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ON M-PROJECTIVE CURVATURE TENSOR OF PARA-KENMOTSU MANIFOLDS ADMITTING ZAMKOVOY CONNECTION

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Abstract: In this paper, relation between curvature tensors of Levi-Civita connection and Zamkovoy connection on para-Kenmotsu manifolds have been obtained.Quasi M-projectively flat, M-projectively flat and ϕ-M-projectively flat para-Kenmotsu manifolds admitting Zamkovoy connection have been studied. Also, para-Kenmotsu manifolds admitting Zamkovoy connction satisfying $\bar{M}(\xi, U).\bar{R} = 0$ and $\bar{M}(\xi, U).\bar{S} = 0$ have been developed.

Keywords and Phrases: Para-Kenmotsu manifold, M-projective curvature tensor, Zamkovoy connection, Quasi M-projectively flat, $\phi-$ M-projectively flat, Bianchi's identity.

2020 Mathematics Subject Classification: 53C15, 53D15.

1. Introduction

In 2008, the notion of Zamkovoy connection was introduced by S. Zamkovoy [21] for paracontact manifold. Also this is known as canonical paracontact connection whose torsion is the obstruction of paracontact manifold to be para-Sasakian manifold. For an n-dimensional almost contact metric manifold M^n consisting of $(1,1)$ tensor field ϕ, a 1-form η, a vector field ξ and a Riemannian metric g endowed with an almost contact metric structure (ϕ, ξ, η, g), Zamkovoy connection is defined by

$$\bar{\nabla}_U V = \nabla_U V + (\nabla_U \eta)(V)\xi - \eta(V)\nabla_U \xi + \eta(U)\phi V, \tag{1.1}$$

for all $U,\ V \in \chi(M)$. Further Zamkovoy connection have been studied by many authors such as ([2], [3], [5], [9]).

In 1969, Tanno [20] classified connected almost contact Riemannian manifolds whose automorphism groups have the maximum dimension. In this classification, the almost contact Riemannian manifolds are divided into three classes: (i) homogeneous normal contact Riemannian manifolds with constant $\phi-$holomorphic sectional curvature if the sectional curvature for 2-planes which contains ξ, $K(X,\xi) > 0$, (ii) global Riemannian products of a line or a circle and a Kahlerian manifold with constant holomorphic sectoinal curvature if $K(X,\xi) = 0$ and (iii) a warped product space $L \times_f CE^n$, if $K(X,\xi) < 0$ [8]. In 1976, Sato [16] introduced the notion of paracontact manifolds on a Riemannian manifold. On the other hand Kaneyuki and Willams [7] introduced almost paracontact structure on semi-Riemannian manifold. Adati and Matsumoto [1] defined and studied P-Sasakian and SP-Sasakian manifolds which are regarded as special kind of an almost contact Riemannian manifold. In 1995, Sinha and Saiprasad [19], have defined a class of almost paracontact metric manifold namely para-Kenmotsu and special para-Kenmotsu manifolds. Para-Kenmotsu manifolds have been studied by several authors ([4], [5], [13], [15], [21]) and many others.

The notion of M-projective curvature tensor on a Riemannian manifold was introduced by Pokhariyal and Mishra [14] as

$$\begin{aligned} M(U,V)X &= R(U,V)X \\ &\quad - \frac{1}{2(n-1)}[S(V,X)U - S(U,X)V + g(U,X)QV - g(V,X)QU], \end{aligned} \tag{1.2}$$

where R denotes the Riemannian curvature tensor of type (1,3), S denotes the Ricci tensor of type (0,2) and Q denotes the Ricci operator with respect to Levi-Civita connection. R. H. Ojha ([10], [11]) studied the curvature properties of M-projective curvature tensor on Sasakian manifolds. Moreover the M-projective curvature tensor was studied by several authors ([6], [12], [18]).

In a para-Kenmotsu manifold M^n of dimension $n > 2$, the M-projective curvature tensor $\bar{M}$ with respect to Zamkovoy connection ($\bar{\nabla}$) is given by

$$\begin{aligned}\bar{M}(U,V)X &= \bar{R}(U,V)X \\ &\quad - \frac{1}{2(n-1)}[\bar{S}(V,X)U - \bar{S}(U,X)V + g(U,X)\bar{Q}V - g(V,X)\bar{Q}U],\end{aligned} \tag{1.3}$$

where $\bar{R}$ denotes the Riemannian curvature tensor, $\bar{S}$ denotes the Ricci tensor and $\bar{Q}$ denotes the Ricci operator with respect to Zamkovoy connection $\bar{\nabla}$ respectively.

