*South East Asian J. of Mathematics and Mathematical Sciences*
*Vol. 18, No. 2 (2022), pp. 185-196*

DOI: 10.56827/SEAJMMS.2022.1802.17 **ISSN (Online): 2582-0850**

**ISSN (Print): 0972-7752**

# ON THE COMPLETE PRODUCT OF FUZZY GRAPHS

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(**Received:** *Jan. 02, 2022* **Accepted:** *Aug. 05, 2022* **Published:** *Aug. 30, 2022)*

**Abstract:** Strong fuzzy graph and complete fuzzy graph are two special kinds of fuzzy graphs in which each edge membership value equals the least of its vertex membership values incidental to the edge. In this research study, we obtain some interesting results on the complete product of a pair of strong fuzzy graphs as well as the complete product of a pair of complete fuzzy graphs. In precise, we prove that the complete product of two strong fuzzy graphs is again a strong fuzzy graph and the complete product of two complete fuzzy graphs is again a complete fuzzy graph. Also, we discuss the conditions under which the property of regularity will be mutually transmitted between the complete product of two fuzzy graphs and one of its factor fuzzy graphs.

**Keywords and Phrases:** Strong fuzzy graph, Complete fuzzy graph, Regularity of a fuzzy graph, The complete product of fuzzy graphs.

**2020 Mathematics Subject Classification:** 05C72, 05C76.

## 1. Introduction

A graph is a convenient way of representing information involving relationships between objects. The objects are represented by vertices and relations by edges.

When there is vagueness in the description of the objects or their relationships or both, it is natural that we need to design a 'Fuzzy Graph Model'. The application of fuzzy relations is widespread and important in the field of clustering analysis, neural networks, computer networks, pattern recognition, decision making, and expert systems. In each of these, the basic mathematical structure is that of a fuzzy graph. Kaufmann [3] defined the fuzzy graph for the first time in 1973, based on Zadeh's fuzzy relations [12]. But it was Azriel Rosenfeld [9] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. Mordeson and Chang-Shyh [4] defined operations on fuzzy graphs. Nagoor Gani and Radha [8] described some properties of regular fuzzy graphs and the degree of a vertex in some fuzzy graphs. Muhammad Akram and Saira Nawaz [7] introduced the concepts of soft graphs and described some operations on soft graphs.

The complete product of fuzzy graphs introduced by Chaitanya and Pradeep Kumar [1] is a binary operation on the set of all fuzzy graphs defined on a nonempty set S. The set of all fuzzy graphs on a non-empty set S is closed under the operation of the complete product and also satisfies commutative law ($G \times_P H = H \times_P G$). It allows us to create a new fuzzy graph from the given pair of fuzzy graphs and paves the way to develop some operational properties over the set of all fuzzy graphs. Prior to the introduction of the complete product of fuzzy graphs, different types of product operations like the Cartesian product of fuzzy graphs, Tensor product of fuzzy graphs, Normal product of fuzzy graphs, Modular product of fuzzy graphs, etc. are defined by various mathematicians on a pair of fuzzy graphs. But all of them are defined on restricted domains (proper subsets of $U \times V$), not on the whole Cartesian product of $U \times V$, where U and V are the vertex sets of the two fuzzy graphs. This is the reason why one could consider the complete product stronger than the other products of fuzzy graphs. Fuzzy-graph theory has a large number of applications in modeling various real-time systems where the information inherent in the system is vague and uncertain. The complete product of fuzzy graphs is a new framework to handle fuzzy information by combining a pair of fuzzy graphs.

In this research study, we obtain some interesting results on the complete product of a pair of strong fuzzy graphs as well as the complete product of a pair of complete fuzzy graphs. In precise, we prove that the complete product of two strong fuzzy graphs is again a strong fuzzy graph and the complete product of two complete fuzzy graphs is again a complete fuzzy graph. Also, we discuss the conditions under which the property of regularity will be mutually transmitted between the complete product of two fuzzy graphs and one of its factor fuzzy graphs.

## 2. Preliminaries

In this section we have given the definitions and examples related to the concepts which are discussed in the introduction.

