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ON CERTAIN IDENTITIES INVOLVING BASIC (q) HYPERGEOMETRIC SERIES

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Abstract: In this paper we establish certain identities by making use of Bailey's ${}_2\Psi_2$ transformation formula. Special cases of these identities have also been discussed.

Keywords and Phrases: Transformation formula, identity, basic bilateral hypergeometric series, summation formula.

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1. Introduction, Notations and Definitions

Throughout the present paper, we adopt the following notations and definitions. For a and q complex numbers with $|q| < 1$ the q-shifted factorial is defined as,

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} = (1-a)(1-aq)...(1-aq^{n-1}),$$

$$(a;q)_0 = 1$$

and

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r).$$

For brevity we write,

$$(a_1;q)_n(a_2;q)_n...(a_r;q)_n = (a_1,a_2,...,a_r;q)_n.$$

Also,

$$(a;q)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{a^n(q/a;q)_n}.$$

Following [Gasper and Rahman [5]] the basic hypergeometric series is defined as,

$$ {}_r\Phi_s\left[\begin{array}{l} a_1,a_2,...,a_r;q;z \\ b_1,b_2,...,b_s \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n}{(q,b_1,b_2,...,b_s;q)_n}z^n\left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r}, \quad (1.1)$$

which converges for $|z|<\infty$ if $r\le s$ and for $|z|<1$ if $r=s+1$.
The basic bilateral hypergeometric series is defined as

$$ {}_r\Psi_s\left[\begin{array}{l} a_1,a_2,...,a_r;q;z \\ b_1,b_2,...,b_s \end{array}\right] = \sum_{n=-\infty}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n}{(b_1,b_2,...,b_s;q)_n}z^n\left\{(-1)^n q^{n(n-1)/2}\right\}^{s-r}, \quad (1.2)$$

which converges for $\left|\frac{b_1b_2...b_s}{a_1a_2...a_r}\right| < |z| < 1$ if $r=s$ and for $s>r$ it converges in the whole complex-plane i.e. for $|z|<\infty$.
A great deal of literature is available on special functions of two and more variables, transformations formulas and identities [1, 2, 5]. However, the literature on basic multiple hypergeometric functions seems to be a lot less extensive. Apart from the aforementioned work on basic (q) series identities have developed [6, 7, 8, 9, 10, 11] various interesting properties of basic (q) series and their generalizations and special cases. In the present paper we prove a number of general bilateral q-series identities and transformations which are shown to be applicable in the derivation of continued fraction and partition theoretic interpretation and its generating functions. We also consider several other interesting consequences of some of the results presented here.

2. Main Results

In this section we establish following results

$$(b-c)\ {}_3\Psi_3\left[\begin{matrix} a,b,c;q;\frac{1}{a} \\ d,bq,cq \end{matrix}\right] = \frac{(q;q)_\infty^2}{(q/a,d;q)_\infty} \times\left\{b^2(1-c)\frac{(bq/a,d/b;q)_\infty}{(bq,q/b;q)_\infty} - c^2(1-b)\frac{(cq/a,d/c;q)_\infty}{(cq,q/c;q)_\infty}\right\}. \tag{2.1}$$

$$(b-c)\ {}_3\Psi_3\left[\begin{matrix} a,b,c;q;\frac{q}{a} \\ d,bq,cq \end{matrix}\right] = \frac{(q;q)_\infty^2}{(q/a,d;q)_\infty} \times\left\{b(1-c)\frac{(bq/a,d/b;q)_\infty}{(bq,q/b;q)_\infty} - c(1-b)\frac{(cq/a,d/c;q)_\infty}{(cq,q/c;q)_\infty}\right\}. \tag{2.2}$$

$${}_4\Psi_4\left[\begin{matrix} a,b,cq,\lambda q;q;\frac{1}{a} \\ d,bq,c,\lambda \end{matrix}\right] = \frac{(b-c)(b-\lambda)}{b(1-\lambda)(1-c)}\frac{(q;q)_\infty^2(bq/a,d/b;q)_\infty}{(q/a,q/b,d,bq;q)_\infty}. \tag{2.3}$$

Proof of (2.1)-(2.3)

Let us consider the Bailey's transform,

$${}_2\Psi_2\left[\begin{matrix} a,b;q;z \\ d,c \end{matrix}\right] = \frac{(az,d/a,c/b,dq/abz;q)_\infty}{(z,d,q/b,cd/abz;q)_\infty}\ {}_2\Psi_2\left[\begin{matrix} a,abz/d;q;d/a \\ az,c \end{matrix}\right]. \tag{2.4}$$