Definition 1.1. *An n-dimensional para-Kenmotsu manifold M^n is said to be Einstein manifold if its Ricci tensor is of the form $S(X,Y) = ag(X,Y)$, for all $U, V \in \chi(M)$, where a is scalar function.*

2. Para-Kenmotsu Manifolds

An odd dimensional smooth manifold M^n equipped with structure (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is $1-$form such that

$$\eta(\xi) = 1, \tag{2.1}$$

$$\phi^2 X = X - \eta(X)\xi. \tag{2.2}$$

Then M^n is an almost paracontact manifold. Let g be the Riemannian metric satisfying for all vector fields X and Y on M^n

$$g(X,\xi) = \eta(X), \tag{2.3}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.5}$$

Then the manifold M^n is said to admit an almost paracontact metric structure (ϕ, ξ, η, g). A manifold of dimension n with Riemannian metric g admitting a tensor field ϕ of type $(1,1)$, a vector field ξ and $1-$form η satisfying (2.1) and (2.3) along with

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0, \tag{2.6}$$

$$(\nabla_X \nabla_Y \eta)(U) = [-g(X,U) + \eta(X)\eta(U)]\eta(Y) + [-g(X,Y) + \eta(X)\eta(Y)]\eta(U), \tag{2.7}$$

$$\nabla_X \xi = \phi^2 X = X - \eta(X)\xi, \tag{2.8}$$

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)X \tag{2.9}$$

is called para-Kenmotsu manifold [17]. A para-Kenmotsu manifold admitting a 1-form η satisfies

$$(\nabla_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.10}$$

It is known that [17] in a para-Kenmotsu manifold, the following relation hold

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$
$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.12}$$
$$g(R(X,Y)U,\xi) = \eta(R(X,Y)U) = g(X,U)\eta(Y) - g(Y,U)\eta(X), \tag{2.13}$$
$$S(X,\xi) = -(n-1)\eta(X), \tag{2.14}$$
$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y). \tag{2.15}$$

The Zamkovoy connection on para-Kenmotsu manifold is given as

$$(\bar{\nabla}_X Y) = \nabla_X Y + g(X,Y)\xi - \eta(Y)X + \eta(X)\phi Y. \tag{2.16}$$

Example.
We consider the three-dimensional manifold
$M^3 = \{(u,v,w) \in R^3, w \neq 0\}$, where (u,v,w) are the standard coordinates in R^3. The vector fields

$$f_1 = \frac{\partial}{\partial u}, \quad f_2 = \frac{\partial}{\partial v}, \quad f_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + \frac{\partial}{\partial w},$$

are linearly independent at each point of M. Furthermore, by direct calculations, we have

$$[f_1,f_2] = 0, [f_1,f_3] = f_1, [f_2,f_3] = f_2.$$

Let η be the 1 -form defined by $\eta(Z) = g(Z,f_3)$ for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M.

Let φ be the $(1,1)$ tensor field defined by $\varphi(f_1) = f_2, \varphi(f_2) = f_1, \varphi(f_3) = 0$. Then using the linearity of φ and g we have $\eta(f_3) = 1$, $\varphi^2(Z) = Z - \eta(Z)f_3$, $g(\varphi Z,\varphi W) = -g(Z,W) + \eta(Z)\eta(W)$, for any $Z,W \in \chi(M)$. Thus for $f_3 = \xi, (\varphi,\xi,\eta,g)$ defines an almost contact metric structure on M.

Let the Levi-Civita connection with respect to g be ∇, then using Koszul's formula we get the following:

$$\nabla_{f_1} f_3 = f_1, \quad \nabla_{f_1} f_2 = 0, \quad \nabla_{f_1} f_1 = -f_3,$$
$$\nabla_{f_2} f_3 = f_2, \quad \nabla_{f_2} f_2 = f_3, \quad \nabla_{f_2} f_1 = 0$$
$$\nabla_{f_3} f_3 = 0, \quad \nabla_{f_3} f_2 = 0, \quad \nabla_{f_3} f_1 = 0.$$

From above relations we see that the manifold satisfies (2.13) for $f_3 = \xi$. Therefore the structure $M^3(\varphi,\xi,\eta,g)$ is a three-dimensional para-Kenmotsu manifold.

