)$ .

We put

$$D_n(f, x) := D_n(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k) := \quad (3.26)$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \delta_{n, k_1, k_2, \dots, k_N}(f) \left(\prod_{i=1}^N \psi(nx_i - k_i)\right),$$

$\forall x \in \mathbb{R}^N$ .

Let  $f \in C^m\left(\prod_{i=1}^N [a_i, b_i]\right)$ ,  $m, N \in \mathbb{N}$ . Here  $f_{\alpha}$  denotes a partial derivative of  $f$ ,  $\alpha := (\alpha_1, \dots, \alpha_N)$ ,  $\alpha_i \in \mathbb{Z}_+$ ,  $i = 1, \dots, N$ , and  $|\alpha| := \sum_{i=1}^N \alpha_i = l$ , where  $l = 0, 1, \dots, m$ . We write also  $f_{\alpha} := \frac{\partial^{\alpha} f}{\partial x^{\alpha}}$  and we say it is of order  $l$ .

We denote

$$\omega_{1,m}^{\max}(f_{\alpha}, h) := \max_{\alpha: |\alpha|=m} \omega_1(f_{\alpha}, h).$$

Call also

$$\|f_{\alpha}\|_{\infty, m}^{\max} := \max_{\alpha: |\alpha|=m} \{\|f_{\alpha}\|_{\infty}\},$$

where  $\|\cdot\|_{\infty}$  is the supremum norm.

In [21], [23], we studied the basic approximation properties of  $A_n, B_n, C_n, D_n$  neural network operators and as well of their iterates for Banach space valued functions. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator  $I$ .

We need

**Theorem 3.6.** *Let  $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ ,  $0 < \beta < 1$ ,  $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ ,  $N, n \in \mathbb{N}$  with  $n^{1-\beta} > 2$ . Then*

1)

$$|A_n(f, x) - f(x)| \leq \gamma(N) \left[ \omega_1 \left( f, \frac{1}{n^\beta} \right) + 2c(\beta, n) \|f\|_\infty \right] =: \lambda_1, \quad (3.27)$$

and

2)

$$\|A_n(f) - f\|_\infty \leq \lambda_1. \quad (3.28)$$

We notice that  $\lim_{n \rightarrow \infty} A_n(f) = f$ , pointwise and uniformly.

**Proof.** Similar to [23], p. 118.

We need

**Theorem 3.7.** Let  $f \in C_B(\mathbb{R}^N)$ ,  $0 < \beta < 1$ ,  $x \in \mathbb{R}^N$ ,  $N, n \in \mathbb{N}$  with  $n^{1-\beta} > 2$ .

Then

1)

$$|B_n(f, x) - f(x)| \leq \omega_1 \left( f, \frac{1}{n^\beta} \right) + 2c(\beta, n) \|f\|_\infty =: \lambda_2, \quad (3.29)$$

2)

$$\|B_n(f) - f\|_\infty \leq \lambda_2. \quad (3.30)$$

Given that  $f \in (C_U(\mathbb{R}^N) \cap C_B(\mathbb{R}^N))$ , we obtain  $\lim_{n \rightarrow \infty} B_n(f) = f$ , uniformly.

**Proof.** Similar to [23], p. 128.

We also need

**Theorem 3.8.** Let  $f \in C_B(\mathbb{R}^N)$ ,  $0 < \beta < 1$ ,  $x \in \mathbb{R}^N$ ,  $N, n \in \mathbb{N}$  with  $n^{1-\beta} > 2$ .

Then

1)

$$|C_n(f, x) - f(x)| \leq \omega_1 \left( f, \frac{1}{n} + \frac{1}{n^\beta} \right) + 2c(\beta, n) \|f\|_\infty =: \lambda_3, \quad (3.31)$$

2)

$$\|C_n(f) - f\|_\infty \leq \lambda_3. \quad (3.32)$$

Given that  $f \in (C_U(\mathbb{R}^N) \cap C_B(\mathbb{R}^N))$ , we obtain  $\lim_{n \rightarrow \infty} C_n(f) = f$ , uniformly.

**Proof.** Similar to [23], p. 129.

We also need

**Theorem 3.9.** Let  $f \in C_B(\mathbb{R}^N)$ ,  $0 < \beta < 1$ ,  $x \in \mathbb{R}^N$ ,  $N, n \in \mathbb{N}$  with  $n^{1-\beta} > 2$ .

