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ON CERTAIN CONTINUED FRACTIONS INVOLVING BASIC BILATERAL HYPERGEOMETRIC FUNCTION $_2\Psi_2$

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Abstract: In this paper we establish certain continued fractions representation for the ratio of two $_2\Psi_2$'s. We also discuss certain continued fraction representation for the ratio of two $_3\Psi_2$'s with two bases and $_4\Psi_3$'s with one base.

Keywords and Phrases: Basic bilateral hypergeometric series, continued fraction, basic hypergeometric series and poly-basic q-series.

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1. Introduction

Since ancient time continued fractions have been playing a very important role in Number theory and Classical Analysis. The Indian mathematician Arya Bhatt (475-550 AD) used a continued fraction to solve a linear equation [9]. In the beginning of 20^{th} century, the theory of continued fractions got advancement due to the Indian genius Srinivasa Ramanujan. Chapter 12 of Ramanujan's second notebook [11] is entirely devoted to the study of continued fractions.

Various continued fractions representations for the ratio of two $_2\Psi_2$'s are known in the literature. A good number of them are established by Bhagirathi [2], Denis [5], Gupta [7], Pathak and Srivastava [10] and Srivastava [12]. The region of convergence of some of these collapses unless one of the denominator parameters is of the form $q^n (n \in N)$.

In the present paper, by making use of a known transformation established by Bailey [1], we attempt to establish the continued fraction representation for the ratio of two ${}_2\Psi_2$'s, where above problem does not arise. Also, by making use of a known transformation of a poly-basic q-series established by Denis and Singh [6], we established the continued fraction representation for the ratio of two ${}_3\Psi_2$'s with two bases q and q^2.

2. Definitions and Notations

We shall use the following definitions and notations throughout the paper. A generalized bi-basic hypergeometric function of one variable is defined as,

$$ {}_{A+B}\Phi_{C+D}\left[\begin{array}{l}(a);(b);q,q_1;z\\(c);(d);q^i,q_1^j\end{array}\right]=\sum_{n=0}^{\infty}\frac{[(a);q]_n[(b);q_1]_n z^n q^{i\binom{n}{2}}q_1^{j\binom{n}{2}}}{[q;q]_n[(c);q]_n[(d);q_1]_n} \tag{2.1}$$

where (a) stands for the sequence of A parameters $a_1, a_2..., a_A$.
Also, for $|q| < 1$ and arbitrary a
$[a;q]_n \equiv (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), \quad n > 0$ and $[a;q]_0 = 1$.
Further $\binom{n}{2} \equiv \frac{n(n-1)}{2}$.
The series on the right of (2.1) converges for $|q|, |q_1| < 1, |z| < \infty$, when $i, j > 0$ and $|q|, |q_1|, |z| < 1$ when $i = 0 = j$, in which case we drop q^i and q_1^j, from the notation.

We also define a generalized basic hypergeometric series with one base as,

$$ {}_A\Phi_B\left[\begin{array}{l}(a);q;z\\(b)\end{array}\right]=\sum_{n=0}^{\infty}\frac{[(a);q]_n z^n}{[q;q]_n[(b);q]_n} \tag{2.2}$$

valid for $|z| < 1, |q| < 1$.
We further, define a generalized basic bilateral hypergeometric series,

$$ {}_r\Psi_r\left[\begin{array}{l}(a_r);q;z\\(b_r)\end{array}\right]=\sum_{n=-\infty}^{\infty}\frac{[(a);q]_n z^n}{[(b);q]_n} \tag{2.3}$$

for, $\left|\frac{b_1 b_2 ... b_r}{a_1 a_2 ... a_r}\right| < |z| < 1$ and the parameters (a_r) stands for the sequence of parameters $a_1, a_2, ..., a_r$. (2.3) reduces to ${}_r\Phi_{r-1}$ when any of the denominator parameters is q. also,

$$ \prod\left[\begin{array}{l}a;q\\b\end{array}\right]=\frac{[a;q]_\infty}{[b;q]_\infty}. \tag{2.4}$$

3. Main Results

In this paper we shall establish the following results,

$$\frac{(abz-c)(abz-q)}{(1-az)(1-bz)a^2b^2z^2}\times\frac{{}_2\Psi_2\left[\begin{matrix}abzq/c, abz; q; \frac{c}{abz}\\ azq, bzq\end{matrix}\right]}{{}_2\Psi_2\left[\begin{matrix}abz/c, abz/q; q; \frac{cq}{abz}\\ az, bz\end{matrix}\right]}$$
$$=\frac{1}{1+}\,\frac{(1-a)(1-b)z}{1-z+}\,\frac{abzq-c}{1+}\,\frac{(1-aq)(1-bq)z}{1-z+}\,\frac{abzq^3-c}{1+\ldots} \tag{3.1}$$

