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ON COMPACTNESS IN BI-GENERALIZED TOPOLOGICAL SPACES

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Abstract: In this paper, we define compactness for all open sets defined in bi-generalized topological spaces such as: $\mu_{(m,n)}$-semi compactness, $\mu_{(m,n)}$-pre compactness, $\mu_{(m,n)}$-regular compactness, $\mu_{(m,n)}$-α-compactness, $\mu_{(m,n)}$-β-compactness, $\overline{\mu}_{(m,n)}$-compactness and (m,n)-compactness. For our investigation, we choose $\mu_{(m,n)}$-semi compactness as a base space and studies the relationships between the $\mu_{(m,n)}$-semi compactness and other compactness in bi-generalized topological spaces.

Keywords and Phrases: Bi-generalized topological spaces, Open sets, Compactness.

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1. Introduction

Kelly [15] initiated the concept of bi-topological space (briefly, Bi-TS) in 1963 and thereafter many mathematicians generalized the topological ideas into bi-topological settings. Cs*á*sz*á*r [6] introduced the concepts of generalized neighborhood systems and generalized topological space (briefly, GTS). Research in GTS is still a hot area of research in which researchers introduced several types of continuity, compactness, homogeneity, and sets are extended from ordinary topological spaces to include GTS. As a generalization of Bi-TS, Boonpok [3] introduced the concept of bi-generalized topological space (briefly, Bi-GTS) and studied (m,n)-closed sets and (m,n)-open sets in Bi-GTS. Also, several authors [2, 7, 10, 12, 14, 21, 24] further extended the concept of various types of closed sets in Bi-GTS.

Different types of open sets in Bi-GTS were defined by several authors [4, 13, 19]. M. Murugalingam and S. Sompong [20, 25] introduced the boundary set on Bi-GTS. S. Sompong [26] introduced the exterior set on Bi-GTS. Further, S. Sompong [27] defined the dense set in Bi-GTS. S. Sompong and B. Rodjanadid [28] defined the neighbourhood and accumulation points in Bi-GTS. A. H. Zakari [31] defined the almost homeomorphism on Bi-GTS. Also the M. K. V. Donesa, L. L. L. Lusanta and W. K. Min [8, 16, 18] introduced various types of continuous functions in Bi-GTS. R. G. D. Gnanam and S. Sompong [11, 29] introduced a new kind of connectedness in Bi-GTS. In this Bi-GTS, separation axioms were defined by several authors [9, 17, 22, 29, 30]. Recently, S. AI Ghour and S. Sompong [1, 29] introduced certain covering properties, minimal sets and compact sets in Bi-GTS.

In this paper, we defined compactness for all the open sets in Bi-GTS such as: $\mu_{(m,n)}$-semi compactness, $\mu_{(m,n)}$-pre compactness, $\mu_{(m,n)}$-regular compactness, $\mu_{(m,n)}$-α-compactness, $\mu_{(m,n)}$-β-compactness, $\overline{\mu}_{(m,n)}$-compactness and (m,n)- compactness. We choose the $\mu_{(m,n)}$-semi compactness as a base space and studied its properties. Then, we established the relationships between the $\mu_{(m,n)}$-semi compactness and other compactness in Bi-GTS.

2. Preliminaries

Definition 2.1. [6] *Let X be a non-empty set and let we denote **P(X)** be the power set of X. A subset μ of **P(X)** is said to be a generalized topology (briefly, GT) on X if it satisfying the following axioms:*

1. *$\emptyset \in \mu$.*
2. *An arbitrary union of elements of μ belongs to μ.*

If μ is a GT on X, then (X, μ) is called a generalized topological space (briefly, GTS). The elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets.