[5; (5.20) (i), p.150]

Putting $c=bq$ and $z=q/a$ in (2.4) we have,

$${}_2\Psi_2\left[\begin{matrix} a,b;q;q/a \\ d,bq \end{matrix}\right] = \frac{(q;q)_\infty^2(d/a,d/b;q)_\infty}{(q/a,q/b,d,d;q)_\infty}\ {}_2\Phi_1\left[\begin{matrix} a,bq/d;q;d/a \\ bq \end{matrix}\right]. \tag{2.5}$$

Summing the ${}_2\Phi_1$-series by making use of [5; App. II (II.8)] we have,

$${}_2\Psi_2\left[\begin{matrix} a,b;q;q/a \\ d,bq \end{matrix}\right] = \frac{(q;q)_\infty^2(bq/a,d/b;q)_\infty}{(q/a,q/b,d,bq;q)_\infty}, \tag{2.6}$$

which is a known result [3; (1.1) p. 165].
Now, let us consider

$${}_2\Psi_2\left[\begin{matrix} a,b;q;1/a \\ d,bq \end{matrix}\right] - b\ {}_2\Psi_2\left[\begin{matrix} a,b;q;q/a \\ d,bq \end{matrix}\right] = \sum_{n=-\infty}^{\infty}\frac{(a;q)_n(1-b)}{(d;q)_n}\frac{1}{a^n}$$

$$= (1-b)\ {}_1\Psi_1\left[\begin{array}{c} a;q;1/a \\ d \end{array}\right]. \tag{2.7}$$

If we make use of the summation formula [5; App. II (II.20)] to sum ${}_1\Psi_1$-series in (2.7) we get,

$$ {}_2\Psi_2\left[\begin{array}{c} a,b;q;1/a \\ d,bq \end{array}\right] = b\ {}_2\Psi_2\left[\begin{array}{c} a,b;q;q/a \\ d,bq \end{array}\right]. \tag{2.8}$$

From (2.6) and (2.8) we find,

$$ {}_2\Psi_2\left[\begin{array}{c} a,b;q;1/a \\ d,bq \end{array}\right] = b\frac{(q;q)_\infty^2(bq/a,d/b;q)_\infty}{(q/a,q/b,d,bq;q)_\infty}. \tag{2.9}$$

Now, consider

$$ {}_3\Psi_3\left[\begin{array}{c} a,b,c;q;1/a \\ d,bq,cq \end{array}\right] - c\ {}_3\Psi_3\left[\begin{array}{c} a,b,c;q;q/a \\ d,bq,cq \end{array}\right] = (1-c)\ {}_2\Psi_2\left[\begin{array}{c} a,b;q;1/a \\ d,bq \end{array}\right]. \tag{2.10}$$

From (2.9) and (2.10) we get,

$$ {}_3\Psi_3\left[\begin{array}{c} a,b,c;q;1/a \\ d,bq,cq \end{array}\right] - c\ {}_3\Psi_3\left[\begin{array}{c} a,b,c;q;q/a \\ d,bq,cq \end{array}\right] = (1-c)b\frac{(q;q)_\infty^2(bq/a,d/b;q)_\infty}{(q/a,q/b,d,bq;q)_\infty}. \tag{2.11}$$

Again, interchanging b and c in (2.11) we get,

$$ {}_3\Psi_3\left[\begin{array}{c} a,b,c;q;1/a \\ d,bq,cq \end{array}\right] - b\ {}_3\Psi_3\left[\begin{array}{c} a,b,c;q;q/a \\ d,bq,cq \end{array}\right] = (1-b)c\frac{(q;q)_\infty^2(cq/a,d/c;q)_\infty}{(q/a,q/c,d,cq;q)_\infty}. \tag{2.12}$$