**Definition 2.1.** [2, 7] *A graph G is a triplet consisting of a vertex set V (G), an edge set E(G), and a relation that associates two vertices with each edge. The two vertices are called its endpoints (not necessarily distinct). Graphically, we represent a graph by drawing a point for each vertex and representing each edge by a curve joining its endpoints.*

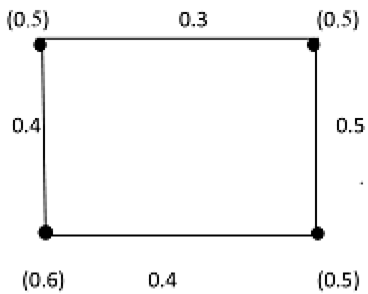
**Definition 2.2.** [13] *Let S be a non - empty set. Then any Mapping* $\sigma : S \rightarrow [0,1]$ *is called a fuzzy set on S. The reference set S is called universe of discourse, and for each* $x \in S$*, the value* $\sigma(x)$ *is called the grade of membership of x in* $(S.\sigma)$*.*

**Definition 2.3.** [9, 12] *Let S be a non - empty set. Let* $\mu : S \times S \rightarrow [0,1]$ *and* $\sigma : S \rightarrow [0,1]$ *be any two fuzzy sets.Then* $\mu$ *is said to be a fuzzy relation on* $\sigma$ *if* $\mu(x,y) \leq \sigma(x) \cap \sigma(y) \quad \forall x,y \in S$*, where* $\sigma(x) \cap \sigma(y)$ = *minimum of* $(\sigma(x), \sigma(y))$*.*

**Definition 2.4.** [9, 12] *i) A fuzzy relation is called Symmetric if* $\mu(x,y) = \mu(y,x) \quad \forall x,y \in S$*. ii) A fuzzy relation is called Reflexive if* $\mu(x,y) = \sigma(x) \quad \forall x \in S$*.*

**Definition 2.5.** [3, 5, 6, 9] *A fuzzy graph G is a pair of functions* $G : (\sigma, \mu)$ *where* $\sigma$ *is a fuzzy subset on a non-empty set V and* $\mu$ *is a symmetric fuzzy relation on* $\sigma$*. The underlying crisp graph of* $G : (\sigma, \mu)$ *is denoted by* $\hat{G^*}(V,E)$ *where* $E \subseteq V \times V$*.*

**2.5 Example:** Here we have given an example of fuzzy graph.




Here 'σ' values are written in the brackets at the vertices and 'μ' values are written in the middle of the edges

Figure 2.5.1.

**Note 2.6.** [11] *In any fuzzy graph* $G : (\sigma, \mu)$ *on* $\hat{G^*}(V,E)$ *, we assume that V is finite and* $\mu$ *is reflexive on* $\sigma$*. Also, we ignore the loops and parallel edges. Let* $\sigma^* = \{x \in V / \sigma(x) > 0\}$ *and let* $\mu^* = \{(x,y) \in V \times V / \mu(x,y) > 0\}$*.*

**Definition 2.7.** [11] *A fuzzy graph* $G : (\sigma, \mu)$ *is said to be a strong fuzzy graph if* $\mu(x,y) = \sigma(x) \cap \sigma(y) \quad \forall x,y \in \hat{\mu^*}$*.*

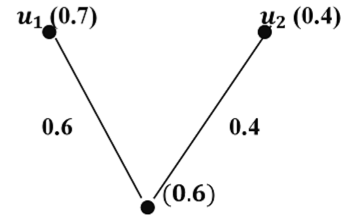



Figure 2.7.1.

**Definition 2.8.** [11] *A fuzzy graph G:$(\sigma, \mu)$ is said to be a complete fuzzy graph if* $\mu(x,y) = \sigma(x) \cap \sigma(y) \quad \forall x,y \in \hat{\sigma^*}$.

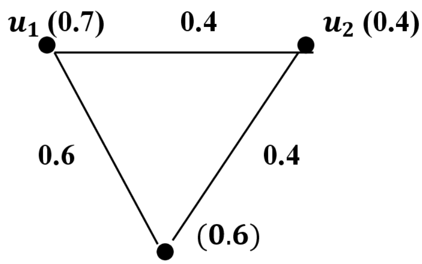



Figure 2.8.1.

