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FOURIER TYPE TRANSFORMS AND THEIR CONVOLUTIONS ON $\mathbb{R}^n$

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Abstract: In this paper the operational properties of two integral transforms of Fourier type are defined. The purpose of the study is to define the convolution of the Fourier type transform on $L_1(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n)$. Also we obtained the Inversion, Uniqueness and Plancherel's theorem of these two transform. Lastely we have applied these transform to differential equation of higher order for the solution.

Keywords and Phrases: Plancherel's theorem, Convolution, Hermite function.

2020 Mathematics Subject Classification: 42A38, 44A35.

1. Introduction

In literature we have studied the Fourier-sine and Fourier-cosine integral transforms([8], [9]). Along with these transforms Fourier transform were also studied and applied in many fields of Mathematics and Physics ([7], [9]). The Fourier transform plays an important role in engineering and science. It has vide applications in signal processing and communication theory. B. T Giang, N. M. Tuan [4] has given the operational properties of two integral transforms of Fourier type and their convolution. We consider here the following transforms which are known as integral transforms of Fourier type [4] and are defined as

$$(H_1 f)(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cos(zt + \frac{\pi}{4})\, f(t)\, \mathrm{d}t \tag{1.1}$$

$$(H_2 f)(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sin(zt + \frac{\pi}{4})\, f(t)\, \mathrm{d}t \tag{1.2}$$

where f is real or complex valued function defined on $\mathbb{R}^n$. The main difference between Fourier sine and Fourier cosine transform and H_1, H_2 is that kernel are changed from $\cos(xy)$, $\sin(xy)$ to $\cos(xy + \frac{\pi}{4})$, $\sin(xy + \frac{\pi}{4})$.

We investigate definition and operational properties and convolution of H_1, H_2 on $S(\mathbb{R}^n)$, $L_1(\mathbb{R}^n)$, $L_2(\mathbb{R}^n)$.

We have given the properties of H_1, H_2 on $\mathbb{R}^n$ so that H_1,H_2 becomes bounded linear operators on $L_1(\mathbb{R}^n)$, $L_2(\mathbb{R}^n)$.

2. Operational Properties

Let $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ be the set of natural numbers. Let $\mathscr{S}$or $\mathscr{S}_n$ denote the set of all K valued functions on $\mathbb{R}^n$ which are infinitely differentiable such that

$$q_m(f) \;=\; \sup_{|n| \,\leq\, |m|} \; \sup_{x \in \mathbb{R}^n} \, (1 + |x|^2)^m \; |(D_n f)(x)| \; < \infty \tag{2.1}$$

Here $|x|^2 \;=\; \sum x_i^2$ Here $K \;=\; \mathbb{R}^n$ and $D_n f \;=\; f^{(n)} \; for\; n \;\in\; N$, $\mathscr{S}$is a vector space since $P(x)D_n f$ is a bounded function on $\mathbb{R}^n$ for every polynomial P and for every index n which is true if we replace P by $(1 + |x|^2)^N \, P(x)$ which implies that $P(x).D_n f \;\in\; L_1(\mathbb{R}^n)$.

Hence $\mathscr{S}$is Frechet space by taking countable collection of seminorms for which $q_m(f)$ defines a weakly convex topology.

2.1. Transforms of the Hermite Function

The Hermite polynomial of degree m is defined by

$$H_m(x) \;=\; (-1)^m \, e^{x^2} \, \frac{\partial^{(m)}}{\partial x^{(m)}} \, e^{-x^2}$$

where

$$m = m_1 + m_2 + \cdots + m_n$$

$$x^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = \sum_{i=1}^{n} x_i^2$$

The corresponding Hermite Function ϕ_m is given by

$$\phi_m(x) = (-1)^m e^{\frac{x^2}{2}} (\frac{\partial}{\partial x})^m e^{-x^2}$$

Let $n_i = 4m_i + k_i \qquad k_i = 0, 1, 2, 3$

Theorem 2.1. $n = 4m + k \qquad k = 0, 1, 2, 3$

$$\begin{aligned} \textit{where} \quad n &= n_1 + n_2 + \cdots + n_n \\ m &= m_1 + m_2 + \cdots + m_n \\ k &= k_1 + k_2 + \cdots + k_n \end{aligned}$$

Here $x = x_1 x_2 \cdots x_n \qquad y = y_1 y_2 \cdots y_n$ *then*

$$H_1\phi_n = \begin{cases} \phi_n & \textit{if} \quad k{=}0{,}3 \\ -\phi_n & \textit{if} \quad k{=}1{,}2 \end{cases} \tag{2.2}$$

$$H_2\phi_n = \begin{cases} \phi_n & \textit{if} \quad k{=}0{,}1 \\ -\phi_n & \textit{if} \quad k{=}2{,}3 \end{cases} \tag{2.3}$$

Proof. Obliviously all $\phi_n \in \mathscr{S}$ we have for $i = 1, 2, 3, \cdots, n$

$$\cos(x_i x_i' + \frac{\pi}{4}) = \frac{e^{i(x_i x_i' + \frac{\pi}{2})} + e^{-i(x_i x_i' + \frac{\pi}{2})}}{2}$$

$$\frac{\partial^{(n)}}{\partial x^{(n)}} e^{\frac{1}{2}(x+iy)^2} = (\pm i)^n \frac{\partial^{(n)}}{\partial y^{(n)}} e^{\frac{1}{2}(x+iy)^2}$$

integrating by parts n times,

$$\begin{aligned} (H_1\phi_n)(x_1, x_2, \cdots, x_n) &= \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdot \int_{-\infty}^{\infty} \phi_n(x_1', \cdots, x_n') \prod_{i=1}^{n} \cos(x_i x_i' + \frac{\pi}{4})\, \mathrm{d}x_i' \\ &= \sqrt{2}\, \cos(\frac{\pi}{4} + \frac{n\pi}{2})\, \phi_n(x_1, x_2, \cdots, x_n) \end{aligned}$$

