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DECOMPOSITION OF CONTINUITY IN TERMS OF BOTH GENERALIZED TOPOLOGY AND TOPOLOGY

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Abstract: Here decomposition of continuity like notion is explored in terms of generalized topology as well as topology on a set. This concept is used as a new tool to study different characterizations of a given generalized topological space, giving a new dimension in the study of topological spaces. Firstly, more properties of μ^*-open(closed), μ'-open(closed) sets, μ'-continuous and μ^*-continuous functions are studied. Also, a new family of sets μ^*_α-open(closed) and μ'_β-open(closed) sets are introduced. In terms of these sets, the notion of μ^*_α-continuous and μ'_β-continuous are defined. Interrelations, characterizations of these sets and functions are explored.

Keywords and Phrases: μ'-continuous, μ^*-continuous functions, μ^*_α-open, μ'_β-open set, μ^*_α-continuous, μ'_β-continuous functions.

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1. Introduction and Preliminaries

The notion of generalized topology (in short, GT) was initiated in 2002 by Á. Császár [1]. In 2005, Á. Császár introduced semiopen, preopen, α-open, β-open sets in generalized topological space(GTS) in terms of closure and interior taken with respect to GT [2]. Again in 2015, B. Roy and R. Sen used both topology and GT on a nonempty set U to define a new class of sets in terms of their closure and interior taken with respect to topology and GT in different combinations and were termed as μ^* and $\mu^{'}$-open and closed sets. Also, a notion of $\mu^{'}$-continuous function was defined and their basic properties were studied by them [5]. Again in 2020, R.K.Tiwari, J. K. Maitra and R. Vishwakarma explored more aspects of μ^*, $\mu^{'}$-open sets and $\mu^{'}$-continuous functions. Also, the idea of μ^*-continuous function was initiated and studied [6].

In this paper we study more properties of μ^*-open(closed) sets, $\mu^{'}$-open(closed) sets, $\mu^{'}$-continuous and μ^*-continuous functions. Also, a new class of sets μ^*_α-open(closed) and $\mu^{'}_\beta$-open(closed) sets are introduced. In terms of these sets, the notion of μ^*_α-continuous and $\mu^{'}_\beta$-continuous are defined. Interrelations, characterizations of these sets, and functions are explored.

In a nonempty set U let μ be a subset of the power set of U with $\phi \in \mu$ and arbitrary union of elements of μ also is in μ, then μ is called a generalized topology (GT in short) on U and (U, μ), the generalized topological space (GTS in short) [1]. The generalized closure and interior of a set B on U, denoted as $c_\mu(B)$ and $i_\mu(B)$ respectively are $c_\mu(B) = \cap\{P \subseteq U : B \subseteq P, U - P \in \mu\}$ and $i_\mu(B) = \cup\{Q \subseteq U : Q \subseteq B, Q \in \mu\}$ [1], [2]. It can be seen easily that c_μ and i_μ are monotonic and idempotent where $g : \exp U \to \exp U$ is called monotonic if $P \subseteq Q \subseteq U$ then $g(P) \subseteq g(Q)$ and idempotent if $P \subseteq U$ then $g(g(P)) = g(P)$. In a GTS (U, μ), for $P \subseteq U$ we have, $c_\mu(U - P) = U - i_\mu(P)$ and $i_\mu(U - P) = U - c_\mu(P)$ [1].

Throughout this paper by a space (U, μ, σ) we mean a GT μ with also a topology σ on U. Also, for a subset $P \subseteq U$, $i(P)$ and $cl(P)$ denotes usual interior and closure of P with respect to σ. In a topological space, a set P is semiopen [3] (resp. semiclosed [3], β-open [4], β-closed [4], α-open [4], α-closed [4]) if $P \subseteq cl(i(P))$(resp. $i(cl(P)) \subseteq P$, $P \subseteq cl(i(cl(P)))$, $i(cl(i(P))) \subseteq P$, $P \subseteq i(cl(i(P)))$, $cl(i(cl(P))) \subseteq P$). In a space (U, μ, σ) where μ and σ are GT and topology respectively on U, a set P is termed as μ^*-open($\mu^{'}$-open) if $P \subseteq cl(i_\mu(P))$(resp. $P \subseteq i(c_\mu(P))$) and μ^*-closed($\mu^{'}$-closed) if $i(c_\mu(P)) \subseteq P$(resp. $cl(i_\mu(P)) \subseteq P$). In a space (U, μ, σ), $\mu^{'}$-open and $\mu^{'}$-closed sets are complements of each other. Also, μ^*-open and μ^*-closed sets are complements of each other [5]. In a space (U, μ, σ), every μ-open(closed) is μ^*-open(closed) set [6].