Substracting (2.12) from (2.11) we get

$$ (b-c)\ {}_3\Psi_3\left[\begin{array}{c} a,b,c;q;\frac{q}{a} \\ d,bq,cq \end{array}\right] = \frac{(q;q)_\infty^2}{(q/a,d;q)_\infty} \times\left\{b(1-c)\frac{(bq/a,d/b;q)_\infty}{(bq,q/b;q)_\infty} - c(1-b)\frac{(cq/a,d/c;q)_\infty}{(cq,q/c;q)_\infty}\right\} \tag{2.13}$$

which is precisely (2.2).
Multiplying (2.11) by b and (2.12) by c and then Substracting second from first we

have,

$$(b-c)\ {}_3\Psi_3\left[\begin{array}{l} a,b,c;q;\frac{1}{a} \\ d,bq,cq \end{array}\right] = \frac{(q;q)_\infty^2}{(q/a,d;q)_\infty}$$
$$\times\left\{b^2(1-c)\frac{(bq/a,d/b;q)_\infty}{(bq,q/b;q)_\infty} - c^2(1-b)\frac{(cq/a,d/c;q)_\infty}{(cq,q/c;q)_\infty}\right\} \tag{2.14}$$

which is precisely (2.1).
Let us now consider,

$${}_3\Psi_3\left[\begin{array}{l} a,b,cq;q;1/a \\ d,bq,c \end{array}\right] = \sum_{n=-\infty}^{\infty}\frac{(a,b;q)_n}{(d,bq;q)_n}\frac{1}{a^n}\left(\frac{1-cq^n}{1-c}\right)$$
$$= \frac{1}{1-c}\ {}_2\Psi_2\left[\begin{array}{l} a,b;q;1/a \\ d,bq \end{array}\right] - \frac{c}{1-c}\ {}_2\Psi_2\left[\begin{array}{l} a,b;q;q/a \\ d,bq \end{array}\right]. \tag{2.15}$$

Using (2.6) and (2.9) in (2.15) we get,

$${}_3\Psi_3\left[\begin{array}{l} a,b,cq;q;1/a \\ d,bq,c \end{array}\right] = \frac{(b-c)}{(1-c)}\frac{(q;q)_\infty^2(bq/a,d/b;q)_\infty}{(q/a,q/b,d,bq;q)_\infty}. \tag{2.16}$$

which is a known result [4, page 305].
Now, let us consider

$${}_3\Psi_3\left[\begin{array}{l} a,b,cq;q;1/a \\ d,bq,c \end{array}\right] - b\ {}_3\Psi_3\left[\begin{array}{l} a,b,cq;q;q/a \\ d,bq,c \end{array}\right]$$
$$= (1-b)\ {}_2\Psi_2\left[\begin{array}{l} a,cq;q;1/a \\ d,c \end{array}\right]$$
$$= \frac{1-b}{1-c}\ {}_1\Psi_1\left[\begin{array}{l} a;q;1/a \\ d \end{array}\right] - \frac{c(1-b)}{1-c}\ {}_1\Psi_1\left[\begin{array}{l} a;q;q/a \\ d \end{array}\right]. \tag{2.17}$$

Summing ${}_1\Psi_1$ series in (2.17) by using [5; App. II (II.20)] we find,

$${}_3\Psi_3\left[\begin{array}{l} a,b,cq;q;1/a \\ d,bq,c \end{array}\right] = b\ {}_3\Psi_3\left[\begin{array}{l} a,b,cq;q;q/a \\ d,bq,c \end{array}\right]. \tag{2.18}$$

Thus from (2.16) and (2.18) we have

$$ {}_3\Psi_3\left[\begin{array}{l} a, b, cq; q; q/a \\ d, bq, c \end{array}\right] = \frac{(b-c)}{b(1-c)} \frac{(q;q)_\infty^2 (bq/a, d/b; q)_\infty}{(q/a, q/b, d, bq; q)_\infty}. \tag{2.19} $$

Again, proceeding by taking

$$ {}_4\Psi_4\left[\begin{array}{l} a, b, cq, \lambda q; q; 1/a \\ d, bq, c, \lambda \end{array}\right] = \sum_{n=-\infty}^{\infty} \frac{(a, b, cq; q)_\infty}{(d, bq, c; q)_\infty} \left(\frac{1-\lambda q^n}{1-\lambda}\right) \frac{1}{a^n} $$
$$ = \frac{1}{1-\lambda}\, {}_3\Psi_3\left[\begin{array}{l} a, b, cq; q; 1/a \\ d, bq, c \end{array}\right] - \frac{\lambda}{1-\lambda}\, {}_3\Psi_3\left[\begin{array}{l} a, b, cq; q; q/a \\ d, bq, c \end{array}\right]. \tag{2.20} $$

Making use of (2.16) and (2.19) in (2.20) we have

$$ {}_4\Psi_4\left[\begin{array}{l} a, b, cq, \lambda q; q; 1/a \\ d, bq, c, \lambda \end{array}\right] = \frac{(b-c)(b-\lambda)}{b(1-\lambda)(1-c)} \frac{(q;q)_\infty^2 (bq/a, d/b; q)_\infty}{(q/a, q/b, d, bq; q)_\infty}, \tag{2.21} $$

which is precisely (2.3).