$$\frac{(1-bz)}{(1-bzq)}\times\frac{{}_3\Phi_2\left[\begin{matrix}b, a; aq^2; q, q^2; zq\\ aq/b; a\end{matrix}\right]}{{}_3\Phi_2\left[\begin{matrix}b, a; aq^2; q, q^2; z\\ aq/b; a\end{matrix}\right]}$$
$$=\frac{1}{1+}\,\frac{(1-aq)(1-bq)z}{1-z+}\,\frac{abzq^3-aq/b}{1+}\,\frac{(1-aq^2)(1-bq^2)z}{1-z+}\,\frac{abzq^5-aq^2/b}{1+\ldots} \tag{3.2}$$

$$\frac{(1-bz)}{(1-bzq)}\times\frac{{}_4\Phi_3\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, b; q; zq\\ \sqrt{a}, -\sqrt{a}, aq/b\end{matrix}\right]}{{}_4\Phi_3\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, b; q; z\\ \sqrt{a}, -\sqrt{a}, aq/b\end{matrix}\right]}$$
$$=\frac{1}{1+}\,\frac{(1-aq)(1-bq)z}{1-z+}\,\frac{abzq^3-aq/b}{1+}\,\frac{(1-aq^2)(1-bq^2)z}{1-z+}\,\frac{abzq^5-aq^2/b}{1+\ldots} \tag{3.3}$$

4. Proof. Bailey [1; 3.12] established the following transformation of a ${}_2\Psi_2$ into another ${}_2\Psi_2$,

$${}_2\Psi_2\left[\begin{matrix}a, b; q; z\\ c, d\end{matrix}\right]=\prod\left[\begin{matrix}az, bz, cq/abz, dq/abz; q\\ q/a, q/b, c, d\end{matrix}\right]{}_2\Psi_2\left[\begin{matrix}abz/c, abz/d; q; cd/abz\\ az, bz\end{matrix}\right] \tag{4.1}$$

Taking $d = q$ we get the following transformation of a ${}_2\Psi_2$ into a

$$ {}_2\Psi_2 \left[\begin{matrix} abz/c, abz/q; q; \frac{cq}{abz} \\ az, bz \end{matrix} \right] = \prod \left[\begin{matrix} q, c, q/a, q/b; q \\ az, bz, \frac{cq}{abz}, \frac{q^2}{abz} \end{matrix} \right] {}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ c \end{matrix} \right] \tag{4.2} $$

Also, Denis [4; 3.9] established the following continued fraction for the ratio of two ${}_2\Phi_1$'s,

$$ \frac{{}_2\Phi_1 \left[\begin{matrix} a, b; q; zq \\ c \end{matrix} \right]}{{}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ c \end{matrix} \right]} = \frac{1}{1+} \frac{(1-a)(1-b)z}{1-z+} \frac{abzq - c}{1+} \frac{(1-aq)(1-bq)z}{1-z+\ldots} \tag{4.3} $$

Now, replacing the two ${}_2\Phi_1$'s on the left of (4.3) by their equivalent ${}_2\Psi_2$'s with the help of (4.2), we easily get (3.1).

Denis and Singh [6] established the following transformation of ${}_3\Phi_2$ with three bases q, pq and p into ${}_2\Phi_1$ with two bases q and p,

$$ {}_3\Phi_2 \left[\begin{matrix} b; apq; a; q, pq, p; z \\ -; a; ap/b \end{matrix} \right] = (1 - bz)\ {}_2\Phi_1 \left[\begin{matrix} bq; ap; q, p; z \\ -; ap/b \end{matrix} \right] \tag{4.4} $$

Setting, $p = q$ in (4.4) we get,

$$ {}_3\Phi_2 \left[\begin{matrix} b, a; aq^2; q, q^2; z \\ aq/b, a \end{matrix} \right] = (1 - bz)\ {}_2\Phi_1 \left[\begin{matrix} aq, bq; q; z \\ aq/b \end{matrix} \right] \tag{4.5} $$

By making use of the transformation (4.5) into (4.3) we obtain (3.2).
Also, (4.5) can be rewritten as,

$$ {}_4\Phi_3 \left[\begin{matrix} a, q\sqrt{a}; -q\sqrt{a}, b; q; z \\ \sqrt{a}, -\sqrt{a}, aq/b \end{matrix} \right] = (1 - bz)\ {}_2\Phi_1 \left[\begin{matrix} aq, bq; q; z \\ aq/b \end{matrix} \right] \tag{4.6} $$

By making use of transformation (4.6) into (4.3) we get (3.3).

It is quite difficult to find the continued fraction representation for the ratio of ${}_3\Phi_2$, ${}_4\Phi_3$ and higher order with general argument, but from the above results it is evident that the same can be achieved with the help of suitable transformations.

Also, it has not been possible to establish continued fraction representation for the ratio of basic hypergeometric function with more than one base. In this paper we have showed that this is also possible with the help of certain transformation, which express a q-series with more than one base into another q-series with one base. Thus success of this attempt depends on the existence of suitable transformation.

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