2. More on μ^*-open(closed) and $\mu^{'}$-open (closed) sets

We shall begin this section with an example to show that μ^*-open(closed) and $\mu^{'}$-open(closed) sets do not imply each other.

Example 2.1. Let us consider a space (U, μ, σ), with $U = \{q, m, n, t, e\}$, $\sigma = \{\phi, \{m, n, t\}, U\}$ and $\mu = \{\phi, \{q\}, \{e\}, \{q, e\}\}$. Suppose $P = \{q\}$. Then, $i_\mu(P)) = P$, $cl(i_\mu(P)) = \{q, e\}$. Thus, $P \subseteq cl(i_\mu(P))$. So, P is μ^*-open. Also, $c_\mu(P) =$

$\{q, m, n, t\}$ and $i(c_\mu(P)) = \{m, n, t\}$. So, P is not μ'-open.
Let $Q = \{m, n, t\}$. Then $i(c_\mu(Q)) = Q$, showing Q is μ'-open but it fails to be μ^*-open as $cl(i_\mu(Q)) = \phi$. Further, taking $A = \{q, e\}$ we see that A is μ'-closed as $cl(i_\mu(A)) = \{q, e\}$ but A fails to be μ^*-closed as $i(c_\mu(A)) = U$. Also, taking $B = \{q, m, n, t\}$, $cl(i_\mu(B)) = \{q, e\}$ and $i(c_\mu(B)) = \{m, n, t\}$. So, B is μ^*-closed but fails to be μ'-closed.

Theorem 2.1. *In a space (U, μ, σ), if $\sigma \subseteq \mu$, then for $S \subseteq U$ we have $i(S) \subseteq i_\mu(S)$ and $c_\mu(S) \subseteq cl(S)$.*
Proof. Straightforward.

Theorem 2.2. *In a space (U, μ, σ), if $\sigma \subseteq \mu$, then every semiopen(closed) set is μ^*-open(closed).*
Proof. For any semiopen set P, $P \subseteq cl(i(P))$. By Theorem (2.1), $cl(i(P)) \subseteq cl(i_\mu(P))$ and $P \subseteq cl(i_\mu(P))$, proving P is μ^*-open. Similarly, here every semiclosed set is μ^*-closed.

Theorem 2.3. *Any μ'-open set S in a space (U, μ, σ) is μ^*-open if $i(c_\mu(S)) = i_\mu(S)$.*
Proof. Straightforward.
However the converse doesn't hold .i.e. if μ'-open set is μ^*-open then $i(c_\mu(S)) = i_\mu(S)$ may not hold is shown by an example given below.

Example 2.2. Let us consider a space (U, μ, σ) where $U = \{q, r, s, t\}$, $\sigma = \{\phi, U, \{r, t\}, \{q, s\}\}$ and $\mu = \{\phi, \{s\}, \{r, t\}, \{r, s, t\}\}$.
Let $S = \{q, s\}$. Then, S is both μ'-open and μ^*-open but $i(c_\mu(S)) \neq i_\mu(S)$.

Theorem 2.4. *A μ-open set in a space (U, μ, σ) is μ'-closed iff it is closed.*
Proof. For any μ-open and μ'-closed set S, $cl(i_\mu(S)) \subseteq S$. Also, $S = i_\mu(S)$. Hence, $cl(S) \subseteq S$, proving S is closed.
Conversely, let S be closed. $i_\mu(S) \subseteq S$. So, $cl(i_\mu(S)) \subseteq cl(S) = S$. Thus, S is μ'-closed.

Theorem 2.5. *A μ-closed set in a space (U, μ, σ) is μ'-open iff it is open.*
Proof. Similarly as above.

Theorem 2.6. *A μ^*-open set S in a space (U, μ, σ) is μ'-closed iff $S = cl(i_\mu(S))$.*
Proof. Straightforward.

Theorem 2.7. *A μ'-open set S in a space (U, μ, σ) is μ^*-closed iff $S = i(c_\mu(S))$.*
Proof. Straightforward.

Remark 2.1. *The intersection of any two μ'-open sets in a space (U, μ, σ) need* *not be μ'-open is shown by an example below:*

Example 2.3. Let us consider a space (U, μ, σ) with $U = \{q, n, s, t, u\}$, $\sigma = \{\phi, U, \{q, n, s\}, \{s, t, u\}, \{s\}\}$ and $\mu = \{\phi, \{q, s, t, u\}, \{t, u\}\}$. Let $S = \{q, n\}$ and $N = \{n, s\}$. Then, $i(c_\mu(S)) = \{q, n, s\}$, $i(c_\mu(N)) = \{q, n, s\}$. Hence, S and N are μ'-open. On the other hand, $S \cap N = \{n\}$, $i(c_\mu(\{n\})) = \phi$. So, $\{n\} \nsubseteq i(c_\mu(\{n\}))$, showing $\{n\}$ is not μ'-open.