$$\text{since } \sqrt{2}\, \cos(\frac{\pi}{4} + \frac{n\pi}{2}) = \begin{cases} 1 & \text{if} \quad \text{k=0,3} \\ -1 & \text{if} \quad \text{k=1,2} \end{cases}$$

similarly we can show for H_2 also,

$$\sin(x_i x_i^{'} + \frac{\pi}{4}) = \frac{e^{i(x_i x_i^{'} + \frac{\pi}{4})} - e^{-i(x_i x_i^{'} + \frac{\pi}{4})}}{2i}$$

$$\frac{\partial^{(n)}}{\partial x^{(n)}} e^{\frac{1}{2}(x+iy)^2} = (\pm\, i)^n \frac{\partial^{(n)}}{\partial y^{(n)}} e^{\frac{(x+iy)^2}{2}}$$

$$\frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{\pm ixy - \frac{x^2}{2}} \,\mathrm{d}x = e^{\frac{-y^2}{2}}$$

Integrating by parts n times we get

$$(H_2\, \phi_n)(x_1, x_2, \cdots, x_n) = \frac{1}{\pi^{\frac{n}{2}}} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} \phi_n(x_1^{'}, x_2^{'}, \cdots, x_n^{'}) \sin(x_i x_i^{'} + \frac{\pi}{4}) \,\mathrm{d}x_i^{'}$$

$$= \sqrt{2}\, \sin(\frac{\pi}{4} + \frac{n\pi}{2})\, \phi_n(x_1, x_2, \cdots, x_n)$$

$$\text{but} \quad \sqrt{2}\, \sin(\frac{\pi}{4} + \frac{n\pi}{2}) = \begin{cases} 1 & if \quad k = 0, 1 \\ -1 & if \quad k = 2, 3. \end{cases}$$

2.2. Definition of H_1, H_2 in Space $S(\mathbb{R}^n), L_1(\mathbb{R}^n), L_2(\mathbb{R}^n)$

The space $S(\mathbb{R}^n), L_1(\mathbb{R}^n), L_2(\mathbb{R}^n)$ are defined in [7]. Let $C_0(\mathbb{R}^n)$ denote the supremum - normed Banach space of all continuous functions on $\mathbb{R}^n$ that vanish at infinity.

Theorem 2.2. *If $f \in L_1(\mathbb{R}^n)$ then $(H_1 f), (H_2 f) \in C_0(\mathbb{R}^n)$ and*
$\|H_1 f\|_\infty \leq \frac{1}{\pi^{\frac{n}{2}}} \|f\|_1$, $\|H_2 f\|_\infty \leq \frac{1}{\pi^{\frac{n}{2}}} \|f\|_1$
where $\| \quad \|_1$ is L_1 norm.

Proof. $C_0(\mathbb{R}^n)$ is the supremum normed Banach space of all complex valued continuous functions on $\mathbb{R}^n$ that vanish at infinity. Using Riemann Lebegue lemma [9] we have $H_1 f, H_2 f \in C_0(\mathbb{R}^n)$. We have

$$\left|\cos(x_i x_i^{'} + \frac{\pi}{4})\right| \leq 1, \quad \left|\sin(x_i x_i^{'} + \frac{\pi}{4})\right| \leq 1 \qquad \text{for} \quad i = 1, 2, 3, \cdots, n$$

$$|(H_1\, f(x_1, x_2, \cdots, x_n)| = |\frac{1}{\pi^{\frac{n}{2}}} \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} \prod_{i=1}^{n} \cos(x_i x_i^{'} + \frac{\pi}{4}) f(x_1^{'}, \cdots, x_n^{'})\,| \,\mathrm{d}x_i^{'}$$

$$\leq \frac{1}{\pi^{\frac{n}{2}}} \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} |f(x_1^{'}, \cdots, x_n^{'})| \,\mathrm{d}x_1^{'} \cdots \mathrm{d}x_n^{'}$$

$$ess.\ \sup|H_1\, f(x_1, x_2, \cdots, x_n)| \;\leq\; \frac{1}{\pi^{\frac{n}{2}}}\; \|f\|_1$$

$$\|H_1\, f\|_\infty \;\leq\; \frac{1}{\pi^{\frac{n}{2}}}\; \|f\|_1 \qquad \forall x_i \in \mathbb{R} \quad i = 1, 2, \cdots, n \tag{2.4}$$

Again using Riemann Lebegue lemma [9], we have

$$\begin{aligned}|(H_2\, f(x_1, x_2, \cdots, x_n)| \;&=\; |\frac{1}{\pi^{\frac{n}{2}}} \int\limits_{-\infty}^{\infty} \cdots \prod_{i=1}^{n} \sin(x_i x_i^{'} + \frac{\pi}{4})\, f(x_1^{'}, \cdots, x_n^{'})\,|\, \mathrm{d}x_i^{'} \\ &\leq \frac{1}{\pi^{\frac{n}{2}}} \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} |f(x_1^{'}), \cdots, x_n^{'}|\, \mathrm{d}x_1^{'} \cdots \mathrm{d}x_n^{'}\end{aligned}$$

$$ess.\ \sup|H_2\, f(x_1, x_2, \cdots, x_n)| \;\leq\; \frac{1}{\pi^{\frac{n}{2}}}\; \|f\|_1 \qquad \text{for all} \quad x_i \in \mathbb{R} \quad i = 1, 2, 3, \cdots, n$$

$$\|H_2\, f\|_\infty \;\leq\; \frac{1}{\pi^{\frac{n}{2}}}\, \|f\|_1$$

Let us define $h_m(x) \;=\; x^m\, h(x)$, here $x^{(m)} \;=\; (x_1^{m_1}\, x_2^{m_2} \cdots\, x_n^{m_n})$, $x \in \mathbb{R}^n$
$m = (m_1, m_2, \cdots, m_n) \in N$, $n = (n_1, n_2, \cdots, n_n) \in N$
The function $D^{(n)}h_m$ belongs to $S(\mathbb{R}^n)$.

