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DEGREE OF APPROXIMATION OF FUNCTION OF CLASS $Z^w(\alpha, \gamma)$ BY $(N, p, q)C_1$ MEAN OF FOURIER SERIES

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Abstract: In this paper we have established a result on the degree of approximation of function belonging to the generalized Zygmund class $z^w(\alpha, \gamma)$ by $(N, p, q)C_1$ means of Fourier series.

Keywords and Phrases: Degree of approximation, Generalized Zygmund class, (N, p, q) mean, $(C, 1)$ mean, $(N, p, q)C_1$ mean.

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1. Introduction

The degree of approximation of function belonging to different classes like $Lip\, \alpha$, $Lip(\alpha, p)$, $Lip(\xi(t), p)$, $W(L^p, \xi(t))$ have been studied by many researchers like Shyamlal [8], Dhakal [1] using different summability methods. The error estimation of function in Lipschitz and Zygmund class, using different means of Fourier series and conjugate Fourier series have been great interest among the researchers.The generalized Zygmund class was introduced by Leindler [3], Moricz [4], Moricz and Nemeth [5] etc. Recently Singh et al. [7], Mishra et al. [6], Kim [2] find results in Zygmund class by using different summability means. In this paper we find the degree of approximation of function in the generalized Zygmund class $z^w(\alpha, \gamma)$ by $(N, p, q)C_1$ mean of Fourier series.

2. Definition

Let f be a periodic function of period 2π integrable in the sense of Lebesgue over $[\pi, -\pi]$. Then the Fourier series of f given by

$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{2.1}$$

Zygmund class Z is defined as

$$Z = \{f \in C[-\pi, \pi] : |f(x+t) + f(x-t) - 2f(x)| = O(|t|)\}.$$

In this paper, we introduce a generalized Zygmund $Z^w(\alpha, \gamma)$ defined as

$$Z^w(\alpha,\gamma) = \left\{ f \in C[-\pi,\pi] : (\int_{-\pi}^{\pi} |f(x+t) + f(x-t) - 2f(x)|^{\gamma} dx)^{\frac{1}{\gamma}} = O(|t|^{\alpha}\omega(t)) \right\}, \tag{2.2}$$

where $\alpha \geq= 0$, $\gamma \geq 1$ and ω is continuous, non negative and non decreasing function. If we take $\alpha = 1$, $\omega =$constant and $\gamma \to \infty$, then $Z^w(\alpha, \gamma)$ class reduces to the Z class.
Let$\sum u_n$ be a infinite series such that whose n^{th} partial sum $s_n = \sum_{k=0}^{n} u_k$. Then $\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k$ is $(C, 1)$ mean of the sequence $\{s_n\}$.
If $\sigma_n \to s$ as $n \to \infty$ then the sequence $\{s_n\}$ is said to be summable by Ces$a$ro method $(C, 1)$.
The generalized Nörlund transform (N, p, q) of the sequence $\{s_n\}$ is the sequence

$$t_n^{p,q} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} s_{n-k}.$$

If $t_n^{p,q} \to s$ as $n \to \infty$ then the sequence $\{s_n\}$ is said to be summable by generalized Nörlund method (N, p, q) to s.
The (N, p, q) transform of the $(C, 1)$ transform defines the $(N, p, q)C_1$ transform of the partial sum $\{s_n\}$ of the series $\sum u_n$. Thus, if

$$t_n^{p,q,C_1} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \sigma_{n-k} \to s \quad \text{as} \quad n \to \infty.$$

then the sequence $\{s_n\}$ is said to be summable by $(N, p, q)C_1$ method to s.
We shall use the following notations

$$\emptyset_x(t) = f(x+t) - 2f(x) + f(x-t)$$

$$(NC)_n(t) = \frac{1}{2\pi R_n}\sum_{k=0}^{\infty}\frac{p_n q_{n-k}}{n-k+1}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}$$

3. Known Results

Jaeman Kim [2] find the degree of approximation of function of class $Z_{y,g^{\omega}}(\alpha,\gamma)$ by Ceasaro means of Fourier series and established following result :

Theorem 3.1. *Let f be a 2π periodic continuous function belonging to $Z_{y,g^{\omega}}(\alpha,\gamma)$. Then the degree of approximation of f by Cesaro mean of its Fourier series is given by*

$$\|\sigma_n - f\|_{\gamma} = O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right)$$

4. Main Result

In this paper we prove the following theorem.