Theorem 2.8. *The intersection of any two μ'-closed sets in a space (U, μ, σ) is μ'-closed.*
Proof. For μ'-closed sets S and R, $cl(i_\mu(S \cap R)) \subseteq cl(i_\mu(S)) \cap cl(i_\mu(R)) \subseteq S \cap R$, showing $S \cap R$ is μ'-closed.

Theorem 2.9. *Let σ be discrete topology or indiscrete topology on the underlying set U and μ be generalized topology. Let $S \neq \phi$ and $M \neq \phi$ be μ-open sets then $S \cap M$ is μ'-closed set.*
Proof. Case i. Let σ be discrete topology. So, $i_\mu(S \cap M) = S \cap M$ and $cl(i_\mu(S \cap M)) = cl(S \cap M) \subseteq cl(S) \cap cl(M) = S \cap M$. So, $cl(i_\mu(S \cap M)) = cl(i_\mu(S) \cap i_\mu(M)) = cl(S \cap M) \subseteq cl(S) \cap cl(M) = S \cap M$. Hence, $S \cap M$ is μ'-closed.
Case ii. Let σ be indiscrete topology on U. So, $cl(i_\mu(S \cap M)) = U$, showing $S \cap M$ is μ'-closed.

Remark 2.2. *The intersection of any μ^*-open and μ'-open set is not either of them is shown by an example below:*

Example 2.4. Consider a space (U, μ, σ) where $U = \{a, m, n, i\}$, $\sigma = \{\phi, U, \{i\}\}$ and $\mu = \{\phi, \{a, m, n\}, \{m, n, i\}, U\}$. Every μ-open being μ^*-open, we have $\{a, m, n\}$ is μ^*-open. Now $i(c_\mu(\{m, n, i\})) = U$, $\{m, n, i\}$ is μ'-open. Their intersection $\{m, n\}$ is none of μ^*-open and μ'-open.

3. Further on μ^*-continuous and μ'-continuous Functions

Definition 3.1. *A mapping g between a space (U, μ, σ) and a topological space (V, ρ) is called μ-continuous* [5] *(resp. μ^*-continuous* [6], *μ'-continuous* [5]*) if $g^{-1}(B)$ is μ-open (resp. μ^*-open, μ'-open) $\forall$ $B \in \rho$,.*

Theorem 3.1. *Every continuous function between a space (U, μ, σ) and a topological space (V, ρ) is μ^*-continuous if $\sigma \subseteq \mu$ but not conversely* [6].

Now, let us establish the condition under which the converse of this theorem also holds. For this lets consider an example:

Example 3.1. Let us consider a space (U, μ, σ) and topological space (V, ρ),

where $U = \{m, q, r, n\}$, $\sigma = \{U, \phi, \{m, q\}\}$, $\mu = \{U, \phi, \{m, q\}, \{r, n\}\}$ on U. Let $V = \{a, i, c, d\}$ and $\rho = \{V, \phi, \{a, i\}\}$. Here, $\sigma \subseteq \mu$. Let $g : U \to V$ given as $g(m) = a, g(q) = i, g(r) = c, g(n) = d$. $g^{-1}\{a, i\} = \{m, q\}$. Now, $cl(i_\mu(\{m, q\})) = U$. Hence, g is μ^*-continuous. Also $g^{-1}\{a, i\} = \{m, q\}$ which is open.
Thus, g is also continuous. Here, converse part of the above theorem is also true.

This example motivates us to find in general the condition under which the converse is also true.

Remark 3.1. *The converse of Theorem (3.1) holds if* $cl(g^{-1}(A)) = i(g^{-1}(A))$ *for any* $A \subseteq V$.
Proof. Let $g : U \to V$ be μ^*-continuous. For $Q \in \rho$, $g^{-1}(Q)$ is μ^* open. So, $g^{-1}(Q) \subseteq cl(i_\mu(g^{-1}(Q)) \subseteq cl((g^{-1}(Q)) = i(g^{-1}(Q))$. Thus, $g^{-1}(Q) \in \sigma$, showing g is continuous.