Definition 2.2. [5] *Let (X, μ) be a GTS and $A \subseteq X$. Then, the μ-interior of A, denoted by $int_\mu(A)$, is the union of all μ-open sets contained in A. The μ-closure of A, denoted by $cl_\mu(A)$, is the intersection of all μ-closed sets containing A.*

Definition 2.3. [3] *Let X be a non-empty set and μ_1, μ_2 be GTs on X. Then, the triple (X, μ_1, μ_2) is said to be Bi-generalized topological space (briefly, Bi-GTS).*

Definition 2.4. [3] *Let (X, μ_1, μ_2) be a Bi-GTS and A be a subset of X. Then, the μ_m-interior of A with respect to μ_m, denoted by $int_{\mu_m}(A)$, is the union of all μ_m-open sets contained in A. The μ_m-closure of A with respect to μ_m, denoted by $cl_{\mu_m}(A)$, is the intersection of all μ_m-closed sets containing A.*

Definition 2.5. [28] *Let (X, μ_1, μ_2) be a Bi-GTS and Y be a subset of X. Define*

μ_{1_Y} and μ_{2_Y} as follows: $\mu_{1_Y} = \{Y \cap G : G \in \mu_1\}$ and $\mu_{2_Y} = \{Y \cap H : H \in \mu_2\}$. Then, the triple $(X, \mu_{1_Y}, \mu_{2_Y})$ is called a bi-generalized topological subspace of (X, μ_1, μ_2).

Definition 2.6. *Let (X, μ_1, μ_2) be a Bi-GTS and A be a subset of X. Then, A is said to be*

1. *$\mu_{(m,n)}$-semi open set [3] if $A \subseteq cl_{\mu_n}(int_{\mu_m}(A))$,*
2. *$\mu_{(m,n)}$-pre open set [3,11] if $A \subseteq int_{\mu_m}(cl_{\mu_n}(A))$,*
3. *$\mu_{(m,n)}$-regular open set [3] if $A = int_{\mu_m}(cl_{\mu_n}(A))$,*
4. *$\mu_{(m,n)}$-α-open set [3] if $A \subseteq int_{\mu_m}(cl_{\mu_n}(int_{\mu_m}(A)))$,*
5. *$\mu_{(m,n)}$-β-open set [3] if $A \subseteq cl_{\mu_n}(int_{\mu_m}(cl_{\mu_n}(A)))$,*
6. *(m,n)-open set [3] if $A = int_{\mu_m}(int_{\mu_n}(A))$.*

Where $m, n \in \{1,2\}$ and $m \neq n$. The complement of $\mu_{(m,n)}$-semi open ($\mu_{(m,n)}$-pre open, $\mu_{(m,n)}$-regular open, $\mu_{(m,n)}$-α-open, $\mu_{(m,n)}$-β-open, (m,n)-open) set is called $\mu_{(m,n)}$-semi closed ($\mu_{(m,n)}$-pre closed, $\mu_{(m,n)}$-regular closed, $\mu_{(m,n)}$-α-closed, $\mu_{(m,n)}$-β-closed, (m,n)-closed) set.

Let us denote the collection of all $\mu_{(m,n)}$-semi open sets, $\mu_{(m,n)}$-pre open sets, $\mu_{(m,n)}$-regular open sets, $\mu_{(m,n)}$-α-open sets, $\mu_{(m,n)}$-β-open sets on X by $\sigma_{(m,n)}(\mu)$, $\pi_{(m,n)}(\mu)$, $\gamma_{(m,n)}(\mu)$, $\alpha_{(m,n)}(\mu)$, $\beta_{(m,n)}(\mu)$ respectively.

We note that A is said to be a μ_n-semi open set in (X, μ_1, μ_2) if $A \subseteq cl_{\mu_n}(int_{\mu_n}(A))$, where $n \in \{1,2\}$.

Definition 2.7. [4] *Let (X, μ_1, μ_2) be a Bi-GTS and A be a subset of X. Then, A is said to be a $\overline{\mu}_{(m,n)}$-open set if there exists a μ_m-open set U of X such that $U \subseteq A \subseteq cl_{s_{\mu_n}}(U)$, where $cl_{s_{\mu_n}}(U)$ is the intersection of all μ_n-semi closed sets containing U. That is, the smallest μ_n-semi closed set containing U, where $m, n \in \{1,2\}$ and $m \neq n$.*

The complement of a $\overline{\mu}_{(m,n)}$-open set is called a $\overline{\mu}_{(m,n)}$-closed set.