Theorem 2.3. *Let* $h \in \mathscr{S}_n = S(\mathbb{R}^n)$ *then for all* $m, n \in \mathbb{N}$ $\forall x \in \mathbb{R}^n$

$$x^m \cdot D^{(n)}(H_1 h)(x) \;=\; \begin{cases} H_1\, D^{(m)}\, h_n(x) & \mathit{if} \quad n + m = 0(mod\, 4) \\ -H_2\, D^{(m)}\, h_n(x) & \mathit{if} \quad n + m = 1(mod\, 4) \\ -H_1\, D^{(m)}\, h_n(x) & \mathit{if} \quad n + m = 2(mod\, 4) \\ H_2\, D^{(m)}\, h_n(x) & \mathit{if} \quad n + m = 3(mod\, 4) \end{cases} \tag{2.5}$$

and

$$x^m \cdot D^{(n)}(H_2 h)(x) \;=\; \begin{cases} H_2\, D^{(m)}\, h_n(x) & \mathit{if} \quad n + m = 0(mod\, 4) \\ H_1\, D^{(m)}\, h_n(x) & \mathit{if} \quad n + m = 1(mod\, 4) \\ -H_2\, D^{(m)}\, h_n(x) & \mathit{if} \quad n + m = 2(mod\, 4) \\ -H_1\, D^{(m)}\, h_n(x) & \mathit{if} \quad n + m = 3(mod\, 4) \end{cases} \tag{2.6}$$

Proof. Here $D^{(n)}$ stands for $\dfrac{\partial^{n_1}}{\partial x_1^{n_1}} \dfrac{\partial^{n_2}}{\partial x_2^{n_2}} \cdots \dfrac{\partial^{n_n}}{\partial x_n^{n_n}}$

$\dfrac{\partial^{(k)}}{\partial x_i^{(k)}}\, \cos{(x_i x_i^{'} + \dfrac{\pi}{4})} = (x_i^{'})^k\, \cos(x_i x_i^{'} + \dfrac{\pi}{4} + \dfrac{k\pi}{2})$ $\quad k \in \mathbb{N}$

Now,

$$D^{(n)}(H_1h)(x) = \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \cos(x_i x_i^{'} + \frac{\pi}{4} + \frac{n_i\pi}{2})\, h(x_1^{'}, \cdots, x_n^{'})\, (x_i^{'})^{n_i} \cdot \mathrm{d}x_i^{'}$$

$$= \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \cos(x_i x_i^{'} + \frac{\pi}{4} + \frac{n_i\pi}{2}) \cdot h_n(x_1^{'}, \cdots, x_n^{'}) \cdot \mathrm{d}x_i^{'}$$

Integrating by parts m times,

$$(x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}) \cdot D^{(n)}(H_1h)(x_1, x_2, \cdots, x_n) =$$

$$\frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \cos(x_i x_i^{'} + \frac{\pi}{4} + \frac{n_i\pi}{2}) \cdot x_i^{m_i} \cdot h_n(x_1^{'}, \cdots, x_n^{'}) \cdot \mathrm{d}x_i^{'}$$

$$= \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{\partial^{(m_i)}}{\partial_i^{(m_i)}} \cos(x_i x_i^{'} + \frac{\pi}{4} + \frac{(n_i - m_i)\pi}{2}) \times h_n(x_1^{'}, \cdots, x_n^{'}) \cdot \mathrm{d}x_i^{'}$$

$$= \frac{(-1)^{\sum_{i=1}^{n} m_i}}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \cos(x_i x_i^{'} + \frac{\pi}{4} + \frac{(n_i - m_i)\pi}{2}) \times D^{(m)} \cdot h_n(x_1^{'}, \cdots, x_n^{'}) \cdot \mathrm{d}x_i^{'}$$

$$= \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \cos(x_i x_i^{'} + \frac{\pi}{4} + \frac{(n_i + m_i)\pi}{2}) \times D^{(m)} h_n(x_1^{'}, \cdots, x_n^{'}) \cdot \mathrm{d}x_i^{'}$$

for all $m, n \in \mathbb{N}$ where $n = (n_1, n_2, \cdots, n_n)$, $m = (m_1, m_2, \cdots, m_n)$ which completes the proof.

Now for H_2 we have $\dfrac{\partial^k}{\partial x_i^k} \sin(x_i x_i^{'} + \frac{\pi}{4}) = (x_i^{'})^k \sin(x_i x_i^{'} + \frac{\pi}{4} + \frac{k\pi}{2}) \quad k \in \mathbb{N}$

$$D^{(n)}(H_2h)(x) = \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \sin(x_i x_i^{'} + \frac{\pi}{4} + \frac{n_i\pi}{2})\, h(x_1^{'}, \cdots, x_n^{'})\, (x_i^{'})^{n_i} \cdot \mathrm{d}x_i^{'}$$

$$= \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \sin(x_i x_i^{'} + \frac{\pi}{4} + \frac{n_i\pi}{2}) \cdot h_n(x_1^{'}, \cdots, x_n^{'}) \cdot \mathrm{d}x_i^{'}$$