Theorem 3.2. *For a function g between two spaces (U, μ, σ) and (V, ρ) the following are equivalent:*
(1) *g is μ^*-continuous.*
(2) *For any closed set S in V, $g^{-1}(S)$ is μ^*-closed.*
(3) *$i(c_\mu(g^{-1}(F))) \subseteq g^{-1}(cl(F))$ for any subset F of V.*
(4) *$g(i(c_\mu(N))) \subseteq cl(g(N))$ for any subset N of U.*
Proof. (1)$\Longleftrightarrow$(2) Refer [6].
(2) $\Longrightarrow$ (3) For $F \subseteq V$, $g^{-1}(cl(F))$ is μ^*-closed set in U. So, $i(c_\mu(g^{-1}(F))) \subseteq i(c_\mu(g^{-1}(cl(F)))) \subseteq g^{-1}(cl(F))$. Hence, $i(c_\mu(g^{-1}(F))) \subseteq g^{-1}(cl(F))$.
(3) $\Longrightarrow$ (4) Let $N \subseteq U$ then $g(N) \subseteq V$ and $i(c_\mu(g^{-1}(g(N)))) \subseteq g^{-1}(cl(g(N)))$. Then, $i(c_\mu(N)) \subseteq g^{-1}(cl(g(N)))$. So, $g(i(c_\mu(N))) \subseteq cl(g(N)$.
(4) $\Longrightarrow$ (2) For any closed set S in V, $g^{-1}(S) \subseteq U$ and $g(i(c_\mu(g^{-1}(S)))) \subseteq cl(g(g^{-1}(S))) \subseteq cl(S) = S$. Hence, $i(c_\mu(g^{-1}(S))) \subseteq g^{-1}(S)$, showing $g^{-1}(S)$ is μ^*-closed.

Theorem 3.3. *For an injective function $g : (U, \mu, \sigma) \longrightarrow (V, \rho)$ the following conditions are equivalent.*
(1) *g is $\mu^{'}$-continuous.*
(2) *For any $p \in U$ and $B \in \rho$ with $g(p) \in B$, $\exists$ $\mu^{'}$-open set A that contains p satisfying $g(A) \subseteq B$.*
(3) *$g^{-1}(B)$ is $\mu^{'}$-closed in U for any closed set B in V.*
(4) *$cl(i_\mu(g^{-1}(S))) \subseteq g^{-1}(cl(S))$ for any subset S of V.*
(5) *$g(cl(i_\mu(E))) \subseteq cl(g(E))$ for any subset E of U.*
Proof. (1) $\Longrightarrow$ (2) Let $p \in U$ and $B \in \rho$ with $g(p) \in B$. g being $\mu^{'}$-continuous, $g^{-1}(B)$ is $\mu^{'}$-open which contains p. Taking $g^{-1}(B) = A$, $g(A) \subseteq B$.
(2) $\Longrightarrow$ (3) For a closed set B be on V, $G = V - B \in \rho$. Let $p \in g^{-1}(G)$, then $\exists$

a μ'-open set A of U with $p \in A$ and $g(A) \subseteq G$. Now, $p \in A \subseteq i(c_\mu(A))$ and f being injective, $A = g^{-1}(G) \subseteq i(c_\mu(g^{-1}(G)))$. So $g^{-1}(G)$ is μ'-open set and thus $g^{-1}(B) = U - g^{-1}(V - B) = U - g^{-1}(G)$ is μ'-closed in U.
(3) $\implies$ (4) Let $S \subseteq V$. Then $cl(S)$ is a closed set in V and $g^{-1}(cl(S))$ is μ'-closed set in U. So, $cl(i_\mu(g^{-1}(cl(S)))) \subseteq g^{-1}(cl(S))$. Hence, $cl(i_\mu(g^{-1}(S))) \subseteq g^{-1}(cl(S))$.
(4) $\implies$ (5) For $E \subseteq U$, $g(E) \subseteq V$ and $cl(i_\mu(g^{-1}(g(E)))) \subseteq g^{-1}(cl(g(E)))$. So, $cl(i_\mu(E)) \subseteq g^{-1}(cl(g(E)))$. Hence, $g(cl(i_\mu(E))) \subseteq cl(g(E))$.
(5) $\implies$ (3) For a closed set B in V, $g^{-1}(B) \subseteq U$ and $g(cl(i_\mu(g^{-1}(B)))) \subseteq cl(g(g^{-1}(B))) \subseteq cl(B) = B$. So, $cl(i_\mu(g^{-1}(B))) \subseteq g^{-1}(B)$, proving $g^{-1}(B)$ is μ'-closed.
(3) $\implies$ (1) Let $B \in \rho$ then $V - B = F$ and so $g^{-1}(F)$ is μ'-closed in V. As $g^{-1}(B) = U - g^{-1}(V - B) = U - g^{-1}(F)$, $g^{-1}(B)$ is μ'-open in U.

4. μ^*_α-open (closed) and μ'_β-open (closed) sets

Definition 4.1. *In a space (U, μ, σ), $S \subseteq U$ is called*
(i)μ^*_α-open if $S \subseteq i_\mu(cl(i_\mu(S)))$ and μ^*_α-closed if $c_\mu(i(c_\mu(S))) \subseteq S$.
(ii)μ'_β-open if $S \subseteq c_\mu(i(c_\mu(S)))$ and μ'_β-closed if $i_\mu(cl(i_\mu(S))) \subseteq S$.