Definition 2.8. [19] *Let (X, μ_1, μ_2) be a $Bi-GTS$ and A be a subset of X. Then, A is said to be a quasi generalized open set (briefly, q_μ-open set) if for every $x \in A$, then there exist a μ_1-open set U such that $x \in U \subseteq A$, or a μ_2-open set V such that $x \in V \subseteq A$.*

The complement of a q_μ-open set is called a q_μ-closed set.

The relationships between the $\mu_{(m,n)}$-semi open set and other open sets in *Bi-GTS* were studied in [4] and [23].

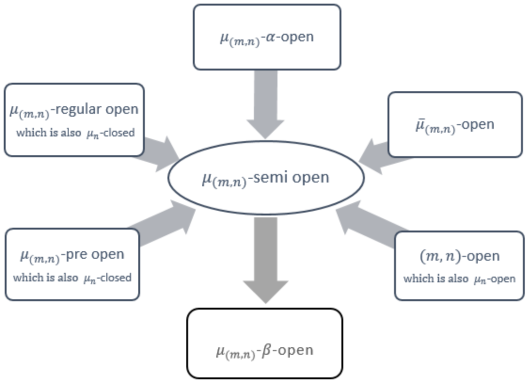

Figure1: Relationships between the $\mu_{(m,n)}$-semi open set and other open sets in *Bi-GTS* ([4],[23]).

3. Results

Definition 3.1. *A Bi-GTS is said to be $\mu_{(m,n)}$-semi compact space if for every $\mu_{(m,n)}$-semi open cover of (X, μ_1, μ_2) has a finite subcover, where $\mu_{(m,n)}$-semi open cover of (X, μ_1, μ_2) is defined to be a collection $\{G_i : i \in I\}$ of $\mu_{(m,n)}$-semi open sets of (X, μ_1, μ_2) such that $X = \bigcup_{i \in I} G_i$.*

Example 3.1. Let $X = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{a, b\}, X\}$. Then, $\sigma_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $X = \{a, b\} \cup \{a, c\}$. Therefore, (X, μ_1, μ_2) is $\mu_{(1,2)}$-semi compact space.

In similar way, we define the other types of compactness for other open sets in *Bi-GTS* (X, μ_1, μ_2).

Definition 3.2. *A Bi-GTS is said to be $\mu_{(m,n)}$-α-compact space if for every $\mu_{(m,n)}$-α-open cover of (X, μ_1, μ_2) has a finite subcover, where $\mu_{(m,n)}$-α-open cover of (X, μ_1, μ_2) is defined to be a collection $\{G_i : i \in I\}$ of $\mu_{(m,n)}$-α-open sets of (X, μ_1, μ_2) such that $X = \bigcup_{i \in I} G_i$.*

Definition 3.3. *A Bi-GTS is said to be $\mu_{(m,n)}$-β-compact space if for every $\mu_{(m,n)}$-β-open cover of (X, μ_1, μ_2) has a finite subcover, where $\mu_{(m,n)}$-β-open cover of (X, μ_1, μ_2) is defined to be a collection $\{G_i : i \in I\}$ of $\mu_{(m,n)}$-β-open sets of (X, μ_1, μ_2) such that $X = \bigcup_{i \in I} G_i$.*

Definition 3.4. *A Bi-GTS is said to be $\mu_{(m,n)}$-pre compact space if for every*

$\mu_{(m,n)}$-pre open cover of (X, μ_1, μ_2) has a finite subcover, where $\mu_{(m,n)}$-pre open cover of (X, μ_1, μ_2) is defined to be a collection $\{G_i : i \in I\}$ of $\mu_{(m,n)}$-pre open sets of (X, μ_1, μ_2) such that $X = \bigcup_{i \in I} G_i$.

