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A STUDY ON G-SETS

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Abstract. This paper concerns the study of G-sets and their basic properties.

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1. Introduction

The idea of groups with operators has been discussed in [2]. This idea leads to a generalization in the form of sets with operators. In the group theory this concept is known as G-sets. In this paper, we obtain some basic properties of G-sets.

2. Preliminaries

Definition 2.1. Let G be a group, X be a set and $\phi : G \times X \to X$ be a mapping. Then, the pair (X, ϕ) is called a G-set (or a set with operator G), if for all $g_1, g_2 \in G$ and $x \in X$, the following conditions are satisfied:

(i) $\phi(g_1g_2, x) = \phi(g_1, \phi(g_2, x))$,

(ii) $\phi(e, x) = x$,

where e is the identity of G.

For the sake of convenience, one can denote $\phi(g, x)$ as gx. Under this notation, above conditions become

(i) $(g_1g_2)x = g_1(g_2x)$,

(ii) $ex = x$.

3. Results on G-Sets

Theorem 3.1. Every normal subgroup H of a group G is a G-set under the mapping $\phi : G \times H \to H$ defined by $\phi(a, h) = aha^{-1}$ for every $a \in G$ and $h \in H$.

Proof. For all elements $a, b \in G$, we have

$$\begin{aligned}\phi(ab, h) = &\ (ab)h(ab)^{-1} \ = \ (ab)h(b^{-1}a^{-1}) = a(bhb^{-1})a^{-1} \\ = &\ a(\phi(b, h))a^{-1} \ = \ \phi(a, \phi(b, h)).\end{aligned}$$

Further, let e be the identity of G. Then, we have $\phi(e, h) = ehe^{-1} = ehe = h$. Therefore, H is a G-set.

Theorem 3.2. Every group G is a G-set under the following mappings

(i) $\phi : G \times G \to G$ such that $\phi(a, g) = ag$ for all $a, g \in G$.
This mapping is known as translation.

(ii) $\phi : G \times G \to G$ such that $\phi(a, g) = aga^{-1}$ for all $a, g \in G$.
This mapping is known as conjugation.

Proof (i). For all elements $a, b \in G$, we have

$$\phi(ab, g) \ = \ (ab)g \ = \ a(bg) \ = \ a(\phi(b, g)) \ = \ \phi(a, \phi(b, g)).$$

Further, let e be the identity element of G. Then, we have $\phi(e, g) = eg = g$.

Therefore, G is a G-set under translation.

Proof (ii). For all elements $a, b \in G$, we have

$$\begin{aligned}\phi(ab, g) = &\ (ab)g(ab)^{-1} \ = \ (ab)g(b^{-1}a^{-1}) \ = \ a(bgb^{-1})a^{-1} \\ = &\ a(\phi(b, g))a^{-1} \ = \ \phi(a, \phi(b, g)).\end{aligned}$$

Further, let e be the identity element of G. Then, we have $\phi(e, g) = ege^{-1} = ege = g$.

Therefore, G is a G-set under conjugation.

Theorem 3.3. Let H be a normal subgroup of a group G. Then, the set G/H of all left cosets of H in G is a G-set under the mapping $\phi : G \times G/H \to G/H$ defined by $\phi(g, aH) = gag^{-1}H$ for all $a, g \in G$.

Proof. First we show that ϕ is well defined. Let $a, b \in G$ and let,

$$\begin{aligned}&(g, aH) = (g, bH) \\ &\Rightarrow \ aH = bH \\ &\Rightarrow \ b^{-1}a \in H. \qquad (3.1)\end{aligned}$$

To show that ϕ is well defined, we prove that $\phi(g, aH) = \phi(g, bH)$. We have, $\phi(g, aH) = gag^{-1}H$ and $\phi(g, bH) = gbg^{-1}H$.

Let us assume $\alpha = gag^{-1}$ and $\beta = gbg^{-1}$. Then, we have

$$\begin{aligned}\beta^{-1}\alpha &= (gbg^{-1})^{-1}(gag^{-1}) = (gb^{-1}g^{-1})(gag^{-1}) \\ &= gb^{-1}(g^{-1}g)ag^{-1} = gb^{-1}eag^{-1} \\ &= g(b^{-1}a)g^{-1} = ghg^{-1} \qquad (\text{ where } h = b^{-1}a \in H \text{ from } (3.1))\end{aligned}$$

i.e., $\beta^{-1}\alpha = ghg^{-1} \in H$ (as H is normal)
$\Rightarrow \beta^{-1}\alpha \in H$
$\Rightarrow \alpha H = \beta H$
$\Rightarrow gag^{-1}H = gbg^{-1}H$
$\Rightarrow \phi(g, aH) = \phi(g, bH)$
Therefore, ϕ is well defined.