Then



1)

$$|D_n(f, x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2c(\beta, n) \|f\|_\infty = \lambda_3, \quad (3.33)$$

2)

$$\|D_n(f) - f\|_\infty \leq \lambda_3. \quad (3.34)$$

Given that  $f \in (C_U(\mathbb{R}^N) \cap C_B(\mathbb{R}^N))$ , we obtain  $\lim_{n \rightarrow \infty} D_n(f) = f$ , uniformly.

**Proof.** Similar to [23], p. 131.

We finally mention (similar to [21], p. 481)

**Theorem 3.10.** Let  $f \in C^m\left(\prod_{i=1}^N [a_i, b_i]\right)$ ,  $0 < \beta < 1$ ,  $n, m, N \in \mathbb{N}$ ,  $n^{1-\beta} > 2$ ,  $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ . Then

$$\begin{aligned} & i) \left| A_n(f, x) - f(x) - \sum_{j_*=1}^m \left( \sum_{|\alpha|=j_*} \left( \frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) A_n\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x\right) \right) \right| \leq \quad (3.35) \\ & \gamma(N) \cdot \left\{ \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left( \frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2c(\beta, n) \right\}, \end{aligned}$$

ii)

$$|A_n(f, x) - f(x)| \leq \gamma(N). \quad (3.36)$$

$$\begin{aligned} & \left\{ \sum_{j_*=1}^m \left( \sum_{|\alpha|=j_*} \left( \frac{|f_\alpha(x)|}{\prod_{i=1}^N \alpha_i!} \right) \left[ \frac{1}{n^{\beta j_*}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) c(\beta, n) \right] \right) \right. \\ & \left. + \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left( \frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2c(\beta, n) \right\}, \end{aligned}$$

iii)

$$\|A_n(f) - f\|_\infty \leq \gamma(N). \quad (3.37)$$

$$\begin{aligned} & \left\{ \sum_{j_*=1}^m \left( \sum_{|\alpha|=j_*} \left( \frac{\|f_\alpha\|_\infty}{\prod_{i=1}^N \alpha_i!} \right) \left[ \frac{1}{n^{\beta j_*}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) c(\beta, n) \right] \right) \right. \\ & \left. + \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left( \frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2c(\beta, n) \right\}, \end{aligned}$$

iv) additionally assume  $f_\alpha(x_0) = 0$ , for all  $\alpha : |\alpha| = 1, \dots, m; x_0 \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ , then

$$|A_n(f, x_0) - f(x_0)| \leq \gamma(N) \left\{ \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right) + \left( \frac{\|b - a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2c(\beta, n) \right\}, \quad (3.38)$$

notice in the last the extremely high rate of convergence at  $n^{-\beta(m+1)}$ .

#### 4. Main Results: Fuzzy multivariate Neural Network Approximation based on a general sigmoid function

We define the following General Fuzzy multivariate Neural Network Operators  $A_n^{\mathcal{F}}, B_n^{\mathcal{F}}, C_n^{\mathcal{F}}, D_n^{\mathcal{F}}$ , based on a general sigmoid activation function. These are analogs of the real  $A_n, B_n, C_n, D_n$ , see (3.22), (3.23), (3.24) and (3.26), respectively.

Let  $f \in C_{\mathcal{F}} \left( \prod_{i=1}^N [a_i, b_i] \right)$ ,  $N \in \mathbb{N}$ , we set

$$A_n^{\mathcal{F}}(f, x_1, \dots, x_N) := A_n^{\mathcal{F}}(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor^*} f\left(\frac{k}{n}\right) \odot Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}$$

$$= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor^*} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor^*} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \odot \left(\prod_{i=1}^N \psi(nx_i - k_i)\right)}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i)\right)}, \quad (4.1)$$

$x \in \prod_{i=1}^N [a_i, b_i]$ ,  $n \in \mathbb{N}$ .

Let  $f \in C_B(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}})$ , we put

$$B_n^{\mathcal{F}}(f, x) := B_n^{\mathcal{F}}(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty^*} f\left(\frac{k}{n}\right) \odot Z(nx - k)$$

$$:= \sum_{k_1=-\infty}^{\infty^*} \dots \sum_{k_N=-\infty}^{\infty^*} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \odot \left(\prod_{i=1}^N \psi(nx_i - k_i)\right), \quad (4.2)$$

$x \in \mathbb{R}^N$ ,  $n \in \mathbb{N}$ .