Theorem 4.1. *μ^*_α-open and μ^*_α-closed sets are complements of each other. Also, μ'_β-open and μ'_β-closed sets are complements of each other.*
Proof. Straightforward.

Remark 4.1. *We have the following relation among the sets:*
μ-open(closed) $\implies$ μ^*_α-open (closed) $\implies$ μ^*-open(closed) and
open(closed) $\implies$ μ'-open(closed) $\implies$ μ'_β-open(closed)
However the converses do not hold is shown below.

Example 4.1. Consider a space (U, μ, σ) with $U = \{e, w, r, x\}$, $\mu = \{\phi, \{x\}\}$ and $\sigma = \{U, \phi, \{w, r\}\}$. Taking $A = \{e, x\}$, $cl(i_\mu(A)) = A$, proving A is μ^*-open. Also, $i_\mu(cl(i_\mu(A))) = \{x\}$. So, $A \nsubseteq i_\mu(cl(i_\mu(A)))$. Hence, μ^*-open$\not\Rightarrow$ μ^*_α-open. Also, A is not μ-open. Thus, μ^*-open $\not\Rightarrow$ μ-open.
Similarly, $\{w, r\}$ is μ^*-closed but fails to be μ^*_α-closed. Hence, μ^*-closed $\not\Rightarrow$ μ^*_α-closed. Also,$\{w, r\}$ is not μ-closed. Thus, μ^*-closed $\not\Rightarrow$ μ-closed.

Example 4.2. Consider a space (U, μ, σ) with $U = \{e, w, r, x\}$, $\mu = \{\phi, \{e\}, \{e, r, x\}\}$ and $\sigma = \{U, \phi, \{w\}\}$. Let $A = \{e, x\}$. Then, $i_\mu(cl(i_\mu(A))) = \{e, r, x\}$ and $A \subseteq i_\mu(cl(i_\mu(A)))$. So, A is μ^*_α-open but is not μ-open. Hence, μ^*_α-open$\not\Rightarrow$ μ-open. Further, taking $B = \{w, r\}$, $c_\mu(i(c_\mu(B))) = \{w\} \subseteq B$. So, B is μ^*_α-closed but it fails to be μ-closed. Thus, μ^*_α-closed $\not\Rightarrow$ μ-closed.

Example 4.3. Let (U, μ, σ) be a space with $U = \{e, w, r, x, y\}$, $\mu = \{\phi, \{e, w\}, \{r, x\}, \{e, w, r, x\}\}$ and $\sigma = \{\phi, \{e, w\}, U\}$. Taking $A = \{e, x\}$, $i(c_\mu(A)) = U$. So, A is μ'-open but not open. Hence, μ'-open$\not\Rightarrow$ open. Taking $B = \{w, r\}$, $cl(i_\mu(B)) = \phi$. Hence B is μ'-closed but is not closed. So, μ'-closed$\not\Rightarrow$ closed.

Example 4.4. Consider a space (U, μ, σ) where $U = \{e, w, r, x\}$, $\mu = \{\phi, \{r, x\}\}$ and $\sigma = \{\phi, U, \{e\}\}$. Let $A = \{e, w\}$, then $c_\mu(i(c_\mu(A))) = A$ but $i(c_\mu(A)) = \{e\}$. Therefore, A is μ'_β-open but fails to be μ'-open. Hence, μ'_β-open$\not\Rightarrow$ μ'-open. Also, A is not open, μ'_β-open$\not\Rightarrow$open. Now, taking $B = \{r, x\}$, $i_\mu(cl(i_\mu(B))) = B$ and $cl(i_\mu(B)) = \{w, r, x\} \nsubseteq B$. Hence, B is μ'_β-closed but not μ'-closed. So, μ'_β-closed$\not\Rightarrow$ μ'-closed. Also, B is not closed, μ'_β-closed $\not\Rightarrow$ closed.

Theorem 4.2. *In a space (U, μ, σ), for any $A \in U$, $i_\mu(A)$ is μ^*_α-open iff there exists μ-open set B with $B \subseteq i_\mu(A) \subseteq cl(B)$.*

Proof. Let $i_\mu(A)$ be μ^*_α-open. Now, $i_\mu(A) \subseteq i_\mu(cl(i_\mu(i_\mu(A)))) \subseteq cl(i_\mu(i_\mu(A))) = cl(i_\mu(A))$. Hence, $i_\mu(A) \subseteq cl(i_\mu(A))$. Taking $B = i_\mu(A)$, $B \subseteq i_\mu(A) \subseteq cl(B)$. Conversely, let $\exists$ μ-open set B with $B \subseteq i_\mu(A) \subseteq cl(B)$. Now $B \subseteq i_\mu(A)$, $i_\mu(B) = B \subseteq i_\mu(i_\mu(A))$. This gives, $cl(B) \subseteq cl(i_\mu(i_\mu(A)))$. As, $i_\mu(A) \subseteq cl(B)$, $i_\mu(A) \subseteq cl(i_\mu(i_\mu(A)))$. Now, $i_\mu(i_\mu(A)) \subseteq i_\mu(cl(i_\mu(i_\mu(A))))$ then, $i_\mu(A) \subseteq i_\mu(cl(i_\mu(i_\mu(A))))$. So, $i_\mu(A)$ *is* μ^*_α-open.