Definition 3.5. *A Bi-GTS is said to be $\mu_{(m,n)}$-regular compact space if for every $\mu_{(m,n)}$-regular open cover of (X, μ_1, μ_2) has a finite subcover, where $\mu_{(m,n)}$-regular open cover of (X, μ_1, μ_2) is defined to be a collection $\{G_i : i \in I\}$ of $\mu_{(m,n)}$-regular open sets of (X, μ_1, μ_2) such that $X = \bigcup_{i \in I} G_i$.*

Definition 3.6. *A Bi-GTS is said to be $\overline{\mu}_{(m,n)}$-compact space if for every $\overline{\mu}_{(m,n)}$-open cover of (X, μ_1, μ_2) has a finite subcover, where $\overline{\mu}_{(m,n)}$-open cover of (X, μ_1, μ_2) is defined to be a collection $\{G_i : i \in I\}$ of $\overline{\mu}_{(m,n)}$-open sets of (X, μ_1, μ_2) such that $X = \bigcup_{i \in I} G_i$.*

Definition 3.7. *A Bi-GTS is said to be (m, n)-compact space if for every (m, n)-open cover of (X, μ_1, μ_2) has a finite subcover, where (m, n)-open cover of (X, μ_1, μ_2) is defined to be a collection $\{G_i : i \in I\}$ of (m, n)-open sets of (X, μ_1, μ_2) such that $X = \bigcup_{i \in I} G_i$.*

The compactness for quasi generalized open sets may not be defined in similar manner since the definition of quasi generalized open set is not defined similar to the definitions of other open sets in *Bi-GTS*.

Theorem 3.1. *Let Y be a subspace of a Bi-GTS (X, μ_1, μ_2). Then, Y is said to be a $\mu_{(m,n)}$-semi compact if and only if every covering of Y by sets $\mu_{(m,n)}$-semi open in (X, μ_1, μ_2) contains a finite sub collection covering Y.*
Proof. Suppose that Y is $\mu_{(m,n)}$-semi compact and $G = \{G_\alpha : \alpha \in J\}$ be a covering of Y by sets $\mu_{(m,n)}$-semi open in (X, μ_1, μ_2). Then, the collection $\{G_\alpha \cap Y : \alpha \in J\}$ is a covering of Y by sets $\mu_{(m,n)}$-semi open in Y. Hence, a finite sub collection $\{G_{\alpha_1} \cap Y, G_{\alpha_2} \cap Y, \cdots, G_{\alpha_n} \cap Y\}$ covers Y. Hence $\{G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_n}\}$ is a finite sub collection of G that covers Y.

Conversely, assume that every covering of Y by sets $\mu_{(m,n)}$-semi open in (X, μ_1, μ_2) contains a finite sub collection covering Y. Let $G' = \{G'_\alpha\}$ be a covering of Y by sets $\mu_{(m,n)}$-semi open in Y. For each α, choose a $\mu_{(m,n)}$-semi open set G_α in (X, μ_1, μ_2) such that $G'_\alpha = G_\alpha \cap Y$. Then, the collection $G = \{G_\alpha\}$ is covering of Y by sets $\mu_{(m,n)}$-semi open in (X, μ_1, μ_2). By hypothesis, some finite sub collection $\{G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_n}\}$ covers Y. Hence $\{G'_{\alpha_1}, G'_{\alpha_2}, \cdots, G'_{\alpha_n}\}$ is a finite sub collection of G' that covers Y.

Example 3.2. Let $X = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{a, b\}, X\}$. Then, $\sigma_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$. Now let $Y = \{a, b\} \subseteq X$. So $\mu_{1_Y} = \{\emptyset, \{a\}\}$ and $\mu_{2_Y} = \{\emptyset, \{a, b\}\}$. Then, $\sigma_{(1,2)}(\mu_Y) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Therefore, Y is $\mu_{(1,2)}$-semi compact.

Theorem 3.2. *Every $\mu_{(m,n)}$-semi closed subspace of a $\mu_{(m,n)}$-semi compact space is also a $\mu_{(m,n)}$-semi compact.*

Proof. Let A be a $\mu_{(m,n)}$-semi closed subspace of the $\mu_{(m,n)}$-semi compact space (X, μ_1, μ_2) and let $\mathscr{A} = \{G_i : i \in I\}$ be a covering of A by $\mu_{(m,n)}$-semi open sets in (X, μ_1, μ_2). Let $\mathscr{B}$ be a $\mu_{(m,n)}$-semi open cover of X. Then, $\mathscr{B} = \mathscr{A} \cup \{X - A\}$. Since (X, μ_1, μ_2) is $\mu_{(m,n)}$-semi compact, we get $\mathscr{B}$ has a finite subcover $\mathscr{B}_{finite}$ of X. If $\mathscr{B}_{finite}$ contains the set X - A, discard X - A. Otherwise, leave $\mathscr{B}_{finite}$ alone. Then, $\mathscr{B}_{finite}$ is a finite sub collection of $\mathscr{A}$ that covers A. Therefore, A is a $\mu_{(m,n)}$-semi compact.