Theorem 4.1. *Let f be a 2π periodic continuous function belonging to $Z^{\omega}(\alpha,\gamma)$, then the degree of approximation of f by $(N,p,q)C_1$ mean of its Fourier series is given by*

$$\left\|t_n^{p,q,C_1} - f\right\|_{\gamma} = O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right),$$

provided $\{p_n\}$ and $\{q_n\}$ are two sequence of positive real constant of regular Nörlund method (N,p,q) such that

$$\sum_{k=0}^{n}\frac{p_k q_{n-k}}{n-k+1} = O\left(\frac{R_n}{n+1}\right)\forall n \geq 0.$$

Proof. Following Titchmarch [9], n^{th} partial sum of s_n of Fourier series is given by

$$s_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\emptyset(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

The $(C,1)$ transform of s_n is given by

$$\frac{1}{n+1}\sum_{k=0}^{n}\{s_n(x) - f(x)\} = \frac{1}{2\pi(n+1)}\int_0^{\infty}\frac{\emptyset(t)}{\sin\frac{t}{2}}\sum_{k=1}^{n}\sin\left(n+\frac{1}{2}\right)t\ dt$$

$$\sigma_n(x) - f(x) = \frac{1}{2\pi(n+1)}\int_0^{\pi}\frac{\sin^2\left(n+\frac{1}{2}\right)t}{\sin^2\frac{t}{2}}\emptyset(t)\ dt$$

The $(N,p,q)C_1$ transform of s_n by t_n^{p,q,C_1} is

$$\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\{\sigma_{n-k}(x)-f(x)\} = \int_0^{\pi} \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k q_{n-k}}{n-k+1}\frac{\sin^2{(n-k+1)\frac{t}{2}}}{\sin^2{\frac{t}{2}}}\emptyset(t)\ dt$$

then,

$$t_n^{p,q,C_1}(x)-f(x) = \int_0^{\pi}(NC)_n(t)\emptyset(t)dt$$

where $\emptyset(t)=f(x+t)-2f(x)+f(x-t)$
By generalised Minkowski inequality for integral

$$\begin{aligned}
\left\|t_n^{p,q,C_1}-f\right\| &= \left(\int_{-\pi}^{\pi}\left|t_n^{p,q,C_1}(f,x)-f(x)\right|^{\gamma}dx\right)^{\frac{1}{\gamma}}\\
&= \left(\int_{-\pi}^{\pi}\left|\int_0^{\pi}(NC)_n(t)\emptyset(x,t)dt\right|^{\gamma}dx\right)^{\frac{1}{\gamma}}\\
&\leq= \int_0^{\pi}\left(\int_{-\pi}^{\pi}|\emptyset(x,t)dx|\right)^{\frac{1}{\gamma}}|(NC)_n(t)dt|\\
&= \int_0^{\pi}O(t^{\alpha}\omega(t))|(NC)_n(t)dt|\\
&= O\left(\int_0^{\frac{1}{n+1}}(t^{\alpha}\omega(t))|(NC)_n(t)|dt\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}(t^{\alpha}\omega(t))|(NC)_n(t)|dt\right)\\
&= I_1+I_2. \qquad (4.1)
\end{aligned}$$

For $0<t\leq\dfrac{1}{n+1}$

$$\begin{aligned}
(NC)_n(t) &= \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_n q_{n-k}}{n-k+1}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\\
&\leq \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_n q_{n-k}}{n-k+1}(n-k+1)^2\frac{\sin^2\frac{t}{2}}{\sin^2\frac{t}{2}}\\
&= \frac{1}{2\pi R_n}\sum_{k=0}^{n}p_n q_{n-k}n-k+1\\
&= \frac{1}{2\pi R_n}\sum_{k=0}^{n}p_n q_{n-k}
\end{aligned}$$

$$
\begin{aligned}
&= O(1) \\
&\leq O(n+1) \qquad (4.2)
\end{aligned}
$$

We have

$$I_1 = \int_0^{\frac{1}{n+1}} (t^{\alpha}\omega(t))|(NC)_n(t)|dt$$

then

$$I_1 = O\left(\int_0^{\frac{1}{n+1}} (t^{\alpha}\omega(t))(n+1)dt\right).$$

Since $t^{\alpha}\omega(t)$ is non – negative and increasing function, applying mean value theorem for integral, we have

$$
\begin{aligned}
I_1 &= O\left(\left(\frac{1}{n+1}\right)^{\alpha} \omega\left(\frac{1}{n+1}\right) \int_0^{\frac{1}{n+1}} (n+1)dt\right) \\
&= O\left(\left(\frac{1}{n+1}\right)^{\alpha} \omega\left(\frac{1}{n+1}\right)\right). \qquad (4.3) \\
&\leq \frac{1}{n+1} \sum_{k=1}^{n+1} \left(\frac{1}{k}\right)^{\alpha} \omega\left(\frac{1}{k}\right).
\end{aligned}
$$

So,

$$I_1 = O\left(\frac{1}{n+1} \sum_{k=1}^{n+1} \left(\frac{1}{k}\right)^{\alpha} \omega\left(\frac{1}{k}\right)\right). \qquad (4.4)$$

For $\dfrac{1}{n+1} < t \leq \pi$.