3. Curvature Properties of Para-Kenmotsu Manifolds Admitting Zamkovoy Connection

Let $\bar{R}$ denotes the Riemannian curvature tensor with respect to Zamkovoy connection defined as

$$\bar{R}(U,V)X = \bar{\nabla}_U\bar{\nabla}_V X - \bar{\nabla}_V\bar{\nabla}_U X - \bar{\nabla}_{[U,V]}X. \tag{3.1}$$

In view of equation (2.16), we have

$$\bar{R}(U,V)X = R(U,V)X - g(U,X)V - g(V,X)U, \tag{3.2}$$

where

$$R(U,V)X = \nabla_U\nabla_V X - \nabla_V\nabla_U X - \nabla_{[U,V]}X \tag{3.3}$$

is the Riemannian curvature tensor of Levi-Civita connection ∇. Equation (3.2) is the relation between Riemannian curvature tensors with respect to Zamkovoy connection $\bar{\nabla}$ and Levi-Civita connection ∇. Transvection of Y in equation (3.2), gives

$${}^{\backprime}\bar{R}(U,V,X,Y) = {}^{\backprime}R(U,V,X,Y) - g(U,X)g(V,Y) - g(V,X)g(U,Y), \tag{3.4}$$

where

$${}^{\backprime}\bar{R}(U,V,X,Y) = g(\bar{R}(U,V)X,Y)$$

and

$${}^{\backprime}R(U,V,X,Y) = g(R(U,V)X,Y)$$

Putting $V = X = e_i$ in equation (3.4) and taking summation over $i, 1 \leq i \leq n$, we get

$$\bar{S}(U,Y) = S(U,Y) + (n-1)g(U,Y), \tag{3.5}$$

where $\bar{S}$ and S denotes the Ricci tensors with respect to the connections $\bar{\nabla}$ and ∇ respectively. Again putting $U = Y = e_i$ in equation (3.5) and taking summation over i, $1 \leq i \leq n$, we get

$$\bar{r} = r + (n-1)n, \tag{3.6}$$

where $\bar{r}$ and r denotes the scalar curvatures with respect to the connections $\bar{\nabla}$ and ∇ respectively. From equation (3.5), we have

$$\bar{Q}U = QU + (n-1)U, \tag{3.7}$$

where $\bar{Q}$ and Q denotes the Ricci operators with respect to connections $\bar{\nabla}$ and ∇ respectively. Also

$$\bar{S}(U,\xi) = 0. \tag{3.8}$$

Now from equation (3.2), we have

$$\bar{R}(U,V)\xi = \bar{R}(\xi,V)X = \bar{R}(U,\xi)X = 0. \tag{3.9}$$

Theorem 3.1. *A para-Kenmotsu manifold M^n equipped with Zamkovoy connection satisfies Bianchi's first identity.*

Proof. Writing two more equations by the cyclic permutation of U, V, X in equation (3.2), we get

$$\bar{R}(V,X)U = R(V,X)U - g(V,U)X - g(X,U)V, \tag{3.10}$$

and

$$\bar{R}(X,U)V = R(X,U)V - g(V,X)U - g(U,V)X. \tag{3.11}$$

Adding equations (3.2), (3.10) and (3.11) with the fact that $R(U,V)X + R(V,X)U + R(X,U)V = 0$, we get

$$\bar{R}(U,V)X + \bar{R}(V,X)U + \bar{R}(X,U)V = 0. \tag{3.12}$$

This shows that a para-Kenmotsu manifold M^n equipped with Zamkovoy connection satisfies Bianchi's first identity.

Theorem 3.2. *The curvature tensor of para-Kenmotsu manifold M^n admitting Zamkovoy connection is*
(i) skew-symmetric in first two slots,
(ii) skew-symmetric in last two slots,
(iii) symmetric in pair of slots.

