South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 2 (2022), pp. 01-12

DOI: 10.56827/SEAJMMS.2022.1802.1 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

A STUDY ON q-ANALOGUE OF DEGENERATE $\frac{1}{2}$-CHANGHEE NUMBERS AND POLYNOMIALS

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(**Received:** *Sep. 17, 2021* **Accepted:** *Aug. 25, 2022* **Published:** *Aug. 30, 2022)*

Abstract: The aim of the paper is to introduce q-analogue of degenerate $\frac{1}{2}$-Changhee numbers $Ch_{n,q,\lambda,\frac{1}{2}}$ with the help of a p-adic q-integral on $\mathbb{Z}_p$ and derive explicit expressions and some identities for those numbers. In more detail, we deduce explicit expressions of $Ch_{n,q,\lambda,\frac{1}{2}}$, as a rational function in terms of Euler number and Stirling numbers of the first kind, as a fermionic p-adic q-integral on $\mathbb{Z}_p$.

Keywords and Phrases: Degenerate Catalan numbers, q-analogue of degenerate $\frac{1}{2}$-Changhee numbers, p-adic q-integral on $\mathbb{Z}_p$.

2020 Mathematics Subject Classification: 11B83, 11B73, 05A19.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the filed of p-adic rational numbers and the completion of an algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\ .\ |_p$ is normalized by $|\ p\ |_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous function on $\mathbb{Z}_p$. Let q be an indeterminate in $\mathbb{C}_p$ with $|\ 1-q\ |_p < 1$ and q-extension of x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Then the fermionic p-adic q-integral of f on $\mathbb{Z}_p$ is defined by Kim as follows

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-q}(x+p^N\mathbb{Z}_p),$$

$$= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{(see [24])}. \tag{1.1}$$

Let $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{1.2}$$

It is well known that the Euler numbers are defined by

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \text{(see [6-31])}. \tag{1.3}$$

Let q be an indeterminate in $\mathbb{C}_p$ with $\mid 1-q \mid_p < 1$. The q-analogues of Euler numbers are given by

$$\frac{[2]_q}{qe^t+1} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \text{(see [20, 24, 25, 26])}. \tag{1.4}$$

Note that $\lim_{q\to 1} E_{n,q} = E_n, (n \geq 0)$.

The q-analogues of Changhee numbers are given by

$$\frac{[2]_q}{[2]_q+t} = \sum_{n=0}^{\infty} Ch_{n,q} \frac{t^n}{n!}, \text{(see [17, 26, 31])}. \tag{1.5}$$

Kim-Kim [16] introduced the λ-Changhee polynomials are defined by

$$\frac{2}{(1+t)^\lambda+1}(1+t)^{\lambda x} = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x) \frac{t^n}{n!}, \tag{1.6}$$

where $\lambda \in \mathbb{Z}_p$.

When $x=0$, $Ch_{n,\lambda} = Ch_{n,\lambda}(0)$ are called the λ-Changhee numbers.

For $n \geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l, \text{(see [1-5, 16-20])}, \tag{1.7}$$

where $(x)_0 = 1$, and $(x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$. From (1.7), it is easy to see that

$$\frac{1}{r!}(\log(1+t))^r = \sum_{n=r}^{\infty} S_1(n,r) \frac{t^n}{n!}, \quad (r \geq 0), \text{(see [6-15, 21-31])}. \tag{1.8}$$

For $n \geq 0$, the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \text{(see [8-15])}. \tag{1.9}$$

From (1.9), we see that

$$\frac{1}{r!}(e^t - 1)^r = \sum_{n=r}^{\infty} S_2(n,r)\frac{t^n}{n!}. \tag{1.10}$$

As is well known, the Catalan numbers are defined by the generating function as follows (see [1, 2, 3, 21, 22, 31])

$$\frac{2}{1+\sqrt{1-4t}} = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n=0}^{\infty} C_n t^n, \tag{1.11}$$

where $C_n = \binom{2n}{n}\frac{1}{n+1}, (n \geq 0)$.