Integrating by parts m times,

$$(x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}) \cdot D^{(n)}(H_2 h)(x_1, x_2, \cdots, x_n) =$$

$$\frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \sin(x_i x_i' + \frac{\pi}{4} + \frac{n_i \pi}{2}) \cdot x_i^{m_i} \cdot h_n(x_1', \cdots, x_n') \cdot \mathrm{d}x_i'$$

$$= \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{\partial^{m_i}}{\partial_i^{m_i}} \sin(x_i x_i' + \frac{\pi}{4} + \frac{(n_i - m_i)\pi}{2}) \times h_n(x_1', \cdots, x_n') \cdot \mathrm{d}x_i'$$

$$= \frac{(-1)^{\sum_{i=1}^{n} m_i}}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \int_{-\infty}^{\infty} \sin(x_i x_i' + \frac{\pi}{4} + \frac{(n_i - m_i)\pi}{2}) \times D^{(m)} \cdot h_n(x_1', \cdots, x_n') \mathrm{d}x_i'$$

$$= \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} \sin(x_i x_i' + \frac{\pi}{4} + \frac{(n_i + m_i)\pi}{2}) \times D^{(m)} h_n(x_1', \cdots, x_n') \cdot \mathrm{d}x_i'$$

for all $m, n \in \mathbb{N}$ where $n = (n_1, n_2, \cdots, n_n)$, $\quad m = (m_1, m_2, \cdots, m_n)$ which completes the proof.

Theorem 2.4. *The operators H_1 and H_2 are continuous linear maps of the Frechet Space $\mathscr{S}$ onto itself.*
Proof. $f \in \mathscr{S}$ then $H_1 f, H_2 f$ are smooth on $\mathbb{R}^n$, we have

$$\|x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \cdot D^{(n)}(H_1 f)(x_1 x_2 \cdots x_n)\|_\infty \leq \frac{1}{\pi^{\frac{n}{2}}} \|D^m \cdot h_n\|_1 < \infty$$

and

$$\|x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \cdot D^{(n)}(H_2 f)(x_1 x_2 \cdots x_n)\|_\infty \leq \frac{1}{\pi^{\frac{n}{2}}} \|D^m \cdot h_n\|_1 < \infty$$

$\Rightarrow H_1 f,\ H_2 f \in \mathscr{S}$
We prove that H_1 is closed operator in $\mathscr{S}$
Let $f, g \in \mathscr{S}$. *Let* $\{f_j\}_{j=1}^{\infty}$ be a sequence in $\mathscr{S}$ which converges to f and $H_1 f_j$ converges to g in $\mathscr{S}$ as $j \to \infty$. To prove that $H_1 f = g$ we have convergence in $\mathscr{S}$ implies convergence in $L_1(\mathbb{R}^n)$

$$\|H_1 f_j - H_1 f\| = \|H_1(f_j - f)\| \leq \|f_j - f\|_1 \to 0 \quad as \quad j \to \infty$$

Hence $H_1 f_j$ uniformly converges on $\mathbb{R}^n$ to $H_1 f$ as well as to g. Hence $H_1 f = g$. Similarly by closed graph theorem H_1 is continuous linear operator on $\mathscr{S}$.

Theorem 2.5. *Let $h \in L_1(\mathbb{R}^n)$. If h is function of bounded variation on an interval in the point $x \in \mathbb{R}^n$ then*

$$\frac{1}{2}\{h(x_1+0, x_2+0, \cdots, x_n+0) + h(x_1-0, x_2-0, \cdots, x_n-o)\}$$
$$= \frac{1}{\pi^n}\prod_{i=1}^{n}\int_0^\infty \cdots \int_0^\infty \mathrm{d}x_i \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(y_1, \cdots, y_n) \times \cos(x_i - y_i)x_i' \cdot \mathrm{d}y_i$$

and if h is continuous and is of bounded variation on some interval (δ_1, δ_2) then

$$h(x_1, x_2, \cdots, x_n) = \frac{1}{\pi^n}\prod_{i=1}^{n}\int_0^\infty \cdots \int_0^\infty \mathrm{d}x_i' \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(y_1, \cdots, y_n) \times \cos(x_i - y_i)x_i' \cdot \mathrm{d}y_i$$

Theorem 2.6. [Inversion theorem] *If $f \in \mathscr{S}$ or $f \in S_n$ then*

$$f(x_1, x_2, \cdots, x_n) = \frac{1}{\pi^{\frac{n}{2}}}\prod_{i=1}^{n}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (H_1 f)(x_i', \cdots, x_n') \times \cos(x_i x_i' + \frac{\pi}{4})\, \mathrm{d}x_i' \quad (2.7)$$

$$f(x_1, x_2, \cdots, x_n) = \frac{1}{\pi^{\frac{n}{2}}}\prod_{i=1}^{n}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (H_2 f)(x_1', \cdots, x_n') \times \sin(x_i x_i' + \frac{\pi}{4})\, \mathrm{d}x_i' \quad (2.8)$$

Proof. Given that $f \in \mathscr{S}$ the the R.H.S. of (2.7) is clearly member of $S(\mathbb{R}^n)$ using above theorem (2.5) and Fubini's theorem we get,

$$\frac{1}{\pi^{\frac{n}{2}}}\prod_{i=1}^{n}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (H_1 f)(x_1', , \cdots, x_n') \times \cos(x_i x_i' + \frac{\pi}{4})\, \mathrm{d}x_i'$$
$$= \left[\lim_{\mu_i \to \infty} \frac{1}{\pi^{\frac{n}{2}}}\prod_{i=1}^{n}\int_{-\mu_1}^{\mu_1} \cdots \int_{-\mu_n}^{\mu_n} \cos(x_i x_i' + \frac{\pi}{4}) \times (H_1 f)(x_1', \cdots, x_n')\, \mathrm{d}x_i'\right]$$
$$= \lim_{\mu_i \to \infty} \frac{1}{\pi^n}\prod_{i=1}^{n}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y_1', \cdots, y_n')\, \mathrm{d}y_i' \int_{-\mu_1}^{\mu_1} \cdots \int_{-\mu_n}^{\mu_n} \cos(x_i x_i' + \frac{\pi}{4}) \times \cos(x_i' y_i' + \frac{\pi}{4})\mathrm{d}x_i'$$
$$= \lim_{\mu_i \to \infty} \frac{1}{(2\pi)^n}\prod_{i=1}^{n}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(y_i') \frac{2\sin\, \mu_i(x_i - y_i')}{x_i - y_i}\mathrm{d}y_i'$$
$$= f(x_1, x_2, \cdots, x_n)$$

Similarly we can prove the Inversion formula for H_2.