Proof. (i) Interchanging U and V in equation (3.4), we get

$$`\bar{R}(V,U,X,Y) = `R(V,U,X,Y) - g(V,X)g(U,Y) + g(U,X)g(V,Y). \tag{3.13}$$

Adding equations (3.4) and (3.14) with the fact that $`R(U,V,X,Y) + `R(V,U,X,Y) = 0$, we get

$$`\bar{R}(U,V,X,Y) + `\bar{R}(V,U,X,Y) = 0, \tag{3.14}$$

which shows that $`\bar{R}$ is skew-symmetric in first two slots.

(ii) Interchanging X and Y in equation (3.4), we get

$$`\bar{R}(U,V,Y,X) = `R(U,V,Y,X) - g(U,Y)g(V,X) + g(V,Y)g(U,X). \tag{3.15}$$

Adding equations (3.4) and (3.15) with the fact that $`R(U,V,X,Y) + `R(U,V,Y,X) = 0$, we get

$$`\bar{R}(U,V,X,Y) + `\bar{R}(U,V,Y,X) = 0, \tag{3.16}$$

which shows that $`\bar{R}$ is skew-symmetric in last two slots.
(iii) Interchanging pair of slots in equation (3.4), we get

$$`\bar{R}(X,Y,U,V) = `R(X,Y,U,V) - g(X,U)g(Y,V) + g(Y,U)g(X,V). \quad (3.17)$$

Subtracting equations (3.4) to (3.17) with the fact that $`R(U,V,X,Y) - `R(X,Y,U,V) = 0$, we get

$$`\bar{R}(U,V,X,Y) - `\bar{R}(X,Y,U,V) = 0, \quad (3.18)$$

which shows that $\bar{R}$ is symmetric in pair of slots.

Theorem 3.3. *If a para-Kenmotsu manifold M^n is Ricci flat with respect to Zamkovoy connection then the manifold is an Einstein manifold.*
Proof. Suppose that para-Kenmotsu manifold M^n is Ricci flat with respect to Zamkovoy connection, then from equation (3.5), we have

$$S(U,Y) = -(n-1)g(U,Y), \quad (3.19)$$

which shows that M^n is an Einstein manifold.

Theorem 3.4. *If the curvature tensor of para-Kenmotsu manifold admitting Zamkovoy connection vanishes then the manifold M^n is of constant curvature with respect to Levi-Civita connection.*
Proof. Consider $\bar{R}(U,V)X = 0$, then from equation (3.2), we have

$$R(U,V)X = g(U,X)V - g(V,X)U \quad (3.20)$$

which shows that M^n is constant curvature with respect to Levi-Civita connection.

4. Quasi M-Projectively Flat Para-Kenmotsu Manifolds with respect to Zamkovoy Connection

Definition 4.1. *A para-Kenmotsu manifold M^n is said to be Quasi-M-projectively flat with respect to Zamkovoy connection* [21] *if*

$$g(\bar{M}(\phi U,V)X,\phi Y) = 0, \quad (4.1)$$

where $\bar{M}$ is the M-projective curvature tensor with respect to Zamkovoy connection $\bar{\nabla}$.

Theorem 4.1. *A Quasi-M-projectively flat para-Kenmotsu manifold M^n with respect to Zamkovoy connection is an Einstein manifold.*

Proof. In the view of equation (1.3), we have

$$\begin{aligned} g(\bar{M}(U,V)X,Y) &= g(\bar{R}(U,V)X,Y) \\ &- \frac{1}{2(n-1)}[\bar{S}(V,X)g(U,Y) - \bar{S}(U,X)g(V,Y) \\ &+ g(U,X)\bar{S}(V,Y) - g(V,X)\bar{S}(U,Y)]. \end{aligned} \tag{4.2}$$

Replacing U by ϕU and Y by ϕY in equation (4.2), we get

$$\begin{aligned} g(\bar{M}(\phi U,V)X,\phi Y) &= g(\bar{R}(\phi U,V)X,\phi Y) \\ &- \frac{1}{2(n-1)}[\bar{S}(V,X)g(\phi U,\phi Y) - \bar{S}(\phi U,X)g(V,\phi Y) \\ &+ g(\phi U,X)\bar{S}(V,\phi Y) - g(V,X)\bar{S}(\phi U,\phi Y)]. \end{aligned} \tag{4.3}$$