The Catalan polynomials are defined by the generating function as follows (see [23])

$$\begin{aligned}\int_{\mathbb{Z}_p} (1-4t)^{\frac{x+y}{2}} d\mu_{-1}(y) &= \frac{2}{1+\sqrt{1-4t}}(1-4t)^{\frac{x}{2}} \\ &= \sum_{n=0}^{\infty} C_n(x) t^n. \end{aligned} \tag{1.12}$$

When $x = 0$, $C_n = C_n(0)$ are called the Catalan numbers.

Thus, by (1.11) and (1.12), we have

$$C_n(x) = \sum_{m=0}^{n}\sum_{j=0}^{m}\left(\frac{x}{2}\right)^j S_1(m,j)(-4)^m \frac{C_{n-m}}{m!}.$$

Kim introduced the $\frac{1}{2}$-Changhee polynomials which are given by the generating function (see [22])

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+t)^{\frac{x+y}{2}} d\mu_{-1}(y) &= \frac{2}{1+\sqrt{1+t}}\sqrt{(1+t)^x} \\ &= \sum_{n=0}^{\infty} Ch_{n,\frac{1}{2}}(x)\frac{t^n}{n!}. \end{aligned} \tag{1.13}$$

When $x = 0$, $Ch_{n,\frac{1}{2}} = Ch_{n,\frac{1}{2}}(0)$ are called the $\frac{1}{2}$-Changhee numbers.

On replacing t by $-4t$ in (1.13) and by using (1.12), we have

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{x+y}{2}}d\mu_{-1}(y) = \frac{2}{1+\sqrt{1-4t}}\sqrt{(1-4t)^x}$$

$$= \sum_{n=0}^{\infty} Ch_{n,\frac{1}{2}}(x)(-4)^n\frac{t^n}{n!}.$$

$$\sum_{n=0}^{\infty} C_n(x)t^n = \sum_{n=0}^{\infty} Ch_{n,\frac{1}{2}}(x)(-4)^n\frac{t^n}{n!}. \tag{1.14}$$

Comparing the coefficients of t, we get

$$C_n(x) = \frac{(-1)^n}{n!}Ch_{n,\frac{1}{2}}(x)2^{2n}.$$

Recently, Kim *et al.* [21] introduced the q-analogues of Catalan polynomials which are given by

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{x+y}{2}}d\mu_{-q}(y) = \frac{[2]_q}{1+q\sqrt{1-4t}}(1-4t)^{\frac{x}{2}}$$

$$= \sum_{n=0}^{\infty} C_{n,q}(x)t^n. \tag{1.15}$$

When $x = 0$, $C_{n,q} = C_{n,q}(0)$ are called the q-Catalan numbers.

In this paper, we study q-analogue of degenerate Catalan numbers associated with p-adic q-integral on $\mathbb{Z}_p$ and derive some identities of these numbers and polynomials. Also, we define q-analogue of degenerate $\frac{1}{2}$-Changhee numbers by using p-adic q-integral on $\mathbb{Z}_p$ and deduce some properties of them.

2. q-analogue of Degenerate $\frac{1}{2}$-Changhee Numbers and Polynomials

For $\lambda, t, q \in \mathbb{C}_p$ with $\mid 1-q \mid < 1$ and $\mid \lambda t \mid < p^{-\frac{1}{p-1}}$. Now, we define the q-analogue of degenerate $\frac{1}{2}$-Changhee numbers which are given by the generating function

$$\int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{x}{2}}d\mu_{-q}(x) = \frac{[2]_q}{1+q\sqrt{1+\log(1+\lambda t)^{\frac{1}{\lambda}}}}$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\lambda,\frac{1}{2}}\frac{t^n}{n!}. \tag{2.1}$$

Note that

$$\lim_{\lambda\to 0} Ch_{n,q,\lambda,\frac{1}{2}} = Ch_{n,q,\frac{1}{2}}, (n \geq 0).$$