**Definition 2.9.** [8, 10] *Let G:$(\sigma, \mu)$ be a fuzzy graph. The degree of a vertex v is defined as* $d_G\ (v) = \sum \mu((u,v))$, $\forall\ u \in U$ *such that* $u \neq v$. *Since* $\mu\ (u, v) > 0$ *for* $(u, v) \in E$ *and* $\mu\ (u, v) = 0$ *for* $(u, v) \notin E$, *this is equivalent to*

$$d_G(u) = \sum_{uv \in E} \mu(u, v)$$

**Definition 2.10.** [8, 10] *Let G:$(\sigma, \mu)$ be a fuzzy graph . If* $d_G\ (v) = k\ \forall\ v \in V$, *(i.e.) if each vertex has the same degree k, then G is said to be a regular fuzzy graph of degree k or a k-regular fuzzy graph.*

**Definition 2.11. The Complete Product of Two Fuzzy Graphs** [1, 4] *Let G:$(\sigma, \mu)$ and H: $(\tau, \vartheta)$ are any two fuzzy graphs then their complete product is defined as* $\mathrm{G} \times_p \mathrm{H} : (\sigma \times_p \tau, \mu \times_p \vartheta) = (U \times V, \mathrm{E})$ *where* $\mathrm{E} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8$ *Such that*

$E_1 = \{((u_1, v_1), (u_2, v_2)) : u_1 = u_2 \ \ and \ \ (v_1, v_2) \in E_V\}$ *and*

$E_2 = \{((u_1, v_1), (u_2, v_2)) : u_1 = u_2 \ \ and \ \ (v_1, v_2) \notin E_V\} \ \ and$

$E_3 = \{((u_1, v_1), (u_2, v_2)) : v_1 = v_2 \ \ and \ \ (u_1, u_2) \in E_U\} \ \ and$

$E_4 = \{((u_1, v_1), (u_2, v_2)) : \ v_1 = v_2 \ \ and \ \ (u_1, u_2) \notin E_u\} \ \ and$

$E_5 = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \in E_U \ \ and \ \ (v_1, v_2) \ \notin E_V\} \ \ and$

$E_6 = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \notin E_v \ \ and \ \ (v_1, v_2) \in E_V\} \ \ and$

$E_7 = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \in E_U \ \ and \ \ (v_1, v_2) \in E_V\}$

$E_8 = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \notin E_U \ \ and \ \ (v_1, v_2) \notin E_V\}$

*With* $\sigma \times_P \tau(u, v) = \sigma(u) \cap \tau(v) \quad \forall (u, v) \in U \times V$

$$and \ \ \mu \times_p \vartheta)(\mathrm{w}) = \begin{cases} \sigma(u_1) \cap \vartheta(v_1, v_2), & if \ w \in E_1 \\ \sigma(u_1) \cap \tau(v_1) \cap \tau(v_2), & if \ w \in E_2 \\ \mu(u_1, u_2) \cap \tau(v_1), & if \ w \in E_3 \\ \sigma(u_1) \cap \sigma(u_2) \cap \tau(v_1), & if \ w \in E_4 \\ \mu(u_1, u_2) \cap \tau(v_1) \cap \tau(v_2), & if \ w \in E_5 \\ \sigma(u_1) \cap \sigma(u_2) \cap \vartheta(v_1, v_2), & if \ w \in E_6 \\ \mu(u_1, u_2) \cap \vartheta(v_1, v_2) & if \ w \in E_7 \\ \sigma(u_1) \cap \sigma(u_2) \cap \tau(v_1) \cap \tau(v_2), & if \ w \in E_8 \end{cases}$$

