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ON THE EXTENSION OF A CLASS OF BILATERAL GENERATING FUNCTION INVOLVING MODIFIED BESSEL POLYNOMIALS

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Abstract: In this note, we have obtained an extension of a general result on bilateral generating function of modified Bessel polynomials from the existence of a quasi-bilateral generating function.

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1. Introduction

In [1], Chatterjea and Chakraborty defined quasi-bilateral generating relation as follows :

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n p_n^{(\alpha)}(x) q_m^{(n)}(z),$$

where the coefficients a_n's are arbitrary and $p_n^{(\alpha)}(x)$, $q_m^{(n)}(z)$ are two special functions of orders n and m and of parameters α and n respectively.

In [3], A. K. Chongdar obtained the following theorem on bilateral generating functions for Modified Bessel polynomials by group-theoretic method, same theorem was also found derived in [2] while unifying a class of bilateral generating relation for certain special functions by classical method.

Theorem 1: *If*

$$G(x,w)=\sum_{n=0}^{\infty}a_nY_n^{(\alpha-n)}(x)w^n$$

then

$$exp(\beta w)(1-xw)^{-\alpha+1}G\left(\frac{x}{1-xw},vw\right)=\sum_{n=0}^{\infty}Y_n^{(\alpha-n)}(x)\sigma_n(v)w^n$$

where

$$\sigma_n(v)=\sum_{r=0}^{n}a_r\frac{\beta^{n-r}}{(n-r)!}v^r.$$

In this note, we shall extend the above result in the following form from the existence of a quasi-bilateral generating relation for the modified Bessel polynomials.

2. Group-theoretic Discussion

We first consider the quasi-bilateral generating function for the modified Bessel polynomials

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nY_n^{(\alpha-n)}(x)Y_m^{(n)}(u)w^n.$$

replacing w by $wytv$ on both sides we get

$$G(x,u,wytv)=\sum_{n=0}^{\infty}a_n\left(Y_n^{(\alpha-n)}(x)y^n\right)\left(Y_m^{(n)}(u)t^n\right)(wv)^n. \tag{2.1}$$

We now consider the following operators [2, 3]

$$R_1=x^2y\frac{\partial}{\partial x}+y\{\beta+(\alpha-1)x\}$$
$$R_2=ut\frac{\partial}{\partial u}+t^2\frac{\partial}{\partial t}+t(m-1)$$

such that

$$R_1\left(Y_n^{(\alpha-n)}(x)y^n\right)=\beta Y_{n+1}^{(\alpha-n-1)}(x)y^{n+1}$$
$$R_2\left(Y_m^{(n)}(u)t^n\right)=(m+n-1)Y_m^{(n+1)}(u)t^{n+1},$$

and

$$e^{wR_1}f(x,y)=exp(\beta yw)(1-xyw)^{-(\alpha-1)}f\left(\frac{x}{1-xyw},y\right)$$

$$e^{wR_2}f(u,t)=(1-wt)^{-m+1}f\left(\frac{u}{1-wt},\frac{t}{1-wt}\right).$$

Operating $e^{wR_1}e^{wR_2}$ on both sides of (2.1), we get

$$e^{wR_1}e^{wR_2}G(x,u,wytv)=e^{wR_1}e^{wR_2}\sum_{n=0}^{\infty}a_n\left(Y_n^{(\alpha-n)}(x)y^n\right)\left(Y_m^{(n)}(u)t^n\right)(wv)^n. \tag{2.2}$$

Left member of (2.2) is

$$e^{wR_1}e^{wR_2}G(x,u,wvty)$$
$$=exp(\beta yw)(1-xyw)^{-(\alpha-1)}(1-wt)^{-m+1}G\left(\frac{x}{1-xyw},\frac{u}{1-wt},\frac{wytv}{1-wt}\right). \tag{2.3}$$

The right member of (2.2) is

$$\begin{aligned}
&e^{wR_1}e^{wR_2}\sum_{n=0}^{\infty}a_n\left(Y_n^{(\alpha-n)}(x)y^n\right)\left(Y_m^{(n)}(u)t^n\right)(wv)^n\\
&=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}a_n\frac{w^{r+k}}{r!k!}R_1^r\left(Y_n^{(\alpha-n)}(x)y^n\right)\\
&\times R_2^k\left(Y_m^{(n)}(u)t^n\right)(vw)^n\\
&=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}a_n\frac{w^{n+r+k}}{r!k!}v^n.\beta^rY_{n+r}^{(\alpha-n-r)}(x)y^{n+r}\\
&\times(m+n-1)_kY_m^{(n+k)}(u)t^{n+k}\\
&=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}a_n\beta^r\frac{w^{n+r+k}}{r!k!}(m+n-1)_kv^nY_{n+r}^{(\alpha-n-r)}(x)y^{n+r}\\
&\times Y_m^{(n+k)}(u)t^{n+k}.
\end{aligned} \tag{2.4}$$

Equating (2.3) and (2.4) and then putting $y=t=1$, we get

$$exp(\beta w)(1-xw)^{-(\alpha-1)}(1-w)^{-m+1}G\left(\frac{x}{1-xw},\frac{u}{1-w},\frac{wv}{1-w}\right)$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}a_n\beta^r\frac{w^{n+r+k}}{r!k!}(m+n-1)_kv^nY_{n+r}^{(\alpha-n-r)}(x)Y_m^{(n+k)}(u).$$

which is our derived result.

Corollary.
Putting $m = 0$, we get

$$exp(\beta w)(1-xw)^{-(\alpha-1)}(1-w)G\left(\frac{x}{1-xw}, \frac{wv}{1-w}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} a_n \beta^r \frac{w^{n+r}}{r!} v^n Y_{n+r}^{(\alpha-n-r)}(x) \sum_{k=0}^{\infty} \frac{(n-1)_k}{k!} w^k$$

$$= (1-w)\sum_{n=0}^{\infty}\sum_{r=0}^{n} a_{n-r}\beta^r \frac{w^n}{r!} v^{n-r} Y_n^{(\alpha-n)}(x)\left(\frac{1}{1-w}\right)^{n-r}$$

$$= (1-w)\sum_{n=0}^{\infty} Y_n^{(\alpha-n)}(x)\left(\sum_{r=0}^{n} a_{n-r}\frac{\beta^r}{(r)!}\left(\frac{v}{1-w}\right)^{n-r}\right)w^n$$

$$= (1-w)\sum_{n=0}^{\infty} Y_n^{(\alpha-n)}(x)\left(\sum_{r=0}^{n} a_r\frac{\beta^{n-r}}{(n-r)!}\left(\frac{v}{1-w}\right)^{r}\right)w^n$$

Therefore,

$$exp(\beta w)(1-xw)^{-(\alpha-1)}G\left(\frac{x}{1-xw}, \frac{wv}{1-w}\right) = \sum_{n=0}^{\infty} Y_n^{(\alpha-n)}(x)\left(\sum_{r=0}^{n} a_r\frac{\beta^{n-r}}{(n-r)!}\left(\frac{v}{1-w}\right)^{r}\right)w^n$$

Replacing $\frac{v}{1-w}$ by v on both sides

$$exp(\beta w)(1-xw)^{-\alpha+1}G\left(\frac{x}{1-xw}, vw\right) = \sum_{n=0}^{\infty} Y_n^{(\alpha-n)}(x)\sigma_n(v)w^n$$

where

$$\sigma_n(v) = \sum_{r=0}^{n} a_r \frac{\beta^{n-r}}{(n-r)!} v^r.$$

which is theorem 1.

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