Let  $f \in C_B(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}})$ , we define the multivariate fuzzy Kantorovich type neural network operator,

$$\begin{aligned}
 C_n^{\mathcal{F}}(f, x) &:= C_n^{\mathcal{F}}(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty*} \left( n^N \odot \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt \right) \odot Z(nx - k) := \\
 &\sum_{k_1=-\infty}^{\infty*} \dots \sum_{k_N=-\infty}^{\infty*} \left( n^N \odot \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, \dots, t_N) dt_1 \dots dt_N \right) \\
 &\odot \left( \prod_{i=1}^N \psi(nx_i - k_i) \right), \tag{4.3}
 \end{aligned}$$

$x \in \mathbb{R}^N$ ,  $n \in \mathbb{N}$ .

Let  $f \in C_B(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}})$ , we define the multivariate fuzzy quadrature type neural network operator. Let here

$$\begin{aligned}
 \delta_{nk}^{\mathcal{F}}(f) &:= \delta_{n, k_1, \dots, k_N}^{\mathcal{F}}(f) := \sum_{\bar{r}=0}^{\theta*} w_{\bar{r}} \odot f\left(\frac{k}{n} + \frac{\bar{r}}{n\theta}\right) := \tag{4.4} \\
 &\sum_{r_1=0}^{\theta_1*} \dots \sum_{r_N=0}^{\theta_N*} w_{r_1, \dots, r_N} \odot f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \dots, \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right).
 \end{aligned}$$

We put

$$\begin{aligned}
 D_n^{\mathcal{F}}(f, x) &:= D_n^{\mathcal{F}}(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty*} \delta_{nk}^{\mathcal{F}}(f) \odot Z(nx - k) := \\
 &\sum_{k_1=-\infty}^{\infty*} \dots \sum_{k_N=-\infty}^{\infty*} \delta_{n, k_1, \dots, k_N}^{\mathcal{F}}(f) \odot \left( \prod_{i=1}^N \psi(nx_i - k_i) \right), \tag{4.5}
 \end{aligned}$$

$x \in \mathbb{R}^N$ ,  $n \in \mathbb{N}$ .

We can put together all  $B_n^{\mathcal{F}}$ ,  $C_n^{\mathcal{F}}$ ,  $D_n^{\mathcal{F}}$  fuzzy operators as follows:

$$L_n^{\mathcal{F}}(f, x) := \sum_{k=-\infty}^{\infty*} l_{nk}^{\mathcal{F}}(f) \odot Z(nx - k), \tag{4.6}$$

where

$$l_{nk}^{\mathcal{F}}(f) := \begin{cases} f\left(\frac{k}{n}\right), & \text{if } L_n^{\mathcal{F}} = B_n^{\mathcal{F}}, \\ n^N \odot \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, & \text{if } L_n^{\mathcal{F}} = C_n^{\mathcal{F}}, \\ \delta_{nk}^{\mathcal{F}}(f), & \text{if } L_n^{\mathcal{F}} = D_n^{\mathcal{F}}, \end{cases} \tag{4.7}$$

$x \in \mathbb{R}^N, n \in \mathbb{N}$ .

Similarly, we can put together all  $B_n, C_n, D_n$  real operators as

$$L_n(f, x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx - k), \quad (4.8)$$

where

$$l_{nk}(f) := \begin{cases} f\left(\frac{k}{n}\right), & \text{if } L_n = B_n, \\ n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, & \text{if } L_n = C_n, \\ \delta_{nk}(f), & \text{if } L_n = D_n, \end{cases} \quad (4.9)$$

$x \in \mathbb{R}^N, n \in \mathbb{N}$ .

Let  $r \in [0, 1]$ , we observe that

$$\begin{aligned} [A_n^{\mathcal{F}}(f, x)]^r &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left[ f\left(\frac{k}{n}\right) \right]^r \left( \frac{Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} \right) = \\ &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left[ f_-^{(r)}\left(\frac{k}{n}\right), f_+^{(r)}\left(\frac{k}{n}\right) \right] \left( \frac{Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} \right) = \end{aligned} \quad (4.10)$$

$$\begin{aligned} &= \left[ \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f_-^{(r)}\left(\frac{k}{n}\right) \left( \frac{Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} \right), \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f_+^{(r)}\left(\frac{k}{n}\right) \left( \frac{Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} \right) \right] \\ &= [A_n(f_-^{(r)}, x), A_n(f_+^{(r)}, x)]. \end{aligned} \quad (4.11)$$

We have proved that

$$(A_n^{\mathcal{F}}(f, x))_{\pm}^{(r)} = A_n(f_{\pm}^{(r)}, x), \quad (4.12)$$

$$\forall r \in [0, 1], \forall x \in \left( \prod_{i=1}^N [a_i, b_i] \right).$$

Similarly, as in [21], pp. 485-489, a lengthy proof (see Remark 21.31 and proof of (21.76) there) it holds that

$$(L_n^{\mathcal{F}}(f, x))_{\pm}^{(r)} = L_n(f_{\pm}^{(r)}, x), \quad (4.13)$$

$\forall r \in [0, 1], \forall x \in \mathbb{R}^N$ .