$$
\begin{aligned}
(NC)_n(t) &= \frac{1}{2\pi R_n} \sum_{k=0}^{n} \frac{p_n q_{n-k}}{n-k+1} \frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}} \\
&\leq \frac{1}{2\pi R_n} \sum_{k=0}^{n} \frac{p_n q_{n-k}}{n-k+1} \frac{\pi^2}{t^2} \\
&= \frac{\pi}{2t^2 R_n} \sum_{k=0}^{n} \frac{p_n q_{n-k}}{n-k+1} \\
&= \frac{\pi}{2t^2 R_n} O\left(\frac{R_n}{n+1}\right) \\
&= O\left(\frac{1}{(n+1)t^2}\right). \qquad (4.5)
\end{aligned}
$$

Then we have

$$\begin{aligned} I_2 &= \int_{\frac{1}{n+1}}^{\pi} (t^\alpha \omega(t))|(NC)_n(t)|dt \\ &= O\left(\int_{\frac{1}{n+1}}^{\pi} (t^\alpha \omega(t))\frac{1}{t^2(n+1)}dt\right) \\ &= O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} (t^{\alpha-2}\omega(t))dt\right) \end{aligned}$$

Putting $t = \frac{1}{u}$ then we get

$$I_2 = O\left(\frac{1}{(n+1)}\int_{\frac{1}{\pi}}^{n+1} \left(\frac{1}{u^\alpha}\omega\left(\frac{1}{u}\right)\right)du\right)$$

Since $t^\alpha\omega(t)$ is an increasing function, $\left(\frac{1}{u^\alpha}\omega\left(\frac{1}{u}\right)\right)$ is a decreasing function. Hence we have

$$O\left(\frac{1}{(n+1)}\int_{\frac{1}{\pi}}^{n+1} \left(\frac{1}{u^\alpha}\omega\left(\frac{1}{u}\right)\right)du\right) = O\left(\frac{1}{n+1}t^\alpha\omega(\pi) + \frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^\alpha \omega\left(\frac{1}{k}\right)\right)$$

From continuity of $t^\alpha\omega(t)$ we can choose a positive constant C such that $C(1^\alpha\omega(1)) \geq \pi^\alpha\omega(\pi)$ by non- negative property of $t^\alpha\omega(\pi)$ we have

$$\begin{aligned} &O\left(\frac{1}{n+1}t^\alpha\omega(\pi) + \frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^\alpha \omega\left(\frac{1}{k}\right)\right) \\ &= O\left(\frac{1}{n+1}C\left(\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^\alpha \omega\left(\frac{1}{k}\right)\right) + \frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^\alpha \omega\left(\frac{1}{k}\right)\right). \end{aligned}$$

Hence

$$I_2 = O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^\alpha \omega\left(\frac{1}{k}\right)\right) \tag{4.6}$$

From (4.1), (4.4) and (4.6) we obtain

$$\left\|t_n^{p,q,C_1} - f\right\|_\gamma = O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^\alpha \omega\left(\frac{1}{k}\right)\right).$$

This completes the proof.

5. Corollary

Following corollary can be derived from our main theorem.

Corollary 5.1. *Let f be a 2π periodic continuous function belonging to $Z^\omega(\alpha,\gamma)$, If $\omega(t) = t^m$ $(m > 1)$, then the degree of approximation of f by $(N,p,q)C_1$ mean of its Fourier series is given by*

$$\left\|t_n^{p,q,C_1} - f\right\|_\gamma = O\left(\frac{1}{n+1}\right).$$

Proof. We have

$$\left\|t_n^{p,q,C_1} - f\right\|_\gamma = O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^\alpha \omega\left(\frac{1}{k}\right)\right).$$

Putting $\omega(t) = t^m$, we get

$$= O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha+m}\right) = O\left(\frac{1}{n+1}\right).$$

This completes the proof.

Example 5.2. Consider the infinite series

$$1 + 4\sum_{n=1}^{\infty} n(-1)^n. \tag{5.1}$$

The partial sum of (5.2) is given by

$$s_n = 1 + 4\sum_{k=1}^{n} k(-1)^k$$
$$s_n = (2n+1)(-1)^n$$

And so

$$\sigma_n = \sum_{k=0}^{n}(2k+1)(-1)^k = (-1)^n$$

Therefore the series (5.1) is not $(C,1)$ summable. Since $\{(-1)^n\}$ is (N,p,q) summable, therefore the series (5.1) is $(N,p,q)C_1$ summable. Hence the product summability $(N,p,q)C_1$ is more powerful than the individual method (N,p,q) and $(C,1)$. Consequently $(N,p,q)C_1$ means gives better approximation than individual method (N,p,q) and $(C,1)$.

6. Conclusion

Numerous significant finding on the best approximation of function from distinct Lipschitz classes using different summability means have been reviewed. The outcome of this paper in more general than the existing outcomes. The outcome is quite helpful and extensible in future study by the researchers working in this direction.

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