Remark 4.2. *In a space (U, μ, σ), we have*
(1) *μ^*_α-open $\not\Leftrightarrow$ μ'- open.*
(2) *μ'_β-open $\not\Leftrightarrow$ μ^*-open. Also, μ'_β-open $\not\Leftrightarrow$ μ-open.*
(3) *μ'_β-open $\not\Leftrightarrow$ μ^*_α-open*

Proof. (1) Consider a space (U, μ, σ) where $U = \{e, w, r, x\}$, $\mu = \{\phi, \{x\}, \{w, r\}, \{w, r, x\}\}$ and $\sigma = \{\phi, U, \{e, w\}\}$. Let $A = \{e, w\}$. Then $i_\mu(cl(i_\mu(A))) = \phi$, $A \nsubseteq i_\mu(cl(i_\mu(A)))$. Therefore, A is not μ^*_α-open but is μ'-open since it is open. If $B = \{x\}$, then $i_\mu(cl(i_\mu(B))) = B$. Hence, $B \subseteq i_\mu(cl(i_\mu(B))$. Therefore, B is μ^*_α-open. Now, $i(c_\mu(B)) = \phi$. So, $B \nsubseteq i(c_\mu(B))$, which shows B is not μ'-open. So, μ^*_α-open $\not\Leftrightarrow$ μ'-open.

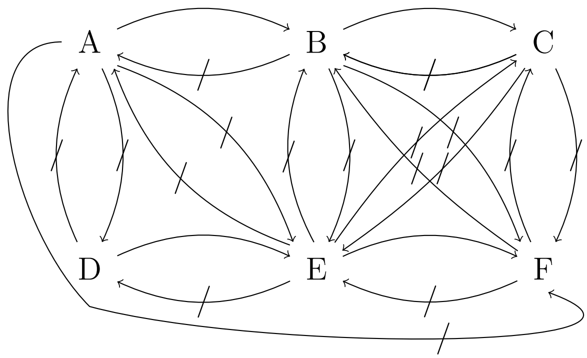
(2) Consider a space (U, μ, σ) with $U = \{e, w, r, x\}$, $\mu = \{\phi, \{e, x\}, \{w\}, \{e, w, x\}\}$ and $\sigma = \{\phi, U, \{r\}\}$. Let $S = \{e, x\}$. Then $c_\mu(i(c_\mu(S)) = \{r\}$. Hence, S is not μ'_β-open but is μ^*-open as it is μ-open.

Now, if $B = \{r\}$, then $cl(i_\mu(B)) = \phi$. So, $B \nsubseteq cl(i_\mu(B))$ i.e. B is not μ^*-open whereas, B is μ'_β-open as it is open. Thus, μ'_β-open $\not\Leftrightarrow$ μ^*-open. Also, B is μ'_β-open and not μ-open and S is μ-open and fail to be μ'_β-open. Thus, μ'_β-open $\not\Leftrightarrow$ μ-open.

(3) Consider a space (U, μ, σ) where $U = \{e, w, r, x\}$, $\mu = \{\phi, \{r, x\}, \{e, r\}, \{e, r, x\},$

$\{x\}\}$ and $\sigma = \{\phi, U, \{e, w\}\}$. Let $A = \{e, w\}$, then $i_\mu(cl(i_\mu(A))) = \phi$. So, A fails to be μ^*_α-open but is $\mu^{'}_\beta$-open since it is open.
Consider $B = \{x\}$. Then, $c_\mu(i(c_\mu(B)) = \phi$ and $B \nsubseteq c_\mu(i(c_\mu(B)))$. Hence, B fails to be $\mu^{'}_\beta$-open but is μ^*_α-open since it is μ-open. Thus, $\mu^{'}_\beta$-open $\not\Leftrightarrow$ μ^*_α*-open.*'

All above relations can be represented by the following arrow diagram



Here, A = μ-open, B = μ^*_α-open, C = μ^*-open, D=open, E = $\mu^{'}$-open, F = $\mu^{'}_\beta$-open.
The same arrow diagram follows for corresponding closed sets.