Example 3.3. Let $X = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{a, b\}, X\}$. Then, $\sigma_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$. Now let $A = \{a, b\}$ be a $\mu_{(1,2)}$-semi closed subspace of (X, μ_1, μ_2). Therefore, A is a $\mu_{(1,2)}$-semi compact.

Definition 3.8. *Let (X, μ_1, μ_2) and $(Y, \lambda_1, \lambda_2)$ are two Bi-GTSs. A function $f : (X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ is said to be $\mu_{(m,n)}$-semi continuous if $f^{-1}(G)$ is a $\mu_{(m,n)}$-semi open set in (X, μ_1, μ_2) for every λ_n-semi open set G in $(Y, \lambda_1, \lambda_2)$. where $m, n \in \{1, 2\}$ and $m \neq n$.*

Theorem 3.3. *Let f:$(X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ be a surjective function. The image of a $\mu_{(m,n)}$-semi compact space under a $\mu_{(m,n)}$-semi continuous function is λ_n-semi compact space.*

Proof. Let f:$(X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ be an onto $\mu_{(m,n)}$-semi continuous function and let $\{G_i : i \in I\}$ is a λ_n-semi open cover for Y. Then, $\{f^{-1}(G_i) : i \in I\}$ is a $\mu_{(m,n)}$-semi open cover for X. Since (X, μ_1, μ_2) is $\mu_{(m,n)}$-semi compact, we get $\mu_{(m,n)}$-semi open cover $\{f^{-1}(G_i) : i \in I\}$ has a finite subcover $\{f^{-1}(G_1), f^{-1}(G_2), \cdots, f^{-1}(G_n)\}$. Since f be an onto, we get $\{G_1, G_2, \cdots, G_n\}$ is a finite λ_n-semi open sub cover for Y. Therefore $(Y, \lambda_1, \lambda_2)$ is λ_n-semi compact.

Example 3.4. Let $X = Y = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{a, b\}, X\}$, $\lambda_1 = \{\emptyset, \{a\}, \{b, c\}, \{a, c\}, Y\}$, $\lambda_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$. Define f:$(X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Then, f be a surjective and (X, μ_1, μ_2) is $\mu_{(1,2)}$-semi compact space under a $\mu_{(1,2)}$-semi continuous function. Therefore, $(Y, \lambda_1, \lambda_2)$ is λ_2-semi compact.

Definition 3.9. *Let (X, μ_1, μ_2) and $(Y, \lambda_1, \lambda_2)$ are two Bi-GTSs. A function $f : (X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ is said to be $\mu_{(m,n)}$-semi irresolute if $f^{-1}(G)$ is a $\mu_{(m,n)}$-semi open set in (X, μ_1, μ_2) for every $\lambda_{(m,n)}$-semi open set G in $(Y, \lambda_1, \lambda_2)$. where $m, n \in \{1, 2\}$ and $m \neq n$.*

Theorem 3.4. *If f:$(X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ is a $\mu_{(m,n)}$-semi irresolute surjective function and if $A \subseteq X$ is a $\mu_{(m,n)}$-semi compact relative to (X, μ_1, μ_2), then the image $f(A)$ is a $\lambda_{(m,n)}$-semi compact relative to $(Y, \lambda_1, \lambda_2)$.*