*Where* $\mathrm{w} = ((u_1, v_1), (u_2, v_2))$

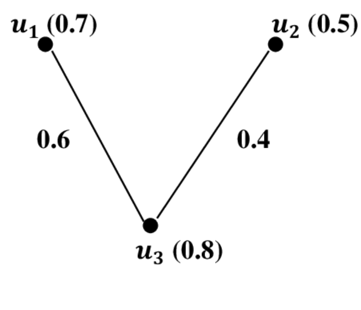



Fig (1): G

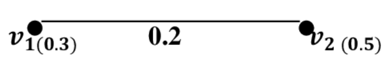



Fig (2): H

**The following diagram represents the complete product of the above two fuzzy graphs G and H.**

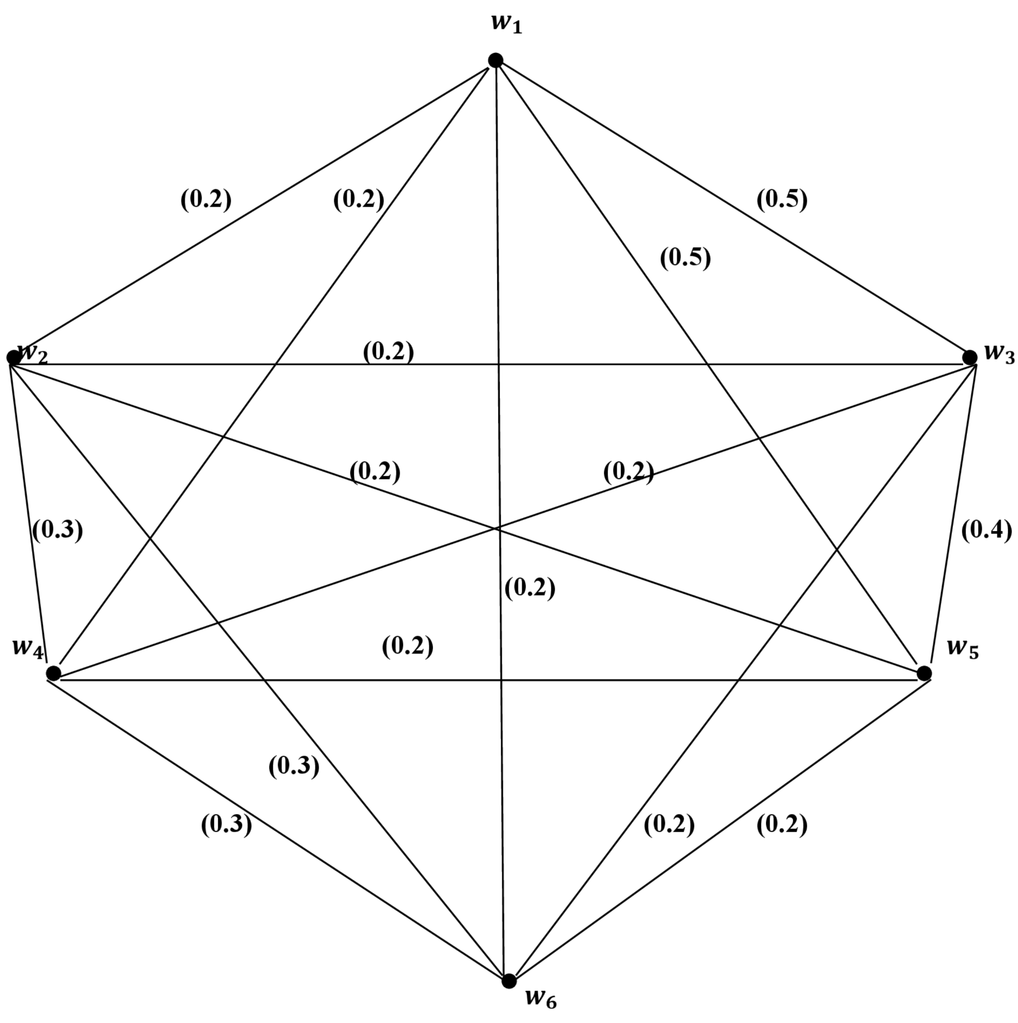



**Fig:(3.2.3) G $\times_P$ H**

where $w_1 = (u_1,v_1) = 0.5$, $w_2 = (u_1,v_2) = 0.3$, $w_3 = (u_2,v_1) = 0.5$, $w_4 = (u_2,v_2) = 0.3$, $w_5 = (u_3,v_1) = 0.5$, $w_6 = (u_3,v_2) = 0.3$. From the diagram it is clear that the crisp graph of G $\times_P$ H **is** a complete graph as there is an edge between every pair of vertices.

## 3. Main Results

**Theorem 3.1.** *If* $G{:}(\sigma,\mu)$ *and* $H{:}\ (\tau,\vartheta)$ *are any two strong fuzzy graphs then their complete product* $G\times_P H$ *is also a strong fuzzy graph.*