Remark 4.3. *In a space (U, μ, σ), we have*
(1) *The intersection of two μ^*_α-open sets may not be μ^*_α-open.*
(2) *The intersection of two $\mu^{'}_\beta$-open sets may not be $\mu^{'}_\beta$-open.*
(3) *The intersection of μ^*_α-open and $\mu^{'}_\beta$-open may not be either of them.*
Proof. (1) Consider a space (U, μ, σ) with $U = \{w, q, m, e\}$, $\sigma = \{\phi, U\}$ and $\mu = \{\phi, \{w, q\}, \{q, m\}, \{w, q, m\}\}$. As every μ-open sets is μ^*_α-open, $\{w, q\}$ and $\{q, m\}$ are μ^*_α-open but their intersection which is $\{q\}$ is not μ^*_α-open.
(2) Consider a space (U, μ, σ) with $U = \{w, q, m, e, t\}$, $\sigma = \{\phi, U, \{e\}\}$ and $\mu = \{\phi, \{w, q, m\}, \{w\}\}$. Let $A = \{w, q, m\}$ and $B = \{w, m, e\}$. Now, $c_\mu(i(c_\mu(A))) = U$ and $c_\mu(i(c_\mu(B))) = U$, hence A and B are both $\mu^{'}_\beta$-open sets but their intersection is $\{m, w\}$ which is not $\mu^{'}_\beta$-open.
(3) Consider a space (U, μ, σ) where $U = \{w, q, m, e\}$, $\sigma = \{\phi, U, \{w, q, m\}\}$ and $\mu = \{\phi, \{q, m, e\}, \{w\}, U\}$. Since any open set is $\mu^{'}_\beta$-open, μ-open is μ^*_α-open, $\{w, q, m\}$ and $\{q, m, e\}$ are $\mu^{'}_\beta$-open and μ^*_α-open respectively. Now intersection of these sets is $\{q, m\}$ which is neither $\mu^{'}_\beta$-open nor μ^*_α-open.

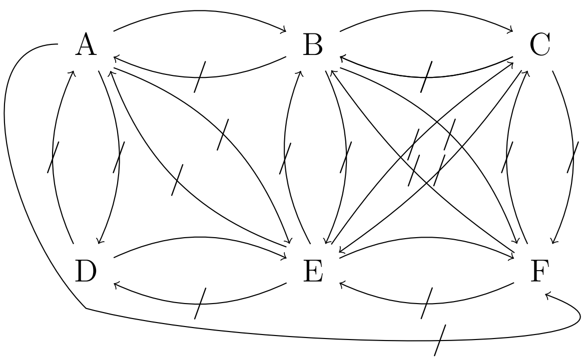
Theorem 4.3. *In a space (U, μ, σ), if $\sigma \subseteq \mu$ then the following are true.*
(1) *Every α-open(closed) is μ^*_α-open(closed).*
(2) *Every $\mu^{'}_\beta$-open(closed) is β-open(closed).*
Proof. (1) For a α-open set P, $P \subseteq i(cl(i(P)))$. By Theorem (2.1), $cl(i(P)) \subseteq$

$cl(i_\mu(P))$ and $i(cl(i(P))) \subseteq i_\mu(cl(i_\mu(P)))$. Therefore, $P \subseteq i_\mu(cl(i_\mu(P))$, proving P is μ^*_α-open. Similarly, here every α-closed set is μ^*_α-closed.
(2) Let P be a μ'_β-open set. Then $P \subseteq c_\mu(i(c_\mu(P)))$. By Theorem (2.1), $c_\mu(P) \subseteq cl(P)$ and $c_\mu(i(c_\mu(P))) \subseteq cl(i(cl(P)))$. So, P is β-open set. Similarly, here every μ'_β-closed set is β-closed set.

5. μ^*_α-continuous and μ'_β-continuous

Definition 5.1. *A function g between a space (U, μ, σ) and a topological space (V, ρ) is termed as*
(i)μ^*_α-continuous if $g^{-1}(B)$ is μ^*_α-open $\forall$ $B \in \rho$.
(ii)μ'_β-continuous if $g^{-1}(B)$ is μ'_β-open $\forall$ $B \in \rho$.

Based on the relationships among different sets, interrelation among different continuities can be established and is given below by the arrow diagram.



Here, A = μ-continuous, B = μ^*_α-continuous, C = μ^*-continuous, D=continuous, E = μ'-continuous, F = μ'_β-continuous.