Proof. Let f:$(X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ be an onto $\mu_{(m,n)}$-semi irresolute map and let $\{G_i : \text{i} \in \text{I}\}$ is a $\lambda_{(m,n)}$-semi open cover of $f(A)$ relative to $(Y, \lambda_1, \lambda_2)$. Then, $\{f^{-1}(G_i) : \text{i} \in \text{I}\}$ is a $\mu_{(m,n)}$-semi open cover for A relative to (X, μ_1, μ_2). Since A is a $\mu_{(m,n)}$-semi compact relative to (X, μ_1, μ_2), we get $\mu_{(m,n)}$-semi open cover has a finite subcover $\{f^{-1}(G_1),\ f^{-1}(G_2),\cdots,f^{-1}(G_n)\}$. Since f be an onto, $\{G_1, G_2,\cdots,G_n\}$ is a finite $\mu_{(m,n)}$-semi open cover for $f(A)$. Therefore, $f(A)$ is a $\lambda_{(m,n)}$-semi compact relative to $(Y, \lambda_1, \lambda_2)$.

Example 3.5. Let $X = Y = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{a, b\}, X\}$, $\lambda_1 = \{\emptyset, \{a\}, \{b, c\}, \{a, c\}, Y\}$, $\lambda_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$. Define f:$(X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ by $f(a) = c$, $f(b) = b$, $f(c) = a$. Then, f be a surjective and $\mu_{(1,2)}$-semi irresolute function. Now let $A = \{a, c\}$ is a $\mu_{(1,2)}$-semi compact relative to (X, μ_1, μ_2). Therefore, $f(A)$ is a $\lambda_{(1,2)}$-semi compact relative to $(Y, \lambda_1, \lambda_2)$.

Definition 3.10. *Let (X, μ_1, μ_2) and $(Y, \lambda_1, \lambda_2)$ are two Bi-GTSs. A function f:$(X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ is said to be $\mu_{(m,n)}$-semi open if $f(G)$ is a $\lambda_{(m,n)}$-semi open set in $(Y, \lambda_1, \lambda_2)$ for every $\mu_{(m,n)}$-semi open set G in (X, μ_1, μ_2). where $m, n \in \{1, 2\}$ and $m \neq n$.*

Theorem 3.5. *If f:$(X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ be a surjective $\mu_{(m,n)}$-semi open function and $(Y, \lambda_1, \lambda_2)$ is $\lambda_{(m,n)}$-semi compact space. Then, (X, μ_1, μ_2) is $\mu_{(m,n)}$-semi compact space.*

Proof. Let f:$(X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ be a surjective $\mu_{(m,n)}$-semi open function and let $\{G_i : \text{i} \in \text{I}\}$ is a $\mu_{(m,n)}$-semi open cover of X. Then, $\{f(G_i) : \text{i} \in \text{I}\}$ is a $\lambda_{(m,n)}$-semi open cover of Y. Since $(Y, \lambda_1, \lambda_2)$ is $\lambda_{(m,n)}$-semi compact space, we get $\lambda_{(m,n)}$-semi open cover has a finite subcover $\{f(G_1),\ f(G_2),\cdots,f(G_n)\}$. Since f be an onto, $\{G_1,\ G_2,\cdots,G_n\}$ is a finite $\mu_{(m,n)}$-semi open cover of X. Therefore, (X, μ_1, μ_2) is $\mu_{(m,n)}$-semi compact space.

Example 3.6. Let $X = Y = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{a,\ b\}, X\}$, $\lambda_1 = \{\emptyset, \{a\}, \{c\}, \{b, c\}, \{a, c\}, Y\}$, $\lambda_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \text{Y}\}$. Define f:$(X, \mu_1, \mu_2) \to (Y, \lambda_1, \lambda_2)$ by $f(a) = c$, $f(b) = b$, $f(c) = a$. Then, f be a surjective, $\mu_{(1,2)}$-semi open function and $(Y, \lambda_1, \lambda_2)$ is $\lambda_{(1,2)}$-semi compact space. Therefore, (X, μ_1, μ_2) is $\mu_{(1,2)}$-semi compact space.