Theorem 2.7. *If* H_1 *and* H_2 *are continuous Linear one to one maps of* $\mathscr{S}$ *onto itself then* $H_1^2 = I,\ H_2^2 = I$ *i.e.* $H_1 = H_1^{-1},\ H_2 = H_2^{-1}$.
Proof. From these two Inversion formulae we see that H_1 *and* H_2 both are one-one and onto $S(\mathbb{R}^n)$. Hence $H_1 = H_1^{-1}$ *and* $H_2 = H_2^{-1} \Rightarrow H_1^2 = I,\ H_2^2 = I$.

Theorem 2.8. *If* $f, H_1 f \in L_1(\mathbb{R}^n)$ *or* $f, H_2 f \in L_1(\mathbb{R}^n)$ *also if*

$$g(x_1, x_2, \cdots, x_n) = \frac{1}{\pi^{\frac{n}{2}}} \prod_{i=1}^{n} \int_{\mathbb{R}n} (H_1 f)(y_1, \cdots, y_n) \cos(x_i y_i + \frac{\pi}{4})\, \mathrm{d}y_i$$

$$\textit{then} \quad g(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n)$$

Proof. Given $f, H_1 f \in L_1(\mathbb{R}^n)$. Let $h \in S_n$ *or* $S(\mathbb{R}^n)$ then by applying Fubini's theorem we get

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x_1, x_2, \cdots, x_n)\, h(y_1, \cdots, y_n) \prod_{i=1}^{n} cos(x_i y_i + \frac{\pi}{4})\, \mathrm{d}x_i\, \mathrm{d}y_i \tag{2.9}$$

$$\begin{aligned} &\int_{\mathbb{R}^n} f(x_1, x_2, \cdots, x_n)\, (H_1 h(y_1, y_2, \cdots, y_n)(x_1, x_2, \cdots, x_n))\, \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \\ &= \int_{\mathbb{R}^n} h(y_1, y_2, \cdots, y_n)\, (H_1 f(x_1, x_2, \cdots, x_n))(y_1, y_2, \cdots, y_n)\, \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_n \end{aligned} \tag{2.10}$$

Since $H_1 f \in L_1(\mathbb{R}^n)$ and $g \in \mathscr{S}$ we have by inversion theorem (2.6) to right side of (2.10) and again applying Fubini's theorem, we get

$$\begin{aligned} &\int_{\mathbb{R}^n} f(x_1, x_2, \cdots, x_n)\, (H_1 h(y_1, y_2, \cdots, y_n)(x_1, x_2, \cdots, x_n))\, \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \\ &= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} H_1 h(y_1, y_2, \cdots, y_n) \prod_{i=1}^{n} \cos(x_i y_i + \frac{\pi}{4})\, \mathrm{d}x_i \right) ((H_1 f(x_1, x_2, \cdots, x_n)) \\ &(y_1, y_2, \cdots, y_n)\, \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_n) \\ &= \int_{\mathbb{R}^n} (H_1 h)(x_1, x_2, \cdots, x_n)\, (\frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} (H_1 f(x_1, x_2, \cdots, x_n)) \\ &(y_1, y_2, \cdots, y_n) \prod_{i=1}^{n} \cos(x_i y_i + \frac{\pi}{4})\, \mathrm{d}y_i \end{aligned}$$

$$= \int_{\mathbb{R}^n} g(x_1, x_2, \cdots, x_n)(H_1 h)(x_1, x_2, \cdots, x_n)\, \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n$$

Let $D(\mathbb{R}^n)$ denote the vector space of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support and $D(\mathbb{R}^n) \subset \mathscr{S}$or $D(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} [g(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)]\, \psi(x_1, x_2, \cdots, x_n) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n = 0$$

for every $\psi \in D(\mathbb{R}^n) \Rightarrow g(x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n) = 0$ almost everywhere $\forall (x_1, x_2, \cdots, x_n) \in \mathbb{R}$

$$g(x_i) = f(x_i) \qquad \forall x_i$$

We prove the theorem for H_2 also.

Theorem 2.9. [Uniqueness theorem for H_1 and H_2] *If $f \in L_1(\mathbb{R}^n)$ and $H_1 f = 0$ in $L_1(\mathbb{R}^n)$ then $f = 0$ a.e. in $L_1(\mathbb{R}^n)$ Similarly $f \in L_1(\mathbb{R}^n)$ and $H_2 f = 0$ in $L_1(\mathbb{R}^n)$ then $f = 0$ a.e. in $L_1(\mathbb{R}^n)$.*

Proof. Given, $H_1 f = 0$

$$\Rightarrow \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \cdots, x_n)\, \cos(x_1 x_1' + \frac{\pi}{4}) \cdots \cos(x_n x_n' + \frac{\pi}{4})\, \mathrm{d}x_1' \cdots \mathrm{d}x_n' = 0$$

$$\text{but} \quad |\cos(x_i x_i' + \frac{\pi}{4})| \leq 1$$

$$\Rightarrow \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \cdots, x_n)\, \mathrm{d}x_1' \cdots \mathrm{d}x_n' = 0$$

$$\text{Hence} f(x_1, x_2, \cdots, x_n) = 0 \quad \text{a.e on } \mathbb{R}^n$$

Similarly

$$H_2 f = 0 \quad \text{then} \quad f(x_1, x_2, \cdots, x_n) = 0 \quad \text{a.e on } \mathbb{R}^n$$