3. Special Cases

In this section we deduce certain special cases of the results established in section 2.

(i) Taking $d = 0$ and $a \to \infty$ in (2.1) we get,

$$ (b-c) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(1-bq^n)(1-cq^n)} $$
$$ = (q;q)_\infty^2 \left\{ \frac{b^2}{(1-b)} \frac{1}{(bq, q/b; q)_\infty} - \frac{c^2}{(1-c)} \frac{1}{(cq, q/c; q)_\infty} \right\}. \tag{3.1} $$

Putting $b = e^{i\theta}$ and $c = e^{-i\theta}$ in (3.1) we get,

$$ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(1-2q^n \cos\theta + q^{2n})} = \frac{(2\cos\theta - 1)}{(1-\cos\theta)} \frac{(q;q)_\infty^2}{\prod_{n=1}^{\infty}(1-2q^n\cos\theta + q^{2n})}. \tag{3.2} $$

Taking $\theta = \pi/2$ in (3.2) we have

$$ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(1+q^{2n})} = -\frac{(q;q)_\infty^2}{\prod_{n=1}^{\infty}(1+q^{2n})} = -\frac{(q;q)_\infty^2}{(-q^2;q^2)_\infty^2}. \tag{3.3} $$

Taking $\theta = \pi$ in (3.2) we get,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(1+q^n)^2} = \frac{-3}{2} \frac{(q;q)_\infty^2}{\prod_{n=1}^{\infty}(1+q^n)^2} = \frac{-3}{2} \frac{(q;q)_\infty^2}{(-q;q)_\infty^2}. \tag{3.4}$$

Putting $\frac{\pi}{2} + \theta$ for θ in (3.1) we have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(1+2q^n \sin\theta + q^{2n})} = -\frac{1+2\sin\theta}{1+\sin\theta} \frac{(q;q)_\infty^2}{\prod_{n=1}^{\infty}(1+2q^n\sin\theta + q^{2n})}. \tag{3.5}$$

(ii) Taking $d = 0$ and $a \to \infty$ in (2.2) we get,

$$\begin{aligned}(b-c) &\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(1-bq^n)(1-cq^n)} \\ &= (q;q)_\infty^2 \left\{ \frac{b}{(1-b)} \frac{1}{(bq, q/b;q)_\infty} - \frac{c}{(1-c)} \frac{1}{(cq, q/c;q)_\infty} \right\}.\end{aligned} \tag{3.6}$$

(3.6) can be expressed as,

$$\begin{aligned}(b-c) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(1-bq^n)(1-cq^n)} &= (q;q)_\infty^2 \left\{ \frac{(1-c)}{(c, 1/c;q)_\infty} - \frac{(1-b)}{(b,1/b;q)_\infty} \right\}. \\ &= (q;q)_\infty \left\{ \frac{(q;q)_\infty}{(1-1/c)(cq,q/c;q)_\infty} - \frac{(q;q)_\infty}{(1-1/b)(bq,q/b;q)_\infty} \right\}.\end{aligned} \tag{3.6A}$$

Comparing (3.6A) with [2; (3.8) p. 34] we have,

$$\begin{aligned}&(q;q)_\infty \left\{ \frac{b}{1-b} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_v(m,n) b^m q^n - \frac{c}{1-c} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_v(m,n) c^m q^n \right\} \\ &= (q;q)_\infty \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_v(m,n) q^n \left\{ \frac{b^{m+1}}{1-b} - \frac{c^{m+1}}{1-c} \right\},\end{aligned} \tag{3.6B}$$

where $N_v(m,n)$ stands for the number vector partitions of n with crank m.