Theorem 3.6. *Let (μ_1, μ_2) and (μ'_1, μ'_2) are two pair of Bi-GTs on the space X, $\mu_1 \subset \mu'_1$ and $\mu_2 \subset \mu'_2$. If (X, μ'_1, μ'_2) is $\mu'_{(m,n)}$-semi compact, then (X, μ_1, μ_2) is $\mu_{(m,n)}$-semi compact.*

Proof. Let $\mathscr{G} = \{G_i : \text{i} \in \text{I}\}$ be a $\mu_{(m,n)}$-semi open cover of X. Since $\mu_1 \subset \mu'_1$ and $\mu_2 \subset \mu'_2$, we get $\sigma_{(m,n)}(\mu) \subset \sigma_{(m,n)}(\mu')$. This implies that $\mathscr{G}$ is a $\mu'_{(m,n)}$-semi open cover of X. Since (X, μ'_1, μ'_2) is $\mu'_{(m,n)}$-semi compact, So $\mathscr{G}$ contains a finite sub covers of X. Therefore, (X, μ_1, μ_2) is $\mu_{(m,n)}$-semi compact.

Example 3.7. Let $X = \{a, b, c\}$, $\mu'_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu'_2 = \{\emptyset, \{c\}, \{a, b\}, X\}$, $\mu_1 = \{\emptyset, \{a\}, \{a, c\}\}$, $\mu_2 = \{\emptyset, \{a, b\}, X\}$. Then, $\sigma_{(1,2)}(\mu') = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, X\}$ and $\sigma_{(1,2)}(\mu) = \{\emptyset, \{c\}, \{a, b\}, \{a, c\}, X\}$. Therefore, (X, μ'_1, μ'_2) is $\mu'_{(1,2)}$-semi compact. Then, (X, μ_1, μ_2) is $\mu_{(1,2)}$-semi compact.

Properties for other compactness can be established in a similar manner.

Now we choose $\mu_{(m,n)}$-semi compact space as a base space and studies the relationships between the $\mu_{(m,n)}$-semi compact space and other compact spaces in *Bi-GTS*.

Lemma 3.1. *Every $\mu_{(m,n)}$-α-compactness is $\mu_{(m,n)}$-semi compactness.*

Proof. Since every $\mu_{(m,n)}$-α-open set is a $\mu_{(m,n)}$-semi open set [23]. Then, the result is immediate.

The converse of the above proposition need not be true in general. This can be seen in the following example:

Example 3.8. Let $X = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then, $\alpha_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$. Therefore, X is a $\mu_{(1,2)}$-semi compact space but not a $\mu_{(1,2)}$-α-compact space.

Lemma 3.2. *Every $\mu_{(m,n)}$-semi compactness is $\mu_{(m,n)}$-β-compactness.*

Proof. Since every $\mu_{(m,n)}$-semi open set is a $\mu_{(m,n)}$-β-open set [23]. Then, the result is immediate.

The converse of the above proposition need not be true in general. This can be seen in the following example:

Example 3.9. Let $X = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ and $\mu_2 = \{\emptyset, \{a, c\}, \{a, b\}, X\}$. Then, $\sigma_{(1,2)}(\mu) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $\beta_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Let $Y = \{a, b\}$ be a subspace of (X, μ_1, μ_2). Then, Y is a $\mu_{(1,2)}$-β-compact but not a $\mu_{(1,2)}$-semi compact.

Lemma 3.3. *Every $\overline{\mu}_{(m,n)}$-compactness is $\mu_{(m,n)}$-semi compactness.*

Proof. Since every $\overline{\mu}_{(m,n)}$-open set is a $\mu_{(m,n)}$-semi open set [4, 23]. Then, the result is immediate.

The converse of the above proposition need not be true in general. This can be seen in the following example:

Example 3.10. Let $X = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{c\}, \{a, b\}, X\}$ and $\mu_2 = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then, $\overline{\mu}_{(m,n)}$-open sets are $\emptyset$, {c}, {a,b}, X and $\sigma_{(1,2)}(\mu) = \{\emptyset, \{c\}, \{a, b\}, \{a, c\}, X\}$. Let $Y = \{a, c\}$ be a subspace of (X, μ_1, μ_2). Then, Y is a $\mu_{(1,2)}$-semi compact but not a $\overline{\mu}_{(1,2)}$-compact.

