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ON ROGERS-RAMANUJAN-SLATER TYPE THETA FUNCTION IDENTITY

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Abstract: The main purpose of this article is to prove two Rogers–Ramanujan–Slater type theta function identities related to $\varphi(q)$ and $\phi_0(q)$, which were earlier investigated by two legendary mathematicians of their time. The results presented in this paper are motivated essentially by recent works of Cao *et al.* (see [5]).

Keywords and Phrases: Theta function, Rogers-Ramanujan-Slater identity, Jacobi's triple-product identity.

2020 Mathematics Subject Classification: Primary 05A30, 11B65, 33D15, 33D45; Secondary 33D60, 39A13, 39B32.

1. Introduction, Definitions and Preliminaries

Throughout this paper, we refer to [5, 6] for definitions and notations. We also suppose that $0 < q < 1$. For complex numbers a, the q-shifted factorials are defined by

$$(a;q)_0 := 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \tag{1}$$

where (see, for example, [6] and [10])

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Here, in our present investigation, we are mainly concerned with the homogeneous version of the Cauchy identity or the following q-binomial theorem (see, for example, [6], [10] and [12]):

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad (|z| < 1). \tag{2}$$

Upon further setting $a = 0$, the relation (2) becomes Euler's identity (see, for example, [6]):

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \qquad (|z| < 1) \tag{3}$$

and its inverse relation given below [6]:

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} z^k = (z;q)_\infty. \tag{4}$$

Based upon the q-binomial theorem (2) and Heine's transformations, Srivastava *et al.* [11] have considered the function (8) and established a set of two presumably new theta-function identities (see, for details, [11]). Ramanujan (see [8] and [9]) defined the general theta function:

$$f(a,b) = 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} (a^n + b^n) \quad |ab| < 1. \tag{5}$$

He also rediscovered Jacobi's famous triple-product identity which, in Ramanujan's notation, is given by (see [4, p.35, Entry 19]):

$$f(a,b) = (-a,ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{6}$$

Equivalently, we have [7]:

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2;q^2)_\infty(-zq;q^2)_\infty\left(-\frac{q}{z};q^2\right)_\infty, \quad (|q|<1,\ z\neq 0). \tag{7}$$

Several q-series identities, which emerge naturally from Jacobi's triple-product identity (6), are worthy of note here (see, for details, [4, pp. 36–37, Entry 22]):

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = (q^2;q^2)_\infty\{(-q;q^2)_\infty\}^2 = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{8}$$

and

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{9}$$

Upon setting $q=-q$ in (8), we obtain, $\varphi(-q) = \frac{(q;q)_\infty}{(-q;q)_\infty}$. In [1, Corollary 7.9, p.113], Andrews proved that for $|q|<1$.

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}. \tag{10}$$

In this paper, we will use one of the mock theta ϕ_0 defined in [13]:

$$\phi_0(q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n. \tag{11}$$

Replacing q by $-q^2$, we obtain:

$$\phi_0(-q^2) = \sum_{n=0}^{\infty}(-1)^n q^{2n^2}(q^2;q^4)_n.$$

Proposition 1. *In [5, Theorem 1, Eq.(14)], if $\varphi(q)$ and $G(q)$ are defined as in (8) and (10), then the following assertion holds true:*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} + \varphi(-q)\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})} = 2G(-q)\varphi(q). \tag{12}$$

Later George E Andrews observed that when we extend the sum to include negative values of n, the first infinite product vanishes, but the second one does not; and he further suggested that the error in the second product can correct by taking into account the new terms when it extend the sum to-infinity. We present the correct form of (12) in the following assertion (13) of Theorem 1. He also pointed out that the formula (13) is precisely the one given by Watson [13, p. 285].

2. Main Theorems

In this section, we establish two Rogers–Ramanujan–Slater type theta function identity.

Theorem 1. *If $\varphi(q)$ and $\phi_0(q^2)$ are defined as in (8) and (11), then each of the following assertion holds true:*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} + \varphi(-q) \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} = 2\phi_0(-q^2), \tag{13}$$

and

$$\sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n} + \varphi(-q) \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} = 2. \tag{14}$$

Proof. We first prove the assertion (13). Let us assume that an empty product is interpreted to be unity. Dividing by $\varphi(-q)$ the left hand side of (13), we get:

$$\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} + \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \text{by}(10)$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

$$= (-q;q)_\infty \sum_{n=0}^{\infty} q^{n^2} \left\{ \frac{1}{(q;q)_\infty(-q;q)_n} + \frac{1}{(-q;q)_\infty(q;q)_n} \right\}. \tag{15}$$

Upon using the fact that

$$(q;q)_\infty = (q;q)_n(q^{1+n};q)_\infty, \; (-q;q)_\infty = (-q;q)_n(-q^{1+n};q)_\infty$$

and

$$(q,-q,q)_n = (q^2,q^2)_n,$$

the right hand side of (15) reads:

$$(-q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} \left\{ \frac{1}{(q^{1+n};q)_\infty} + \frac{1}{(-q^{1+n};q)_\infty} \right\}. \tag{16}$$

Applying (3), equation (16) becomes:

$$(-q;q)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{n^2}(q^{1+n})^k}{(q^2;q^2)_n(q;q)_k}[1+(-1)^k].$$