From (1.2), we note that

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q}{qe^t+1} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}. \tag{2.2}$$

Thus, by (2.3), we get

$$\int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = E_{n,q}, (n \geq 0). \tag{2.3}$$

On the other hand, from (2.3), we note that

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{x}{2}} d\mu_{-q}(x) &= \sum_{m=0}^{\infty}\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x)\frac{1}{2^m}\frac{1}{m!}(\log(1+\lambda t)^{\frac{1}{\lambda}})^m \\ &= \sum_{m=0}^{\infty} E_{m,q}2^{-m}\lambda^{n-m}\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} E_{m,q}2^{-m}\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}. \end{aligned}\tag{2.4}$$

Therefore, by (2.1) and (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$Ch_{n,q,\lambda,\frac{1}{2}} = \sum_{m=0}^{n} E_{m,q}2^{-m}\lambda^{n-m}S_1(n,m).$$

By replacing t with $\frac{1}{\lambda}[e^{-4\lambda t}-1]$ in (2.1), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} (1-4t)^{\frac{x}{2}} d\mu_{-q}(x) &= \sum_{m=0}^{\infty} Ch_{m,q,\lambda,\frac{1}{2}}\frac{(\frac{1}{\lambda}[e^{-4\lambda t}-1])^m}{m!} \\ &= \sum_{m=0}^{\infty} Ch_{m,q,\lambda,\frac{1}{2}}\lambda^{-m}\sum_{n=m}^{\infty} S_2(n,m)(-4)^n\lambda^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} Ch_{m,q,\lambda,\frac{1}{2}}\lambda^{n-m}(-1)^n 2^{2n}S_2(n,m)\right)\frac{t^n}{n!}. \end{aligned}\tag{2.5}$$

On the other hand,

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{x}{2}}d\mu_{-q}(x)=\frac{[2]_q}{1+q\sqrt{1-4t}}=\sum_{n=0}^{\infty}C_{n,q}t^n. \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

Theorem 2.2. *For $n\geq 0$, we have*

$$C_{n,q}=\frac{(-1)^n2^{2n}}{n!}\sum_{m=0}^{n}Ch_{m,q,\lambda,\frac{1}{2}}\lambda^{n-m}S_2(n,m).$$

From (2.6), we have

$$\sum_{n=0}^{\infty}(-1)^n4^n\int_{\mathbb{Z}_p}\binom{\frac{x}{2}}{n}d\mu_{-q}(x)t^n=\sum_{n=0}^{\infty}C_{n,q}t^n. \tag{2.7}$$

From (2.4), we have the following equation

$$\int_{\mathbb{Z}_p}\binom{\frac{x}{2}}{n}d\mu_{-q}(x)=\frac{1}{n!}\sum_{m=0}^{n}E_{m,q}2^{-m}\lambda^{n-m}S_1(n,m). \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.3. *For $n\geq 0$, we have*

$$C_{n,q}=\frac{(-1)^n}{n!}\sum_{m=0}^{n}E_{m,q}2^{2n-m}\lambda^{n-m}S_1(n,m).$$

From (2.1), we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{x}{2}}d\mu_{-q}(x)&=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{x}{2}}{m}d\mu_{-q}(x)\frac{[\log(1+\lambda t)^{\frac{1}{\lambda}}]^m}{m!}\\&=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{x}{2}}{m}d\mu_{-q}(x)\lambda^{n-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\int_{\mathbb{Z}_p}\binom{\frac{x}{2}}{m}d\mu_{-q}(x)\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.9}$$

Therefore, by (2.1) and (2.9), we get the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$Ch_{n,q,\lambda,\frac{1}{2}} = \sum_{m=0}^{n} \int_{\mathbb{Z}_p} \binom{\frac{x}{2}}{m} d\mu_{-q}(x) \lambda^{n-m} S_1(n,m).$$