Putting $b=\lambda e^{i\theta}$ and $c=\lambda e^{-i\theta}$ in (3.6) we have,

$$2\lambda i\sin\theta\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1-2\lambda q^n\cos\theta+\lambda^2q^{2n}}$$

$$=(q;q)_\infty^2\left\{\frac{\lambda e^{i\theta}}{(1-\lambda e^{i\theta})\prod_{n=1}^{\infty}(1-2\lambda q^n\cos\theta+\lambda^2q^{2n})}-\frac{\lambda e^{-i\theta}}{(1-\lambda e^{-i\theta})\prod_{n=1}^{\infty}(1-2\lambda q^n\cos\theta+\lambda^2q^{2n})}\right\} \tag{3.7}$$

which on simplification gives

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(1-2\lambda q^n\cos\theta+\lambda^2q^{2n})}=\frac{(q;q)_\infty^2}{(1-2\lambda\cos\theta+\lambda^2)\prod_{n=1}^{\infty}(1-2\lambda q^n\cos\theta+\lambda^2q^{2n})}. \tag{3.8}$$

Taking $\theta=0$ in (3.8) we get

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(1-\lambda q^n)^2}=\frac{(q;q)_\infty^2}{(1-\lambda)^2\prod_{n=1}^{\infty}(1-\lambda q^n)^2}. \tag{3.9}$$

For $\theta=\pi/2$, (3.8) yields

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(1+\lambda^2 q^{2n})^2}=\frac{(q;q)_\infty^2}{(1+\lambda)^2\prod_{n=1}^{\infty}(1+\lambda^2 q^{2n})}. \tag{3.10}$$

For $\theta=\pi$, (3.8) yields

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(1+\lambda q^n)^2}=\frac{(q;q)_\infty^2}{(1+\lambda)^2\prod_{n=1}^{\infty}(1+\lambda q^n)^2}. \tag{3.11}$$

If we take $\lambda=1$ in (3.11) we get,

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(1+q^n)^2}=\frac{(q;q)_\infty^2}{2(-q;q)_\infty^2}. \tag{3.12}$$

Putting $\lambda=1$ in (3.10) we get

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(1+q^{2n})}=\frac{(q;q)_\infty^2}{2(-q^2;q^2)_\infty^2}. \tag{3.13}$$

(iii) Putting $d = 0$ and $a \to \infty$ in (2.3) we get

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2}(1-cq^n)(1-\lambda q^n)}{(1-bq^n)} = \frac{(b-c)(b-\lambda)}{b(1-b)} \frac{(q;q)_\infty^2}{(bq, q/b; q)_\infty}. \tag{3.14}$$

Comparing (3.14) with [2; (3.8) p. 34] we have,

$$= \frac{(b-c)(b-\lambda)}{b(1-b)}(q;q)_\infty \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_v(m,n) b^m q^n,$$

where $N_v(m,n)$ is the number of vector partitions of n with crank m.
Taking q^5 for q and then putting $b = q^2$ in (3.14) we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n-1)/2}(1-cq^{5n})(1-\lambda q^{5n})}{(1-q^{5n+2})} &= \frac{(q^2-c)(q^2-\lambda)}{q^2(1-q^2)} \frac{(q^5;q^5)_\infty^2}{(q^3, q^7; q^5)_\infty} \\ &= \frac{(q^2-c)(q^2-\lambda)}{q^2} \frac{(q^5;q^5)_\infty^2}{(q^2, q^3; q^5)_\infty}. \end{aligned}\tag{3.15}$$

Taking q^5 for q and then putting $b = q$ in (3.14) we get

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n-1)/2}(1-cq^{5n})(1-\lambda q^{5n})}{(1-q^{5n+1})} &= \frac{(q-c)(q-\lambda)}{q(1-q)} \frac{(q^5;q^5)_\infty^2}{(q^4, q^6; q^5)_\infty} \\ &= \frac{(q-c)(q-\lambda)}{q} \frac{(q^5;q^5)_\infty^2}{(q, q^4; q^5)_\infty}. \end{aligned}\tag{3.16}$$

From (3.15) and (3.16) and corollary [1; (6.2.6) p. 153] we have

$$\begin{aligned}\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n-1)/2}(1-cq^{5n})(1-\lambda q^{5n})}{(1-q^{5n+2})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n-1)/2}(1-cq^{5n})(1-\lambda q^{5n})}{(1-q^{5n+1})}} &= \frac{(q^2-c)(q^2-\lambda)}{q(q-c)(q-\lambda)} \frac{(q, q^4; q^5)_\infty}{(q^2, q^3; q^5)_\infty} \\ &= \frac{(q^2-c)(q^2-\lambda)}{q(q-c)(q-\lambda)} \left\{ \frac{1}{1+} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+\ldots}. \right\} \end{aligned}\tag{3.17}$$

Taking $\lambda = c = 0$ in (3.17) we find,

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n-1)/2}}{(1-q^{5n+2})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n-1)/2}}{(1-q^{5n+1})}} = \frac{q}{1+} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+\ldots}. \tag{3.18}$$