Theorem 2.10. [Plancherel's Theorem] *For every $f \in S(\mathbb{R}^n)$ there exist linear isometric operator $\overline{H_1} f = H_1 f$ and $\overline{H_2} f = H_2 f$. Also $\overline{H_1}^2 = I, \overline{H_2}^2 = I, \quad I \in L_2(\mathbb{R}^n)$ is identity operator.*

Proof. By the inversion theorem, if $h, q \in S(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} h(x_1, x_2, \cdots, x_n)\overline{q}(x_1, x_2, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n$$
$$= \int_{\mathbb{R}^n} \overline{q}(x_1, x_2, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} (H_1 h(x_1, x_2, \cdots, x_n))$$
$$(t_1 t_2 \cdots t_n) \times \cos(x_1 t_1 + \frac{\pi}{4}) \cdots \cos(x_n t_n + \frac{\pi}{4})\, \mathrm{d}t_1 \cdots \mathrm{d}t_n$$
$$= \int_{\mathbb{R}^n} H_1 h(x_1, x_2, \cdots, x_n)\, (t_1 t_2 \cdots t_n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} (\overline{q}(x_1, x_2, \cdots, x_n))$$
$$\times \cos(x_1 t_1 + \frac{\pi}{4}) \cdots \cos(x_n t_n + \frac{\pi}{4})\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

By Parseval theorem

$$\int_{\mathbb{R}^n} h(x_1, x_2, \cdots, x_n)\, \overline{q}(x_1, x_2, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n$$
$$= \int_{\mathbb{R}^n} (H_1 h(x_1, x_2, \cdots, x_n))\, (t_1 t_2 \cdots t_n)\, \overline{H_1 q}\, (t_1 t_2 \cdots t_n)\mathrm{d}t_1 \cdots \mathrm{d}t_n \quad f, g \in S(\mathbb{R}^n)$$
$$\text{if} \quad h = q \text{ then } \quad \|h\|_2 = \|H_1 h\|_2 \qquad h \in S(\mathbb{R}^n) \tag{2.11}$$

Here $S(\mathbb{R}^n)$ is dense in $L_2(\mathbb{R}^n)$. Actually $S(\mathbb{R}^n)$ is dense in $L_2(\mathbb{R}^n)$ by theorem (2.6) the map $f \to H_1 f$ is an isometry of dense subspace $S(\mathbb{R}^n)$ of $L_2(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$ which implies $f \to H_1 f$ has a unique continuous extension $\overline{H_1} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$.$\overline{H_1}$ is linear isometry onto $L_2(\mathbb{R}^n)$ see ([2], [6], [9]).

Corollary 2.11. *$\overline{H_1}$ and $\overline{H_2}$ are unitary operators on the Hilbert space $L_2(\mathbb{R}^n)$.*

Theorem 2.12. [Plancherel's theorem for H_1] *Let $h \in (\mathbb{R}^n$ or $\mathbb{C}^n)$ be a function in $L_2(\mathbb{R}^n)$ and let*

$$H_1(x_1, x_2, \cdots, x_n, k_1, k_2, \cdots, k_n) = \frac{1}{\pi^{\frac{n}{2}}} \int_{-k_1}^{k_1} \cdots \int_{-k_n}^{k_n} \prod_{i=1}^{n}$$
$$\cos(x_i y_i + \frac{\pi}{4})\, h(y_1, \cdots y_n)\, \mathrm{d}y_1 \cdots \mathrm{d}y_n$$

then as $k_i \to \infty, H_1(x_i, k_i)$ *converges over* $\mathbb{R}^n$ *to a function in* $L_2(\mathbb{R}^n)$ *and we call it as* $\overline{H_1}h$ *and*

$$h(x_1, x_2, \cdots, x_n, k_1, k_2, \cdots, k_n) = \frac{1}{\pi^{\frac{n}{2}}} \int_{-k_1}^{k_1} \cdots \int_{-k_n}^{k_n} \prod_{i=1}^{n} \cos(x_i y_i + \frac{\pi}{4}) \, (\overline{H_1}h)(y_1, \cdots y_n) \, \mathrm{d}y_1 \cdots \mathrm{d}y_n$$

converges to h *moreover the functions* $(\overline{H_1}h)$ *and* h *are connected by the formulae.*

$$(\overline{H_1}h)(x_1, \cdots, x_n) = \frac{1}{\pi^{\frac{n}{2}}} D^{(n)} \int_{\mathbb{R}^n} h(y_1, \cdots, y_n) \times \prod_{i=1}^{n} \frac{2\sin\left(x_i y_i + \frac{\pi}{4}\right) - \sqrt{2}}{2y_i} \, \mathrm{d}y_1 \cdots \mathrm{d}y_n$$

$$h(x_1, \cdots, x_n) = \frac{1}{\pi^{\frac{n}{2}}} D^{(n)} \int_{\mathbb{R}^n} (\overline{H_1}h)(y_1, \cdots, y_n) \times \prod_{i=1}^{n} \frac{2\sin\left(x_i y_i + \frac{\pi}{4}\right) - \sqrt{2}}{2y_i} \, \mathrm{d}y_1 \cdots \mathrm{d}y_n$$

for every $x \in \mathbb{R}^n$

Proof. Let $h \in L_2(\mathbb{R}^n)$ then we know that there exists sequence of functions $\{h_n\} \in S(\mathbb{R}^n)$ such that

$$\|h_n - h\| \to 0.$$

But we have already proved that

$$\|h\|_2 = \|H_1 h\|_2$$

so

$$\|H_1 h_m - H_1 h_n\|_2 = \|H_1(h_m - h_n)\|_2 = \|h_m - h_n\|_2 \quad \text{for } m, n \in \mathbb{N}$$