Next, taking the upper signs, so that for $k=2k$, we get:

$$(-q;q)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{n^2}q^{2k+2nk}}{(q^2;q^2)_n(q;q)_{2k}} = 2(-q;q)_\infty \sum_{k=0}^{\infty}\frac{q^{2k}}{(q;q)_{2k}}\sum_{n=0}^{\infty}\frac{q^{n^2-n}}{(q^2;q^2)_n}\left(q^{2k+1}\right)^n$$

$$= 2(-q;q)_\infty \sum_{k=0}^{\infty}\frac{q^{2k}}{(q;q)_{2k}}(-q^{2k+1};q^2)_\infty$$

$$= 2\frac{(-q;q)_\infty}{(q;q^2)_\infty}\sum_{k=0}^{\infty}\frac{q^{2k}}{(q^2;q^2)_k}(q^{2k+1},-q^{2k+1};q^2)_\infty. \tag{17}$$

where we have introduced

$$1 = \frac{(q;q^2)_k(q^{2k+1};q^2)_\infty}{(q;q^2)_\infty}. \tag{18}$$

Further, if we substitute

$$(-q^{2k+1},q^{2k+1};q^2)_\infty = (q^{4k+2};q^4)_\infty$$

in the right hand side of (17), we obtain:

$$2\frac{(-q;q)_\infty}{(q;q^2)_\infty}\sum_{k=0}^{\infty}\frac{q^{2k}}{(q^2;q^2)_k}(q^{4k+2};q^4)_\infty.$$

According to equation (4), the above relation becomes:

$$2\frac{(-q;q)_\infty}{(q;q^2)_\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2-2n+n(4k+2)+2k}}{(q^4;q^4)_n(q^2;q^2)_k}$$

$$= 2\frac{(-q;q)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2}}{(q^4;q^4)_n}\sum_{k=0}^{\infty}\frac{q^{k(4n+2)}}{(q^2;q^2)_k}$$

$$= 2\frac{(-q;q)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2}}{(q^4;q^4)_n}\frac{1}{(q^{4n+2};q^2)_\infty}.$$

Using the identity $(q^2;q^2)_\infty = (q^2;q^2)_{2n}(q^{4n+2};q^2)_\infty$, we obtain:

$$2\frac{(-q;q)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2}}{(q^4;q^4)_n}\frac{1}{(q^{4n+2};q^2)_\infty}$$

$$= 2\frac{(-q;q)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2}(q^2;q^2)_{2n}}{(q^4;q^4)_n}$$

$$= 2\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{2n^2}(q^2;q^4)_n = 2\frac{\phi_0(-q^2)}{\varphi(-q)}.$$

After summarizing the above calculation, we obtain:

$$\frac{1}{\varphi(-q)}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n} + \prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})} = 2\frac{\phi_0(-q^2)}{\varphi(-q)}.$$

Hence we achieves the proof of (13).
Next, we attempt to prove our second assertion (14). The left hand side of (14) reads as;

$$(q;q)_\infty\left\{\sum_{n=0}^{\infty}\frac{q^n}{(q;q)_\infty(-q;q)_n} + \sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_\infty(q;q)_n}\right\}$$

$$= (q;q)_\infty\sum_{n=0}^{\infty}\frac{q^n}{(q^2;q^2)_n}\left(\frac{1}{(q^{1+n};q)_\infty} + \frac{1}{(-q^{1+n};q)_\infty}\right)$$

$$= (q;q)_\infty\sum_{n,k=0}^{\infty}\frac{q^{n+k(1+n)}}{(q;q)_k\,(q^2;q^2)_n}\left(1+(-1)^k\right)$$

$$= 2(q;q)_\infty\sum_{k=0}^{\infty}\frac{q^{2k}}{(q;q)_{2k}}\sum_{n=0}^{\infty}\frac{q^{n(1+2k)}}{(q^2;q^2)_n}$$

$$= 2(q;q)_\infty\sum_{k=0}^{\infty}\frac{q^{2k}}{(q;q)_{2k}}\frac{1}{(q^{1+2k};q^2)_\infty}. \qquad (19)$$

Applying the identity (18), the right hand side of (19) gives:

$$2\frac{(q;q)_\infty}{(q;q^2)_\infty}\sum_{k=0}^{\infty}\frac{q^{2k}}{(q;q)_{2k}}(q;q^2)_k = 2\frac{(q;q)_\infty}{(q;q^2)_\infty}\sum_{k=0}^{\infty}\frac{q^{2k}}{(q^2;q^2)_k} = 2.$$

which completes the proof of the assertion (2.2).
We thus have completed our proof of the above Theorem 1.

Remark 1. *We note that if we simplify our result (14) of Theorem 1, then it reduces as result (2.2) in [2, p.2, Eq.(2.2)] which can be deduced from a special case of Heine's transformation of q-hypergeometric series [1, p.19, Cor. 2.3]. Further, (2.4) in [2] reveals that this identity is especially simple and can be proved by a mathematical induction.*

3. Concluding Remarks and Observations

Our article is motivated by the Rogers-Ramanujan-Slater type theta function identities related to $\varphi(q)$ and $\phi_0(q)$. Here, we have investigated about two identities, which were earlier investigated by two legendary mathematicians of their time. The identity (13) of Theorem 1 was first proved by G.N.Watson in 1937 [13], and here in this article we proposed another proof. The identity (2.2)[2, p.2, Eq.(2.2)] was first proved by G. E. Andrews in 1997, which can be found after simplification of our identity (14) of Theorem 1.

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