(iv) Putting $\lambda = 0$ in (2.3) we get

$$ {}_3\Psi_3 \left[\begin{array}{l} a, b, cq; q; 1/a \\ d, bq, c \end{array} \right] = \frac{(b-c)}{(1-c)} \frac{(q;q)_\infty^2 (bq/a, d/b; q)_\infty}{(q/a, q/b, d, bq; q)_\infty}. \tag{3.19} $$

Taking $d = c = 0$, $a \to \infty$ in (3.19) we get,

$$ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(1 - bq^n)} = \frac{b}{1-b} \frac{(q;q)_\infty^2}{(bq, q/b; q)_\infty}. \tag{3.20} $$

Replacing q by q^2 and putting $b = q$ in (3.20) and using [1; (1.1.7), p. 11] we get,

$$ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)}}{(1 - q^{2n+1})} = q \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} = q\Psi^2(q). \tag{3.21} $$

Replacing q by q^8 and b by q in (3.20) we get,

$$ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{4n(n-1)}}{(1 - q^{8n+1})} = q \frac{(q^8;q^8)_\infty^2}{(q, q^7; q^8)_\infty}. \tag{3.22} $$

Again, replacing q by q^8 and b by q^3 in (3.20) we find,

$$ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{4n(n-1)}}{(1 - q^{8n+3})} = q^3 \frac{(q^8;q^8)_\infty^2}{(q^3, q^5; q^8)_\infty}. \tag{3.23} $$

Taking the ratio of (3.22) and (3.23) and using [1; (6.2.38), p. 154] we get,

$$ \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{4n(n-1)}}{(1 - q^{8n+3})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{4n(n-1)}}{(1 - q^{8n+1})}} = \frac{q^2}{1+} \frac{q+q^2}{1+} \frac{q^4}{1+} \frac{q^3+q^6}{1+\ldots}. \tag{3.24} $$

(v) Taking $d = q$, $a \to \infty$ and $c = 0$ in (3.19) we get,

$$ \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_\infty (1 - bq^n)} = \frac{b}{1-b} \frac{(q;q)_\infty}{(bq;q)_\infty} = \frac{b(q;q)_\infty}{(b;q)_\infty}. \tag{3.25} $$

Replacing q by q^3 and putting $b = q^2$ in (3.25) we get,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n-1)/2}}{(q^3;q^3)_n(1-q^{3n+2})} = q^2 \frac{(q^3;q^3)_\infty}{(q^2;q^3)_\infty}. \tag{3.26}$$

Again, replacing q by q^3 and b by q in (3.25) we have,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n-1)/2}}{(q^3;q^3)_n(1-q^{3n+1})} = q \frac{(q^3;q^3)_\infty}{(q;q^3)_\infty}. \tag{3.27}$$

Taking the ratio of (3.26) and (3.27) and using [1; (7.1.1), p. 179] we get,

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n-1)/2}}{(q^3;q^3)_n(1-q^{3n+1})}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n-1)/2}}{(q^3;q^3)_n(1-q^{3n+2})}} = q^{-1} \frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{q^{-1}}{1-}\,\frac{q}{1+q-}\,\frac{q^5}{1+q^2-}\,\frac{q^5}{1+q^3-\ldots}. \tag{3.28}$$

Putting q^4 for q and $b = q$ in (3.25) we get,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n-1)}}{(q^4;q^4)_n(1-q^{4n+1})} = \frac{q(q^4;q^4)_\infty}{(q;q^4)_\infty}. \tag{3.29}$$

Putting q^4 for q and $b = q^3$ in (3.25) we find,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n-1)}}{(q^4;q^4)_n(1-q^{4n+3})} = \frac{q^3(q^4;q^4)_\infty}{(q^3;q^4)_\infty}. \tag{3.30}$$

Taking the ratio of (3.29) and (3.30) and using [1; (7.1.2) p. 179] we get

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n-1)}}{(q^4;q^4)_n(1-q^{4n+1})}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n-1)}}{(q^4;q^4)_n(1-q^{4n+3})}} = q^{-2} \frac{(q^3;q^4)_\infty}{(q;q^4)_\infty} = \frac{q^{-2}}{1-}\,\frac{q}{1+q^2-}\,\frac{q^3}{1+q^4-}\,\frac{q^5}{1+q^6-\ldots}. \tag{3.31}$$

A number of similar results can also be deduced.

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