Now, let us suppose that M^n is Quasi-M-projectively flat with respect to Zamkovoy connection. Then from equations (4.1) and (4.3), we have

$$\begin{aligned} `\bar{R}(\phi U,V,X,\phi Y) = &- \frac{1}{2(n-1)}[\bar{S}(V,X)g(\phi U,\phi Y) - \bar{S}(\phi U,X)g(V,\phi Y) \\ &+ g(\phi U,X)\bar{S}(V,\phi Y) - g(V,X)\bar{S}(\phi U,\phi Y)]. \end{aligned} \tag{4.4}$$

Using equations (3.2) and (3.5) in above equation, we get

$$\begin{aligned} `R(\phi U,V,X,\phi Y) &= g(\phi U,X)g(V,\phi Y) - g(V,X)g(\phi U,\phi Y) \\ &+ \frac{1}{2(n-1)}[S(V,X)g(\phi U,\phi Y) + (n-1)g(V,X)(\phi U,\phi Y) \\ &- S(\phi U,x)g(V,\phi Y) - (n-1)g(\phi U,X)g(V,\phi Y) \\ &+ g(\phi U,X)S(V,\phi Y) + (n-1)g(V,\phi Y)g(\phi U,X) \\ &- g(V,X)S(\phi U,\phi Y) - (n-1)g(V,X)g(\phi U,\phi Y)]. \end{aligned} \tag{4.5}$$

Let $\{e_1, e_2......e_{n-1}, \xi\}$ be a local orthonormal basis of vector field in M^n, then $\{\phi e_1, \phi e_2, ...\phi e_{n-1}, \xi\}$ is also local orthonormal basis in M^n. Putting $U = Y = e_i$

in equation (4.5) and taking summation over $1 \leq i \leq n-1$, we get

$$\begin{aligned}\sum_{i=1}^{n-1} {}'R(\phi e_i, V, X, \phi e_i) &= \sum_{i=1}^{n-1} g(\phi e_i, X) g(V, \phi e_i) - g(V, X) \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) \\ &+ \frac{1}{2(n-1)}[S(V, X) \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) + (n-1) g(V, X) \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) \\ &- \sum_{i=1}^{n-1} S(\phi e_i, X) g(V, \phi e_i) - (n-1) \sum_{i=1}^{n-1} g(\phi e_i, X) g(V, \phi e_i) \\ &+ \sum_{i=1}^{n-1} g(\phi e_i, X) S(V, \phi e_i) + (n-1) \sum_{i=1}^{n-1} g(V, \phi e_i) g(\phi e_i, X) \\ &- g(X, V) \sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) - (n-1) \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) g(V, X)].\end{aligned} \tag{4.6}$$

Also

$$\sum_{i=1}^{n-1} {}'R(\phi e_i, V, X, \phi e_i) = S(V, X) + g(V, X), \tag{4.7}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = (n-1), \tag{4.8}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = \sum_{i=1}^{n-1} S(e_i, e_i) = r, \tag{4.9}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, V) g(\phi e_i, X) = S(V, X), \tag{4.10}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, X) g(V, \phi e_i) = g(V, X). \tag{4.11}$$

Now using equations (4.7), (4.8), (4.9), (4.10) and (4.11) in equation (4.6), we get

$$S(V, X) = [\frac{-2n^2 + 4n - 2 - r)}{n-1}] g(V, X), \tag{4.12}$$

which shows that M^n is an Einstein manifold.

5. M-projectively Flat Para-Kenmotsu Manifolds admitting Zamkovoy Connection

In this section, we consider $\bar{M}(U,V)X = 0$, where $\bar{M}$ is M-projective curvature tensor with respect to Zamkovoy connection $\bar{\nabla}$.

An n-dimensional para-Kenmotsu manifold M^n is said to be M-projectively flat if the M-projective curvature vanishes identically on it.