Based on (4.12) and (4.13) now one may write

$$D(A_n^{\mathcal{F}}(f, x), f(x)) = \quad (4.14)$$

$$\sup_{r \in [0, 1]} \max \left\{ \left| \left( A_n \left( f_{-}^{(r)}, x \right) \right) - f_{-}^{(r)}(x) \right|, \left| A_n \left( f_{+}^{(r)}, x \right) - f_{+}^{(r)}(x) \right| \right\},$$

and

$$D(L_n^{\mathcal{F}}(f, x), f(x)) = \sup_{r \in [0, 1]} \max \left\{ \left| \left( L_n \left( f_{-}^{(r)}, x \right) \right) - f_{-}^{(r)}(x) \right|, \left| L_n \left( f_{+}^{(r)}, x \right) - f_{+}^{(r)}(x) \right| \right\}. \quad (4.15)$$

We present

**Theorem 4.1.** *Let  $f \in C_{\mathcal{F}} \left( \prod_{i=1}^N [a_i, b_i] \right)$ ,  $0 < \beta < 1$ ,  $x \in \left( \prod_{i=1}^N [a_i, b_i] \right)$ ,  $N, n \in \mathbb{N}$  with  $n^{1-\beta} > 2$ . Then*

1)

$$D(A_n^{\mathcal{F}}(f, x), f(x)) \leq \gamma(N) \left[ \omega_1^{(\mathcal{F})} \left( f, \frac{1}{n^{\beta}} \right) + 2c(\beta, n) D^*(f, \tilde{\delta}) \right] =: \rho_1, \quad (4.16)$$

and

2)

$$D^*(A_n^{\mathcal{F}}(f), f) \leq \rho_1. \quad (4.17)$$

We notice that  $A_n^{\mathcal{F}}(f, x) \xrightarrow{D} f(x)$ , and  $A_n^{\mathcal{F}}(f) \xrightarrow{D^*} f$ , as  $n \rightarrow \infty$ , quantitatively with rates.

**Proof.** Since  $f \in C_{\mathcal{F}} \left( \prod_{i=1}^N [a_i, b_i] \right)$  we have that  $f_{\pm}^{(r)} \in C \left( \prod_{i=1}^N [a_i, b_i] \right)$ ,  $\forall r \in [0, 1]$ . Hence by (3.27) we obtain

$$\left| A_n \left( f_{\pm}^{(r)}, x \right) - f_{\pm}^{(r)}(x) \right| \leq \gamma(N) \left[ \omega_1 \left( f_{\pm}^{(r)}, \frac{1}{n^{\beta}} \right) + 2c(\beta, n) \left\| f_{\pm}^{(r)} \right\|_{\infty} \right]$$

$$\stackrel{\text{(by (2.8), (2.3))}}{\leq} \gamma(N) \left[ \omega_1^{(\mathcal{F})} \left( f, \frac{1}{n^\beta} \right) + 2c(\beta, n) D^*(f, \tilde{\sigma}) \right]. \quad (4.18)$$

By (4.14) now we are proving the claim.

We give

**Theorem 4.2.** *Let  $f \in C_B(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}})$ ,  $0 < \beta < 1$ ,  $x \in \mathbb{R}^N$ ,  $N, n \in \mathbb{N}$ , with  $n^{1-\beta} > 2$ . Then*