This shows that $\{H_1 h_n\}$ is a cauchy sequence which converges to a function $\in L_2(\mathbb{R}^n)$. We denote it by $(\overline{H_1}h)(x_1, \cdots, x_n)$.As $\{h_n\} \in S(\mathbb{R}^n)$ we get,

$$\int_0^{r_1} \cdots \int_0^{r_n} H_1 h_n(x_1, \cdots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{1}{\pi^{\frac{n}{2}}} \prod_{i=1}^{n} \int_0^{r_i} \mathrm{d}x_i \int_{\mathbb{R}^n} h_n(y_1, \cdots, y_n) \cos(x_i y_i + \frac{\pi}{4}) \, \mathrm{d}y_i$$

$$= \frac{1}{\pi^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} h_n(y_1, \cdots, y_n) \prod_{i=1}^{n} \frac{2\sin\left(r_i y_i + \frac{\pi}{4}\right) - \sqrt{2}}{2y_i} \, \mathrm{d}y_1 \cdots \mathrm{d}y_n \tag{2.12}$$

But $\dfrac{2\sin\left(r_i y_i + \dfrac{\pi}{4}\right) - \sqrt{2}}{2y_i} \in L_2(\mathbb{R}^n)$ and $h_n \in S(\mathbb{R}^n)$

We apply Lebegue dominated convergence theorem to the integral in (2.12) and as $n \to \infty$

$$\int\limits_0^{r_1} \cdots \int\limits_0^{r_n} (\overline{H_1} h)(x_1, \cdots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{1}{\pi^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} h(y_1, \cdots, y_n) \prod_{i=1}^{n} \frac{2\sin\left(r_i y_i + \frac{\pi}{4}\right) - \sqrt{2}}{2y_i} \, \mathrm{d}y_1 \cdots \mathrm{d}y_n$$

Hence for every $x \in \mathbb{R}^n$ we get

$$(\overline{H_1} h)(x_1, \cdots, x_n) = \frac{1}{\pi^{\frac{n}{2}}} D^{(n)} \int\limits_{\mathbb{R}^n} h(y_1, \cdots, y_n) \prod_{i=1}^{n} \frac{2\sin\left(r_i y_i + \frac{\pi}{4}\right) - \sqrt{2}}{2y_i} \, \mathrm{d}y_1 \cdots \mathrm{d}y_n \tag{2.13}$$

Now we change h_n to $H_1 h_n$ in (2.12) and by applying theorem (2.6) we get,

$$h(x_1, \cdots, x_n) = \frac{1}{\pi^{\frac{n}{2}}} D^{(n)} \int\limits_{\mathbb{R}^n} (\overline{H_1} h)(y_1, \cdots, y_n) \prod_{i=1}^{n} \frac{2\sin\left(r_i y_i + \frac{\pi}{4}\right) - \sqrt{2}}{2y_i} \, \mathrm{d}y_1 \cdots \mathrm{d}y_n$$

for every $x \in \mathbb{R}^n$. Now we assume that

$$\begin{aligned} h_k(x) &= h(x) \quad \text{for } |x_i| \leq k_i \\ &= 0 \qquad\quad \text{for } |x_i| > k_i \end{aligned}$$

Then

$$h_k \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$$

and

$$\|h_k - h\|_2 \to 0 \quad \text{as} \quad k \to \infty$$

by (2.13) and (2.14) we have

$$
\begin{aligned}
(\overline{H_1}h_k)(x_1,\cdots,x_n) &= \frac{1}{\pi^{\frac{n}{2}}} D^{(n)} \prod_{i=1}^{n} \int_{-k_i}^{k_i} h(y_1,\cdots,y_n)\,\frac{2\sin\left(r_i y_i+\frac{\pi}{4}\right)-\sqrt{2}}{2y_i}\,\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&= \frac{1}{\pi^{\frac{n}{2}}} \prod_{i=1}^{n} \int_{-k_i}^{k_i} h(y_1,\cdots,y_n)\,\cos(x_i y_i+\frac{\pi}{4})\,\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&= H_1(x_1,x_2,\cdots,x_n,k_1,k_2,\cdots,k_n)
\end{aligned}
$$

By Plancherel's theorem and its corollary we have,

$$\|\overline{H_1}h_m-\overline{H_1}h_n\|_2 \;=\; \|h_m-h_n\|_2 \to 0 \quad as \quad m,n\to\infty$$

so $H_1(x_i,k_i)$ converges to $(\overline{H_1}h)(x_1,\cdots,x_n)$ as $k\to\infty$ in $L_2\in(\mathbb{R}^n)$
Similarly we can prove the Plancherel's theorem for H_2.

3. Convolution of H_1

Convolutions: Convolutions played a major role in the development of Mathematics and Physics as well as in many applications in pure and applied Mathematics [11]. Convolution is the way for combining two signals to generate third signal. The generalized convolutions for various integral transform were studied in ([1], [3], [5], [10]).