Theorem 5.1. *A M-projectively flat para-Kenmotsu manifold M^n $(n > 2)$ admitting Zamkovoy connection $\bar{\nabla}$ is an Einstein manifold.*

Proof. Let para-Kenmotsu manifold M^n be M-projectively flat with respect to Zamkovoy connection *i.e.* $\bar{M} = 0$, then from equation (1.3), we have

$$\bar{R}(U,V)X = \frac{1}{2(n-1)}[\bar{S}(V,X)U - \bar{S}(U,X,)V + g(V,X)\bar{Q}U - g(U,X)\bar{Q}U]. \quad (5.1)$$

Taking inner product of above equation with the vector field Y, we get

$$`\bar{R}(U,V,X,Y) = \frac{1}{2(n-1)}[\bar{S}(V,X)g(U,Y) - \bar{S}(U,X,)g(V,Y) + g(V,X)\bar{S}(U,Y) - g(U,X)\bar{S}(U,Y)]. \quad (5.2)$$

Putting $U = Y = e_i$ and taking summation over i, $1 \le i \le n$ in above equation, we have

$$\bar{S}(V,X) = \frac{\bar{r}}{n}g(V,X). \quad (5.3)$$

Using equations (3.5) and (3.6) in equation (5.3), we get

$$S(V,X) = \frac{r}{n}g(V,X). \quad (5.4)$$

This shows that the manifold is an Einstein manifold.

Theorem 5.2. *A ξ- M-projectively flat para-Kenmotsu manifold M^n $(n > 2)$ admitting Zamkovoy connection is an Einsten manifold.*

Proof. Let M^n be ξ- M-projectively flat para-Kenmotsu maniflod with respect to Zamkovoy connection, *i.e.* $\bar{M}(X,Y)\xi = 0$, then from equation (1.3), we have

$$\bar{R}(U,V)\xi - \frac{1}{2(n-1)}[\bar{S}(V,\xi)U - \bar{S}(U,\xi)V - g(U,\xi)\bar{Q}V - g(V,\xi)\bar{Q}U] = 0. \quad (5.5)$$

Using equations (3.8) and (3.9) in equation (5.5), we get

$$\eta(U)\bar{Q}V - \eta(V)\bar{Q}U = 0. \quad (5.6)$$

Taking inner product of above equation with the vector field Y, we get

$$\eta(U)\bar{S}(V,Y) - \eta(V)\bar{S}(U,Y) = 0. \tag{5.7}$$

Putting $U = \xi$ and using equation (3.8) in equation (5.7), we get

$$\bar{S}(V,Y) = 0. \tag{5.8}$$

Using equation (3.5) in equation (5.8), we get

$$S(V,Y) = -(n-1)g(V,Y), \tag{5.9}$$

which shows that the manifold is an Einstein manifold.

Corollary 5.3. *If a para-Kenmotsu manifold M^n admitting Zamkovoy connection $\bar{\nabla}$ is ξ−M-projectively flat then its scalar curvature is constant.*
Proof. Putting $V = Y = e_i$ and taking summation over i, $1 \leq i \leq n$ in equation (5.9), we get

$$r = -n(n-1),$$

which shows that scalar curvature is constant.

Theorem 5.4. *An n-dimensional para-Kenmotsu manifold is ξ−M-projectively flat with respect to Zamkovoy connection iff it is ξ-M-projectively flat with respect to Levi-Civita connection, provided that the vector fields are horizontal vector fields.*
Proof. From equations (1.3), (3.2) and (3.7), we have

$$\begin{aligned}\bar{M}(U,V)X &= R(U,V)X - g(U,X)V + g(V,X)U \\ &\quad - \frac{1}{2(n-1)}[S(V,X)U - S(U,X)V + Q(V)g(U,X) - Q(U)g(V,X)].\end{aligned} \tag{5.10}$$

Using equation (1.2) in equation (5.10), we get

$$\bar{M}(U,V)X = M(U,V)X - g(U,X)V + g(V,X)U. \tag{5.11}$$

Putting $X = \xi$ in equation (5.11), we get

$$\bar{M}(X,Y)\xi = M(X,Y)\xi - \eta(U)V + \eta(V). \tag{5.12}$$

If U and V are horizontal vector fields then from equation (5.12), it follows that

$$\bar{M}(X,Y)\xi = M(X,Y)\xi. \tag{5.13}$$

6. ϕ-M-projectively flat para-Kenmotsu Manifolds admitting Zamkovoy Connection

In this section, we consider a para-Kenmotsu manifold equipped with Zamkovoy connection is $\phi-$M-projectively flat.

Definition 6.1. *A para-Kenmotsu manifold M^n admitting Zamkovoy connection is said to be ϕ-M protectively flat if* $\bar{M}\ (\phi U, \phi V, \phi X, \phi Y) = 0$.