1)

$$D(B_n^{\mathcal{F}}(f, x), f(x)) \leq \omega_1^{(\mathcal{F})} \left( f, \frac{1}{n^\beta} \right) + 2c(\beta, n) D^*(f, \tilde{\sigma}) =: \rho_2, \quad (4.19)$$

and

2)

$$D^*(B_n^{\mathcal{F}}(f), f) \leq \rho_2. \quad (4.20)$$

**Proof.** Similar to Theorem 4.1. We use (3.29) and (4.15), along with (3) and (8).

We also present

**Theorem 4.3.** *All as in Theorem 4.2. Then*

1)

$$D(C_n^{\mathcal{F}}(f, x), f(x)) \leq \omega_1^{(\mathcal{F})} \left( f, \frac{1}{n} + \frac{1}{n^\beta} \right) + 2c(\beta, n) D^*(f, \tilde{\sigma}) =: \rho_3, \quad (4.21)$$

and

2)

$$D^*(C_n^{\mathcal{F}}(f), f) \leq \rho_3. \quad (4.22)$$

**Proof.** Similar to Theorem 4.1. We use (3.31) and (4.15), along with (3) and (8).

We also give

**Theorem 4.4.** *All as in Theorem 4.2. Then*

1)

$$D(D_n^{\mathcal{F}}(f, x), f(x)) \leq \omega_1^{(\mathcal{F})} \left( f, \frac{1}{n} + \frac{1}{n^\beta} \right) + 2c(\beta, n) D^*(f, \tilde{\sigma}) = \rho_3, \quad (4.23)$$

and

2)

$$D^*(D_n^{\mathcal{F}}(f), f) \leq \rho_3. \quad (4.24)$$

**Proof.** Similar to Theorem 4.1. We use (3.33) and (4.15), along with (3) and (8).

**Note 4.5.** By Theorems 4.2, 4.3, 4.4 for  $f \in (C_B(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}}) \cap C_{\mathcal{F}}^U(\mathbb{R}^N))$ , we obtain  $\lim_{n \rightarrow \infty} D(L_n^{\mathcal{F}}(f, x), f(x)) = 0$ , and  $\lim_{n \rightarrow \infty} D^*(L_n^{\mathcal{F}}(f), f) = 0$ , quantitatively with rates, where  $L_n^{\mathcal{F}}$  is as in (4.6) and (4.7).

**Notation 4.6.** Let  $f \in C_{\mathcal{F}}^m\left(\prod_{i=1}^N [a_i, b_i]\right)$ ,  $m, N \in \mathbb{N}$ . Here  $f_{\alpha}$  denotes a fuzzy partial derivative with all related notation similar to the real case, see also Remark 2.15 and Notation 2.16. We denote

$$\omega_{1,m}^{(\mathcal{F})\max}(f_{\alpha}, h) := \max_{\alpha:|\alpha|=m} \omega_1^{(\mathcal{F})}(f_{\alpha}, h), \quad h > 0. \quad (4.25)$$

Call also

$$D_m^{*\max}(f_{\alpha}, \tilde{\delta}) := \max_{\alpha:|\alpha|=m} \{D^*(f_{\alpha}, \tilde{\delta})\}. \quad (4.26)$$

We finally present

**Theorem 4.7.** Let  $f \in C_{\mathcal{F}}^m\left(\prod_{i=1}^N [a_i, b_i]\right)$ ,  $0 < \beta < 1$ ,  $n, m, N \in \mathbb{N}$  with  $n^{1-\beta} > 2$ ,

and  $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ . Then

1)

$$D(A_n^{\mathcal{F}}(f, x), f(x)) \leq \gamma(N).$$

$$\left\{ \sum_{j^*=1}^m \left( \sum_{|\alpha|=j^*} \frac{D(f_{\alpha}(x), \tilde{\delta})}{\prod_{i=1}^N \alpha_i!} \left[ \frac{1}{n^{\beta j^*}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) c(\beta, n) \right] \right) \right. \\ \left. + \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{(\mathcal{F})\max}\left(f_{\alpha}, \frac{1}{n^{\beta}}\right) + \left( \frac{\|b - a\|_{\infty}^m D_m^{*\max}(f_{\alpha}, \tilde{\delta}) N^m}{m!} \right) 2c(\beta, n) \right\}, \quad (4.27)$$

2)

$$D^*(A_n^{\mathcal{F}}(f), f) \leq \gamma(N).$$

$$\left\{ \sum_{j^*=1}^m \left( \sum_{|\alpha|=j^*} \frac{D^*(f_{\alpha}, \tilde{\delta})}{\prod_{i=1}^N \alpha_i!} \left[ \frac{1}{n^{\beta j^*}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) c(\beta, n) \right] \right) \right. \\ \left. + \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{(\mathcal{F})\max}\left(f_{\alpha}, \frac{1}{n^{\beta}}\right) + \left( \frac{\|b - a\|_{\infty}^m D_m^{*\max}(f_{\alpha}, \tilde{\delta}) N^m}{m!} \right) 2c(\beta, n) \right\}, \quad (4.28)$$