First, we note that

$$(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{1}{2}} = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m \frac{[\log(1+\lambda t)^{\frac{1}{\lambda}}]^m}{m!}$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m \lambda^m \sum_{n=m}^{\infty} S_1(n,m) \frac{\lambda^n t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \lambda^{n-m} S_1(n,m) \right) \frac{t^n}{n!}. \quad (2.10)$$

By (2.1) and (2.10), we get

$$[2]_q = \left(\sum_{n=0}^{\infty} Ch_{n,q,\lambda,\frac{1}{2}} \frac{t^n}{n!} \right) \left((1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{1}{2}} \right)$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\lambda,\frac{1}{2}} \frac{t^n}{n!} + \left(\sum_{n=0}^{\infty} Ch_{n,\lambda,\frac{1}{2}} \frac{t^n}{n!} \right) \left(\sum_{m=0}^{k} \left(\frac{1}{2}\right)^m \lambda^{k-m} S_1(k,m) \frac{t^k}{k!} \right)$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\lambda,\frac{1}{2}} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \lambda^{k-m} \left(\frac{1}{2}\right)^m S_1(k,m) Ch_{n-k,q,\lambda,\frac{1}{2}} \right) \frac{t^n}{n!}. \quad (2.11)$$

By comparing the coefficients of t on both sides, we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$Ch_{n,q,\lambda,\frac{1}{2}} + \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \lambda^{k-m} \left(\frac{1}{2}\right)^m S_1(k,m) Ch_{n-k,q,\lambda,\frac{1}{2}} = \begin{cases} [2]_q, & if n = 0 \\ 0, & if n > 1. \end{cases}$$

By replacing t by $-\frac{1}{4}\log(1+\lambda t)^{\frac{1}{\lambda}}$ in (1.15), we get

$$\frac{[2]_q}{1+q\sqrt{1+\log(1+\lambda t)^{\frac{1}{\lambda}}}} = \sum_{m=0}^{\infty} C_{m,q} m! \frac{[-\frac{1}{4}\log(1+\lambda t)^{\frac{1}{\lambda}}]^m}{m!}$$

$$= \sum_{m=0}^{\infty} C_{m,q} \frac{(-1)^m}{4^m} \lambda^{-m} m! \frac{(\log(1+\lambda t))^m}{m!}$$

$$= \sum_{m=0}^{\infty} C_{m,q} \frac{(-1)^m}{4^m} \lambda^{-m} m! \sum_{n=m}^{\infty} S_1(n,m) \frac{\lambda^n t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{(-1)^m}{4^m} C_{m,q} \lambda^{n-m} m! S_1(n,m) \right) \frac{t^n}{n!}. \tag{2.12}$$

Therefore, by (2.1) and (2.12), we get the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$Ch_{n,q,\lambda,\frac{1}{2}} = \sum_{m=0}^{n} \frac{(-1)^m}{4^m} C_{m,q} \lambda^{n-m} m! S_1(n,m).$$

Now, we observe that

$$(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{x}{2}} = \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^m \frac{[\log(1+\lambda t)^{\frac{1}{\lambda}}]^m}{m!}$$

$$= \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^m \lambda^{-m} \sum_{n=m}^{\infty} S_1(n,m) \frac{\lambda^n t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left(\frac{x}{2}\right)^m \lambda^{n-m} S_1(n,m) \right) \frac{t^n}{n!}. \tag{2.13}$$

Now, we consider the q-analogue of degenerate $\frac{1}{2}$-Changhee polynomials which are given by the generating function as follows

$$\int_{\mathbb{Z}_p} (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{x+y}{2}} d\mu_{-q}(y) = \frac{[2]_q}{1+q\sqrt{1+\log(1+\lambda t)^{\frac{1}{\lambda}}}} (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{x}{2}}$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\lambda,\frac{1}{2}}(x) \frac{t^n}{n!}. \tag{2.14}$$