Now, for all elements $a, b \in G$, we have

$$\begin{aligned}\phi(ab, gH) = & (ab)g(ab)^{-1}H = (ab)g(b^{-1}a^{-1})H = a(bgb^{-1})a^{-1}H \\ &= \phi(a, bgb^{-1}H) = \phi(a, \phi(b, gH)).\end{aligned}$$

Further, let e be the identity element of G. Then, we have $\phi(e, gH) = ege^{-1}H = gH$. Therefore G/H is a G-set.

Remark 3.1. If H is not normal subgroup, then G/H is a G-set under the mapping $\phi : G \times G/H \to G/H$ defined by $\phi(g, aH) = gaH$ for all $a, g \in G$.

Theorem 3.4. Let X and Y be two G-sets.Then,

(i) $X \cap Y$ is a G-set

(ii) $X \times Y$ is a G-set

(iii) Disjoint union of X and Y is a G-set.

Proof (i). Let (X, ϕ) and (Y, ψ) be two G-sets. To show that $X \cap Y$ is a G-set, we define a mapping $\eta : G \times (X \cap Y) \to X \cap Y$ such that

$$\eta(a, z) = \phi(a, z) = \psi(a, z) \; \forall \; a \in G \text{ and } z \in X \cap Y.$$

It is easy to show that $X \cap Y$ is a G-set.

(ii). Let (X, ϕ) and (Y, ψ) be two G-sets. To show that $X \times Y$ is a G-set, we define a mapping $\eta : G \times (X \times Y) \to X \times Y$ such that

$$\eta(a, (x, y)) = (\phi(a, x), \psi(a, y)) \; \forall \; a \in G \text{ and } (x, y) \in X \times Y.$$

Now, for all elements $a, b \in G$ and $(x, y) \in X \times Y$, we have

$$\begin{aligned}\eta(ab, (x, y)) &= (\phi(ab, x), \psi(ab, y)) \\ &= (\phi(a, \phi(b, x), \psi(a, \psi(b, y)) \\ &= \eta(a, (\phi(b, x), \psi(b, y)) \qquad (\text{by defintion of } \eta) \\ &= \eta(a, \eta(b, (x, y)).\end{aligned}$$

Further, let e be the identity element of G and $(x, y) \in X \times Y$. Then, we have

$$\begin{aligned} \eta(e,(x,y)) &= (\phi(e,x),\psi(e,y)) \\ &= (x,y). \end{aligned}$$

Therefore, $X \times Y$ is a G-set.

(iii). Let (X, ϕ) and (Y, ψ) be two G-sets such that $X \cap Y = \emptyset$. To show that $X \cup Y$ is a G-set, we define a mapping $\eta : G \times (X \cup Y) \to X \cup Y$ such that

$$\eta(a,z) = \begin{cases} \phi(a,z) & \text{if} \quad z \in X \\ \\ \psi(a,z) & \text{if} \quad z \in Y \quad \text{for all} \quad a \in G \text{ and } z \in X \cup Y. \end{cases}$$

It is easy to verify that $X \cup Y$ is a G-set.

Theorem 3.5. Let X be a G-set. Then, power set of X is also a G-set.

Proof. Let (X, ϕ) be a G-set and let $P(X)$ be the power set of X. To show that $P(X)$ is a G-set, we define a mapping $\eta : G \times P(X) \to P(X)$ such that

$$\eta(a,S) = \{\phi(a,x) \mid x \in S\} \ \forall \ a \in G \ \text{ and } \ S \in P(X).$$

Now, for all elements $a, b \in G$ and $x \in S \in P(X)$, we have

$$\begin{aligned} \eta(ab,S) &= \{\phi(ab,x) \mid x \in S\} \\ &= \{\phi(a,\phi(b,x)) \mid x \in S\} \\ &= \eta(a,\{\phi(b,x) \mid x \in S\}) \\ &= \eta(a,\eta(b,S)). \end{aligned}$$

Further, let e be the identity element of G and $S \in P(X)$. Then, we have

$$\begin{aligned} \eta(e,S) &= \{\phi(e,x) \mid x \in S\} \\ &= \{x \mid x \in S\} \\ &= S. \end{aligned}$$

Therefore, $P(X)$ is a G-set.

References

[1] Curtic, C.W. and Reiner, I., *Methods of Representation Theory with Applications to Finite Groups and Orders*, Vol.I, A Wiley Interscience Publication, New York, 1981.

[2] Jacobson, N., *Lectures in Abstract Algebra*, Vol.I, D. Van Nostrand Company, Inc., Princeton, New Jersey, New York, 1965.

[3] Malik, D.S. and Sen, J.M., *Fundamentals of Abstract Algebra*, the McGraw-Hill Companies, Inc., 1997.