3) additionally assume that  $f_\alpha(x_0) = \tilde{o}$ , for all  $\alpha : |\alpha| = 1, \dots, m; x_0 \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ , then

$$D(A_n^{\mathcal{F}}(f, x_0), f(x_0)) \leq \gamma(N) \left\{ \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{(\mathcal{F})\max} \left( f_\alpha, \frac{1}{n^\beta} \right) + \left( \frac{\|b-a\|_\infty^m D_m^{*\max}(f_\alpha, \tilde{o}) N^m}{m!} \right) 2c(\beta, n) \right\}, \tag{4.29}$$

notice in the last the extremely high rate of convergence at  $n^{-\beta(m+1)}$ .

Above we derive quantitatively with rates the high speed approximation of  $D(A_n^{\mathcal{F}}(f, x), f(x)) \rightarrow 0$ , as  $n \rightarrow \infty$ .

Also we establish with rates that  $D^*(A_n^{\mathcal{F}}(f), f) \rightarrow 0$ , as  $n \rightarrow \infty$ , involving the fuzzy smoothness of  $f$ .

**Proof.** Here  $f_\pm^{(r)} \in C^m \left(\prod_{i=1}^N [a_i, b_i]\right)$ . We observe that

$$\begin{aligned} & \left| A_n \left( f_\pm^{(r)}, x \right) - f_\pm^{(r)}(x) \right| \stackrel{(\text{??})}{\leq} \gamma(N) \cdot \\ & \left\{ \sum_{j^*=1}^m \left( \sum_{|\alpha|=j^*} \left( \frac{\left| \left( f_\pm^{(r)} \right)_\alpha(x) \right|}{\prod_{i=1}^N \alpha_i!} \right) \left[ \frac{1}{n^{\beta j^*}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) c(\beta, n) \right] \right) \right. \\ & \quad + \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{\max} \left( \left( f_\pm^{(r)} \right)_\alpha, \frac{1}{n^\beta} \right) + \\ & \quad \left. \left( \frac{\|b-a\|_\infty^m \left\| \left( f_\pm^{(r)} \right)_\alpha \right\|_{\infty, m}^{\max} N^m}{m!} \right) 2c(\beta, n) \right\} \stackrel{(\text{??})}{=} \tag{4.30} \\ & \gamma(N) \left\{ \sum_{j^*=1}^m \left( \sum_{|\alpha|=j^*} \left( \frac{\left| \left( f_\alpha \right)_\pm^{(r)}(x) \right|}{\prod_{i=1}^N \alpha_i!} \right) \left[ \frac{1}{n^{\beta j^*}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) c(\beta, n) \right] \right) \right. \\ & \quad \left. + \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{\max} \left( \left( f_\alpha \right)_\pm^{(r)}, \frac{1}{n^\beta} \right) + \right\} \end{aligned}$$



$$\left. \left( \frac{\|b - a\|_\infty^m \left\| (f_\alpha)_\pm^{(r)} \right\|_{\infty, m}^{\max} N^m}{m!} \right) 2c(\beta, n) \right\} \stackrel{\text{(by (??), (??))}}{\leq} \quad (4.31)$$

$$\begin{aligned}
 \gamma(N) \left\{ \sum_{j_*=1}^m \left( \sum_{|\alpha|=j_*} \left( \frac{D(f_\alpha(x), \tilde{\delta})}{\prod_{i=1}^N \alpha_i!} \right) \left[ \frac{1}{n^{\beta j_*}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) c(\beta, n) \right] \right) \right. \\
 \left. + \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\mathcal{F} \max} \left( f_\alpha, \frac{1}{n^\beta} \right) + \right. \\
 \left. \left( \frac{\|b - a\|_\infty^m D_m^{* \max}(f_\alpha, \tilde{\delta}) N^m}{m!} \right) 2c(\beta, n) \right\} =: T, \quad (4.32)
 \end{aligned}$$

respectively in  $\pm$ .

We have proved that

$$\left| A_n \left( f_\pm^{(r)}, x \right) - f_\pm^{(r)}(x) \right| \leq T, \quad (4.33)$$

$\forall r \in [0, 1]$ , respectively in  $\pm$ .

Using (4.14) we obtain

$$D \left( A_n^{\mathcal{F}}(f, x), f(x) \right) \leq T, \quad (4.34)$$

proving the theorem.

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