Definition 3.1. [Convolution] *A map* $*: W\times W\to W$ *is called a convolution for* G *if* $G(*(f,g)) = G(f)\cdot G(g)$ *for any* $f,g\in W$*. This bilinear form is denoted by* $*(f,g)$ *with respect to* G*.*

Theorem 3.2. *If* $f_1, f_2\in L_1(\mathbb{R}^n)$ *then*

$$
\begin{aligned}
H_1(*(f_1,f_2))(x_1,x_2,\cdots,x_n) = \frac{1}{2^n(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} &[f_1(x_i-y_i)+f_1(x_i+y_i)+\\
&f_1(-x_i+y_i)-f_1(-x_i-y_i)]\,f_2(y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n
\end{aligned}
\tag{3.1}
$$

This defines a convolution for H_1*.*

Proof. To prove the theorem first we have to prove that, $H_1(*(f_1, f_2)) \in L_1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |H_1(*(f_1,f_2))(x_1,\cdots,x_n)|\,\mathrm{d}x_1\cdots\mathrm{d}x_n \le \frac{1}{2^n(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} |f_2(y_1,\cdots,y_n)|\,\mathrm{d}y_1\cdots\mathrm{d}y_n$$

$$\times\left[\int_{\mathbb{R}^n}|f_1(x_i-y_i)|\mathrm{d}x_i+\int_{\mathbb{R}^n}|f_1(x_i+y_i)|\mathrm{d}x_i+\int_{\mathbb{R}^n}|f_1(-x_i+y_i)|\mathrm{d}x_i+\int_{\mathbb{R}^n}|f_1(-x_i-y_i)|\mathrm{d}x_i\right]$$

$$\le \frac{2^n}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}|f_2(y_1,\cdots,y_n)|\,\mathrm{d}y_1\cdots\mathrm{d}y_n \times \int_{\mathbb{R}^n}|f_1(x_1,\cdots,x_n)|\,\mathrm{d}x_1\cdots\mathrm{d}x_n < \infty$$

Now,

$$(H_1f_1)(x_1,\cdots,x_n)\cdot(H_1f_2)(x_1,\cdots,x_n) = \frac{2^n}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\prod_{i=1}^{n}\cos(x_it_i+\frac{\pi}{4})$$

$$[f_1(t_i-y_i)+f_1(t_i+y_i)+f_1(-t_i+y_i)-f_1(-t_i-y_i)]\;g(y_i)\mathrm{d}y_1\cdots\mathrm{d}y_n\mathrm{d}t_1\cdots\mathrm{d}t_n$$

$$=\frac{1}{\pi^{\frac{n}{2}}}\int_{\mathbb{R}^n}\prod_{i=1}^{n}\cos(x_it_i+\frac{\pi}{4})\,(f_1*f_2)(t_i)\,\mathrm{d}t_1\cdots\mathrm{d}t_n$$

$$=H_1(*(f_1,f_2))$$

Similarly we can prove the convolution theorem for H_2.

4. Applications

Example 1. *Find the solution of the differential equation* $\dfrac{\partial v}{\partial t}=\dfrac{\partial^4 v}{\partial x_1^2\,\partial x_2^2}$ *where* $v(x_1,x_2,0)=h(x_1,x_2)\quad -\infty<x_1<\infty\,,\;-\infty<x_2<\infty\,,\;t>0.$

Solution. We have

$$V(x_1^{'},x_2^{'},t)\;=H_1v(x_1,x_2,t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v(x_1,x_2,t)\prod_{i=1}^{2}\cos(x_ix_i^{'}+\frac{\pi}{4})\,\mathrm{d}x_1\mathrm{d}x_2$$

v and its derivatives becomes zero at ∞ and $-\infty$ on integration by parts we get

$$\frac{\partial V}{\partial t}=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial v}{\partial t}\prod_{i=1}^{2}\cos(x_ix_i^{'}+\frac{\pi}{4})\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^4 v}{\partial x_1^2 \, \partial x_2^2} \prod_{i=1}^{2} \cos(x_i x_i^{'} + \frac{\pi}{4}) \, \mathrm{d}x_1 \, \mathrm{d}x_2$$

$$= -x_1^{'2} \, x_2^{'2} \, V$$

$$\frac{\partial V}{\partial t} = -x_1^{'2} \, x_2^{'2} \, V$$

$$V(x_1^{'}, x_2^{'}, t) \; = \; A(x_1^{'}, x_2^{'}) \, e^{-x_1^{'2} \, x_2^{'2} \, t} \qquad (\star)$$

On putting $t = 0$ we get,

$$V(x_1^{'}, x_2^{'}, 0) \; = \; A(x_1^{'}, x_2^{'})$$

$$V(x_1^{'}, x_2^{'}, 0) = \frac{1}{\pi} \int_{\mathbb{R}^2} v(x_1, x_2, 0) \prod_{i=1}^{2} \cos(x_i x_i^{'} + \frac{\pi}{4}) \, \mathrm{d}x_i$$

$$= \frac{1}{\pi} \int_{\mathbb{R}^2} h(x_1, x_2) \prod_{i=1}^{2} \cos(x_i x_i^{'} + \frac{\pi}{4}) \, \mathrm{d}x_i$$

$$= (H_1 h)(x_1^{'}, x_2^{'})$$

$$But \quad A(x_1^{'}, x_2^{'}) = V(x_1^{'}, x_2^{'}, 0) = H_1 h(x_1^{'}, x_2^{'})$$

putting in $(\star)$ *we get*

$$V(x_1^{'}, x_2^{'}, t) = \; (H_1 h)(x_1^{'} x_2^{'}) \, e^{-x_1^{'2} \, x_2^{'2} \, t}$$

Taking inverse we get

$$v(x_1, x_2, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (H_1 h)(x_1^{'}, x_2^{'}) \, e^{-x_1^{'2} \, x_2^{'2} \, t} \prod_{i=1}^{2} \cos(x_i x_i^{'} + \frac{\pi}{4}) \, \mathrm{d}x_i^{'}$$

5. Conclusion

Here we have defined Fourier type transform and their convolution on $\mathbb{R}^n$ and obtained its inversion for n dimensional space. We have proved some properties like Convolution, Plancherel's Theorem for n dimensional Fourier type transform. We obtained all these properties for two dimensional space and then extended to n dimensional. Lastly an application for two dimensional Fourier type transform for initial value problem is given.

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