Lemma 3.4. *let (X, μ_1, μ_2) be a Bi-GTS in which every $\mu_{(m,n)}$-pre open set is a μ_n-closed set. Then, every $\mu_{(m,n)}$-pre compactness is $\mu_{(m,n)}$-semi compactness in (X, μ_1, μ_2).*

Proof. When a $\mu_{(m,n)}$-pre open set which is also a μ_n-closed set in a *Bi-GTS* (X, μ_1, μ_2) is a $\mu_{(m,n)}$-semi open set [23]. Then, the result is immediate.

The converse of the above proposition need not be true in general. This can be seen in the following example:

Example 3.11. Let $X = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{a, b\}, X\}$. Then, $\sigma_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\pi_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then, (X, μ_1, μ_2) is a $\mu_{(1,2)}$-semi compact but not a $\mu_{(1,2)}$-pre compact.

Lemma 3.5. *let (X, μ_1, μ_2) be a Bi-GTS in which every $\mu_{(m,n)}$-regular open set is a μ_n-closed set. Then, every $\mu_{(m,n)}$-regular compactness is $\mu_{(m,n)}$-semi compactness in (X, μ_1, μ_2).*

Proof. When a $\mu_{(m,n)}$-regular open set which is also a μ_n-closed set in a *Bi-GTS* (X, μ_1, μ_2) is a $\mu_{(m,n)}$-semi open set [23]. Then, the result is immediate.

The converse of the above proposition need not be true in general. This can be seen in the following example:

Example 3.12. Let $X = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{a, b\}, X\}$. Then, $\sigma_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\gamma_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then, (X, μ_1, μ_2) is a $\mu_{(1,2)}$-semi compact but not a $\mu_{(1,2)}$-regular compact.

Lemma 3.6. *let (X, μ_1, μ_2) be a Bi-GTS in which every (m, n)-open set is a μ_n-open set. Then, every (m, n)-compactness is $\mu_{(m,n)}$-semi compactness in (X, μ_1, μ_2).*

Proof. When a (m, n)-open set which is also a μ_n-open set in a *Bi-GTS* (X, μ_1, μ_2) is a $\mu_{(m,n)}$-semi open set [23]. Then, the result is immediate.

The converse of the above proposition need not be true in general. This can be seen in the following example:

Example 3.13. Let $X = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mu_2 = \{\emptyset, \{c\}, \{a, b\},$

$\{a, c\}$,X$\}$. Then, $\sigma_{(1,2)}(\mu) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}$,X$\}$ and (1,2)-open sets are $\emptyset$, $\{$c$\}$, $\{$a,c$\}$. Then, (X, μ_1, μ_2) is a $\mu_{(1,2)}$-semi compact but not a $(1, 2)$-compact.

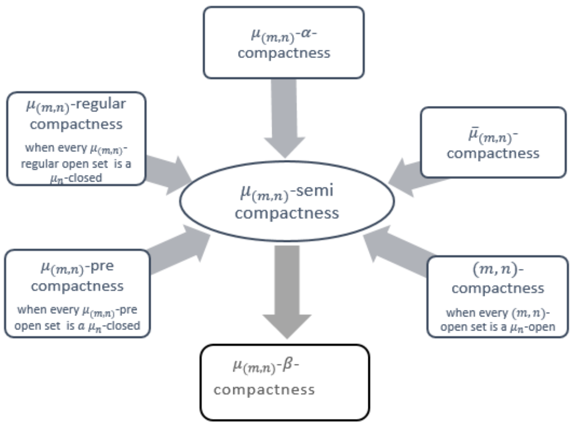

Figure2: Relationships between the $\mu_{(m,n)}$-semi compactness and other compactness in Bi-GTS.

4. Conclusion

In this paper, we defined compactness for all open sets defined in Bi-GTS such as: $\mu_{(m,n)}$-semi compactness, $\mu_{(m,n)}$-pre compactness, $\mu_{(m,n)}$-regular compactness, $\mu_{(m,n)}$-α-compactness, $\mu_{(m,n)}$-β-compactness, $\overline{\mu}_{(m,n)}$-compactness and (m, n)-compactness. For our investigation, we choose $\mu_{(m,n)}$-semi compactness as a base space and studied their properties. Also we studied the relationships between the $\mu_{(m,n)}$-semi compactness and other compactness in Bi-GTS.

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