Theorem 6.1. *A ϕ-M-projectively flat para-Kenmotsu manifold M^n equipped with Zamkovoy Connection $\bar{\nabla}$ then the equation*

$$S(V, X) = -(n-1)\eta(V)\eta(X)$$

is satisfied on M^n.

Proof. Suppose $\bar{M}(\phi X, \phi Y, \phi Z, \phi U) = 0$, then From equation (1.3), we get

$$\begin{aligned}\bar{R}(\phi U, \phi V, \phi X, \phi Y) = \frac{1}{2(n-1)}[&\bar{S}(\phi V, \phi X)g(\phi U, \phi Y) - \bar{S}(\phi U, \phi X)g(\phi V, \phi Y) \\ &+ g(\phi V, \phi X)\bar{S}(\phi U, \phi Y) - g(\phi U, \phi X)\bar{S}(\phi V, \phi Y)].\end{aligned} \tag{6.1}$$

From equation (3.4), we have

$$\begin{aligned}\bar{R}(\phi U, \phi V, \phi X, \phi Y) = {}& R(\phi U, \phi V, \phi X, \phi Y) \\ & - g(\phi U, \phi X,)g(\phi V, \phi Y) + g(\phi V, \phi X,)g(\phi U, \phi Y).\end{aligned} \tag{6.2}$$

Putting $U = Y = e_i$ and taking summation over i, $1 \leq i \leq n-1$ in equation (6.2), we get

$$\bar{S}(\phi V, \phi X) = S(\phi V, \phi X) + (n-1)g(\phi V, \phi X). \tag{6.3}$$

Using equations (6.2) and (6.3) in equation (6.1), we get

$$\begin{aligned}R(\phi U, \phi V, \phi X, \phi Y) = \frac{1}{2(n-1)}[&S(\phi V, \phi X,)g(\phi U, \phi Y) + (n-1)g(\phi V, \phi X)g(\phi U, \phi Y) \\ &- S(\phi U, \phi X)g(\phi V, \phi Y) - (n-1)g(\phi U, \phi X)g(\phi V, \phi Y) \\ &+ S(\phi U, \phi Y)g(\phi V, \phi X) + (n-1)g(\phi U, \phi Y)g(\phi V, \phi X) \\ &- S(\phi V, \phi Y)g(\phi U, \phi X) - (n-1)g(\phi V, , \phi Y)g(\phi U, \phi X)] \\ &+ g(\phi U, \phi X,)g(\phi V, \phi Y) - g(\phi V, \phi X,)g(\phi U, \phi Y).\end{aligned} \tag{6.4}$$

Let $\{e_i\}$ be a local orthonormal basis of the tangent space at any point of the manifold M^n, then $\{\phi e_i, \xi\}$ is also a local orthonormal basis. Putting $U = Y = e_i$ and taking summation over i, $1 \leq i \leq n-1$ in equation (6.4), we get

$$S(\phi V, \phi X) = 0. \tag{6.5}$$

In view of equations (2.15) and (6.5), we have

$$S(V,X) = -(n-1)\eta(V)\eta(X). \tag{6.6}$$

7. Para-Kenmotsu manifolds admitting Zamkovoy Connection satisfying $\bar{M}(\xi,U).\bar{R}=0$

In this section, we consider that $\bar{M}(\xi,U).\bar{R} = 0$ and obtained a relation on para-Kenmotsu manifold, where $\bar{M}$ is M-projective curvature tensor and $\bar{R}$ is Riemannian curvature tensor with respect to Zamkovoy connection $\bar{\nabla}$.

Theorem 7.1. *On an n-dimensional $(n>2)$ para-Kenmotsu manifold M^n admitting Zamkovoy connection if $\bar{M}(\xi,U).\bar{R}=0$ holds then the manifold is an Einstein manifold.*

Proof. Let us assume that a para-Kenmotsu manifold M^n admitting Zamkovoy connection satisfying the condition

$$(\bar{M}(\xi,U).\bar{R})(X,W)V = 0, \tag{7.1}$$

where $(\bar{M})$,$(\bar{R})$ are M-projective curvature tensor and Riemannian curvature tensor with respect to Zamkovoy connection, and $U,V,X,W \in \chi(M)$. Equation (7.1) gives

$$\begin{aligned}\bar{M}(\xi,U)\bar{R}(X,W)V - \bar{R}(\bar{M}(\xi,U)X,W)V - \bar{R}(X,\bar{M}(\xi,U)W)V - \\ \bar{R}(X,W)\bar{M}(\xi,U)V = 0.\end{aligned} \tag{7.2}$$