Theorem 5.1. *For an injective function g between spaces (U, μ, σ) and (V, η, ρ), we have the equivalent statements:*
(1)g is μ^*_α-continuous.
(2) For any $p \in U$ and $B \in \rho$ with $g(p) \in B$, $\exists$ μ^*_α-open set A with $p \in A$ and $g(A) \subseteq B$.
(3) For any closed set Q, $g^{-1}(Q)$ is μ^*_α-closed.
(4)$c_\mu(i(c_\mu(g^{-1}(B)))) \subseteq g^{-1}(cl(B))$ for any subset B of V.
(5) $g(c_\mu(i(c_\mu(A)))) \subseteq cl(g(A))$ for any subset A of U.
Proof. (1) $\Longrightarrow$ (2) For $p \in U$, $B \in \rho$ with $g(p) \in B$, $g^{-1}(B)$ is μ^*_α-open which contains p. Taking $g^{-1}(B) = A$, $g(A) \subseteq B$.
(2) $\Longrightarrow$ (3) For a closed set Q on V, $B = V - Q \in \rho$. Let $p \in g^{-1}(B)$, then $\exists$ a μ^*_α-open set A of U with $p \in A$ and $g(A) \subseteq B$. Now, $p \in A \subseteq i_\mu(cl(i_\mu(A)))$ and g being injective, $g^{-1}(B) = A$. So, $g^{-1}(B) \subseteq i_\mu(cl(i_\mu(g^{-1}(B))))$. So, $g^{-1}(B)$ is

μ^*_α-open set and thus $g^{-1}(Q) = U - g^{-1}(V - Q) = U - g^{-1}(B)$ is μ^*_α-closed in U.
(3) $\implies$ (4) Let $B \subseteq V$. Then $cl(B)$ is a closed set in V and $g^{-1}(cl(B))$ is μ^*_α-closed set in U. So, $g^{-1}(cl(B)) \supseteq c_\mu(i(c_\mu(g^{-1}(cl(B) \supseteq c_\mu(i(c_\mu(g^{-1}(B)$. Hence, $c_\mu(i(c_\mu(g^{-1}(B))) \subseteq g^{-1}(cl(B))$.
(4) $\implies$ (5) Let $A \subseteq U$ then $g(A) \subseteq V$ and $c_\mu(i(c_\mu(g^{-1}(g(A))) \subseteq g^{-1}(cl(g(A)))$. So, $c_\mu(i(c_\mu(A))) \subseteq g^{-1}(cl(g(A)))$ and thus $g(c_\mu(i(c_\mu(A)))) \subseteq cl(g(A))$.
(5) $\implies$ (3) For a closed set B in$V$, $g^{-1}(B) \subseteq U$ and $g(c_\mu(i(c_\mu(g^{-1}(B)))) \subseteq cl(g(g^{-1}(B))) \subseteq cl(B) = B$. So, $c_\mu(i(c_\mu(g^{-1}(B)))) \subseteq g^{-1}(B)$ proving $g^{-1}(B)$ is μ^*_α-closed.
(3) $\implies$ (1) Let $B \in \rho$ then $V - B = F$ and $g^{-1}(F)$ is μ^*_α-closed in V. As $g^{-1}(B) = U - g^{-1}(V - B) = U - g^{-1}(F)$, $g^{-1}(B)$ is μ^*_α-open in U.

Theorem 5.2. *For an injective function g between spaces (U, μ, σ) and (V, η, ρ), we have the equivalent statements:*
(1)*g is μ'_β-continuous.*
(2) *For any $p \in U$ and $B \in \rho$ with $g(p) \in B$, $\exists$ μ'_β-open set A containing p and $g(A) \subseteq B$.*
(3)*For any closed set $Q, g^{-1}(Q)$ is μ'_β-closed.*
(4)*$i_\mu(cl(i_\mu g^{-1}(B))) \subseteq g^{-1}(cl(B))$ for any subset B of V.*
(5)*$g(i_\mu(cl(i_\mu)(A)) \subseteq cl(g(A))$ for any subset A of U.*
Proof. Left to the readers.

Theorem 5.3. *Let (U, μ, σ) be a space and (V, ρ), (Z, ζ) be topological spaces. Then $g \circ h$ is μ'-continuous if $h : U \to V$ and $g : V \to Z$ are μ'-continuous and continuous respectively.*
Proof. Let $B \in \zeta$. Then g being continuous, $g^{-1}(B) \in \rho$. Again by μ'-continuity, $h^{-1}(g^{-1}(B))$ is μ'-open in X. Hence, $h^{-1}(g^{-1}(B)) = (g \circ h)^{-1}(B)$. So, $g \circ h$ is μ'-continuous.

Theorem 5.4. *Let (X, μ, σ) be a space and (V, ρ), (Z, ζ) be two topological spaces. Then $g \circ h$ is μ^*-continuous if $h : X \to V$ and $g : V \to Z$ are μ-continuous and continuous respectively.*
Proof. For $B \in \zeta$, g being continuous on V, $g^{-1}(B) \in \rho$ and $h^{-1}(g^{-1}(B))$ is μ-open in X as h is μ-continuous. Now μ-open implies μ^*-open, $h^{-1}(g^{-1}(B)) = (g\circ h)^{-1}(B)$ is μ^*-open. Thus, $g \circ h$ is μ^*-continuous.

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