When $x = 0$, $Ch_{n,q,\lambda,\frac{1}{2}} = Ch_{n,q,\lambda,\frac{1}{2}}(0)$ are called the q-analogue of degenerate $\frac{1}{2}$-Changhee numbers. From (2.14), we note that

$$\frac{[2]_q}{1+q\sqrt{1+\log(1+\lambda t)^{\frac{1}{\lambda}}}} (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{x}{2}}$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\lambda,\frac{1}{2}} \frac{t^n}{n!} \sum_{m=0}^{\infty} \binom{\frac{x}{2}}{m} m! \frac{(\log(1+\lambda t)^{\frac{1}{\lambda}})^m}{m!}$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\lambda,\frac{1}{2}} \frac{t^n}{n!} \sum_{m=0}^{\infty} \binom{\frac{x}{2}}{m} \lambda^{-m} m! \sum_{l=m}^{\infty} S_1(l,m) \frac{\lambda^l t^l}{l!}$$

$$= \sum_{n=0}^{\infty} Ch_{n,q,\lambda,\frac{1}{2}} \frac{t^n}{n!} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{\frac{x}{2}}{m} m! \lambda^{l-m} m! S_1(l,m) \frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \binom{\frac{x}{2}}{m} \lambda^{l-m} S_1(l,m) Ch_{n-l,q,\lambda,\frac{1}{2}} m! \right) \frac{t^n}{n!}. \quad (2.15)$$

By (2.14) and (2.15), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$Ch_{n,q,\lambda,\frac{1}{2}}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \binom{\frac{x}{2}}{m} \lambda^{l-m} S_1(l,m) Ch_{n-l,q,\lambda,\frac{1}{2}} m!.$$

Replacing t with $\frac{e^{-4\lambda t}-1}{\lambda}$ in (2.14), we have

$$\int_{\mathbb{Z}_p} (1-4t)^{\frac{x+y}{2}} d\mu_{-q}(y) = \frac{[2]_q}{1+q\sqrt{1-4t}} \sqrt{(1-4t)^x} = \sum_{n=0}^{\infty} C_{n,q}(x) t^n. \quad (2.16)$$

On the other hand, we have

$$\sum_{m=0}^{\infty} Ch_{m,q,\lambda,\frac{1}{2}}(x) \frac{[\frac{e^{-4\lambda t}-1}{\lambda}]^m}{m!} = \sum_{m=0}^{\infty} Ch_{m,q,\lambda,\frac{1}{2}}(x) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n,m) \frac{(-4)^n \lambda^n t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} Ch_{m,q,\lambda,\frac{1}{2}}(x) \lambda^{n-m} (-1)^n 2^{2n} S_2(n,m) \right) \frac{t^n}{n!}. \quad (2.17)$$

Therefore, by (2.16) and (2.17), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$C_{n,q}(x) = \frac{(-1)^n}{n!} \sum_{m=0}^{n} Ch_{m,q,\lambda,\frac{1}{2}}(x) \lambda^{n-m} 2^{2n} S_2(n,m).$$

From (1.2), we see that

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{[2]_q}{qe^t+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \quad (2.18)$$

From (2.14), we have

$$\frac{[2]_q}{1+q\sqrt{1+\log(1+\lambda t)^{\frac{1}{\lambda}}}}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{x}{2}} = \int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{\frac{x+y}{2}}d\mu_{-q}(y)$$

$$= \sum_{m=0}^{\infty} 2^{-m}\frac{1}{m!}(\log(1+\lambda t)^{\frac{1}{\lambda}})^m \int_{\mathbb{Z}_p}(x+y)^m d\mu_{-q}(y)$$

$$= \sum_{m=0}^{\infty} 2^{-m}E_{m,q}(x)\lambda^{-m}\sum_{n=m}^{\infty} S_1(n,m)\frac{\lambda^n t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} 2^{-m}E_{m,q}(x)\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}. \tag{2.19}$$

Thus, by (2.14) and (2.19), we get the following theorem.

Theorem 2.9. *For $n \geq 0$, we have*

$$Ch_{n,,q,\lambda,\frac{1}{2}}(x) = \sum_{m=0}^{n} 2^{-m}E_{m,q}(x)\lambda^{n-m}S_1(n,m).$$

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