Putting $X=\xi$ in above eqution, we get

$$\begin{aligned}\bar{M}(\xi,U)\bar{R}(\xi,W)V - \bar{R}(\bar{M}(\xi,U)\xi,W)V - \bar{R}(\xi,\bar{M}(\xi,U)W)V - \\ \bar{R}(\xi,W)\bar{M}(\xi,U)V = 0.\end{aligned} \tag{7.3}$$

Using equation (3.9) in above equation, we get

$$\bar{R}(\bar{M}(\xi,U)\xi,W)V = 0. \tag{7.4}$$

In view of equation (1.3), we have

$$\bar{M}(\xi,U)\xi = \frac{-1}{2(n-1)}[\bar{Q}(U) + (n-1)\eta(U)\xi]. \tag{7.5}$$

Using equation (7.5) in equation (7.4), we get

$$\bar{R}(QU,W)V + (n-1)\bar{R}(U,W)V = 0. \tag{7.6}$$

In view of equation (3.2), we have

$$\begin{aligned} R(QU,W)V - g(QU,V)W + g(W,V)QU \\ + (n-1)[R(U,W)V - g(U,V)W + g(W,V)U] = 0. \end{aligned} \tag{7.7}$$

Taking inner product of above equation with vector field Y, we get

$$\begin{aligned} R(QU,W,V,Y) - S(U,V)g(W,Y) + g(W,V)S(U,Y) \\ + (n-1)[R(U,W,V,Y) - g(U,V)g(W,Y) + g(W,V)g(U,Y)] = 0. \end{aligned} \tag{7.8}$$

Putting $W = V = e_i$ and taking summation over i, $1 \leq i \leq n$ in above equation, we get

$$S(QU,Y) = (2n-2)S(U,Y) - (n-1)^2 g(U,Y). \tag{7.9}$$

Using equation (2.14) in above equation, we get

$$S(U,Y) = \frac{1}{3}(n-1)g(U,Y), \tag{7.10}$$

which shows that the manifold is an Einstein manifold.

8. Para-Kenmotsu manifolds admitting Zamkovoy Connection satisfying $\bar{M}(\xi,U).\bar{S} = 0$

In this section, we consider $\bar{M}(\xi,U).\bar{S} = 0$ and obtained that M^n is an Einstein manifold.

Theorem 8.1. *On an n-dimensional para-Kenmotsu manifold admitting Zamkovoy connection $\bar{\nabla}$, if the condition $\bar{M}(\xi,U).\bar{S} = 0$ holds, then the manifold is an Einstein manifold.*

Proof. Let us assume that a para-Kenmotsu manifold M^n admitting Zamkovoy connection satisfying the condition

$$(\bar{M}(\xi,U).\bar{S}(X,Y) = 0 \tag{8.1}$$

for all $U,\ X,\ Y \in \chi(M)$, then we have

$$\bar{S}(\bar{M}(\xi,U)X,Y) + \bar{S}(X,\bar{M}(\xi,U)Y) = 0. \tag{8.2}$$

In the view of equation (1.3), we have

$$\bar{M}(\xi,U).X = -\frac{1}{2(n-1)}[\bar{S}(U,X)\xi + \eta(X)\bar{Q}U - g(U,X)\bar{Q}\xi]. \tag{8.3}$$

Using equation (8.3) in equation (8.2), we get

$$\eta(X)\bar{S}(\bar{Q}U,Y)-g(U,X)\bar{S}(\bar{Q}\xi,Y)+\eta(Y)\bar{S}(\bar{Q}U,X)-g(U,Y)\bar{S}(\bar{Q}\xi,X)=0. \quad (8.4)$$

Putting $X=\xi$ and using equations (3.5) and (3.7) in equation (8.4), we get

$$S(QU,Y)=-(2n-2)S(U,Y)-(n-1)^2g(U,Y). \quad (8.5)$$

Using equation (2.15) in above equation, we get

$$S(U,Y)=-(n-1)g(U,Y), \quad (8.6)$$

which shows that the manifold is an Einstein manifold.

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