**Proof.** Given G:$(\sigma,\mu)$ and H: $(\tau,\vartheta)$ are two strong fuzzy graphs.
$\Rightarrow \mu(\text{u,v}) = \sigma(\text{u})\cap\sigma(\text{v})\ \forall\text{u,v} \in \mu^*$ and $\vartheta(\text{u,v}) = \tau(\text{u})\cap\tau(\text{v})\ \forall\text{u,v} \in \vartheta^*$.
As per the definition, the edge set of G $\times_p$ H is E $= E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8$ Such that $E_1 = \{((u_1,v_1),(u_2,v_2)) : u_1 = u_2 \text{ and } (v_1,v_2) \in E_V\}$ and
$E_2 = \{((u_1,v_1),(u_2,v_2)) : u_1 = u_2 \text{ and } (v_1,v_2) \notin E_V\}$ and
$E_3 = \{((u_1,v_1),(u_2,v_2)) : v_1 = v_2 \text{ and } (u_1,u_2) \in E_U\}$ and
$E_4 = \{((u_1,v_1),(u_2,v_2)) : v_1 = v_2 \text{ and } (u_1,u_2) \notin E_u\}$ and
$E_5 = \{((u_1,v_1),(u_2,v_2)) : (u_1,u_2) \in E_U \text{ and } (v_1,v_2) \notin E_V\}$ and
$E_6 = \{((u_1,v_1),(u_2,v_2)) : (u_1,u_2) \notin E_v \text{ and } (v_1,v_2) \in E_V\}$ and
$E_7 = \{((u_1,v_1),(u_2,v_2)) : (u_1,u_2) \in E_U \text{ and } (v_1,v_2) \in E_V\}$
$E_8 = \{((u_1,v_1),(u_2,v_2)) : (u_1,u_2) \notin E_U \text{ and } (v_1,v_2) \notin E_V\}$
Let w = $((u_1,v_1),(u_2,v_2))\in$ U$\times$V such that $(\mu\times_P\vartheta)(\text{w}) > 0$.
We prove this theorem by using the case division method.

**Case i:** Suppose that w$\in E_1$ .
Then $(\mu\times_P\vartheta)(\text{w})= \sigma(u_1)\cap \vartheta(v_1,v_2)$ where $u_1 = u_2$ and $(v_1,v_2)\in E_V$ .
Since $(\mu\times_P\vartheta)(\text{w}) > 0$ we have $\vartheta(v_1,v_2)>0$.
Which implies $\vartheta(v_1,v_2) = \tau(v_1)\cap\tau(v_2)$,As H is a strong fuzzy graph.
So $(\mu\times_P\vartheta)(\text{w}) = \sigma(u_1)\cap \vartheta(v_1,v_2) = \sigma(u_1)\cap\tau(v_1)\cap\tau(v_2)$
$=\sigma(u_1)\cap\sigma u_2)\cap\tau(v_1)\cap\tau(v_2)=(\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$
$=(\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.

**Case ii:** Suppose that w$\in E_2$ .
Then $(\mu\times_P\vartheta)(\text{w})= \sigma(u_1)\cap\tau(v_1)\cap\tau(v_2)$ and $u_1= u_2$.
So $(\mu\times_P\vartheta)(\text{w})= \sigma(u_1)\cap\tau(v_1)\cap\tau(v_2) = \sigma(u_1)\cap\sigma(u_2)\cap\tau(v_1)\cap\tau(v_2)$
$= (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2)=(\sigma\times_P\tau)(\text{u}_1,\text{v}_1)\cap(\sigma\times_P\tau)(\text{u}_2,\text{v}_2)$

**Case iii:** Suppose that w$\in$E$_3$.
Then $(\mu\times_P\vartheta)(\text{w})= \mu(\text{u}_1,\text{u}_2)\cap\tau(v_1)$ and $v_1 = v_2$ and $(u_1,u_2)\in E_U$.
Since $(\mu\times_P\vartheta)(\text{w}) > 0$ we have $\mu(u_1,u_2)>0$.
Which implies $\mu(u_1,u_2)= \sigma(u_1)\cap\sigma(u_2)$ as G is strong.
so $(\mu\times_P\vartheta)(\text{w})= \mu(\text{u}_1,\text{u}_2)\cap\tau(v_1) = \sigma(u_1)\cap\sigma(u_2)\cap\tau(v_1)$
$= \sigma(u_1)\cap\sigma(u_2)\cap\tau(v_1)\cap\tau(v_2) = (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$
$= (\sigma\times_P\tau)(\text{u}_1,\text{v}_1)\cap(\sigma\times_P\tau)(\text{u}_2,\text{v}_2)$

**Case iv:** Suppose that w$\in$ E$_4$ .
Then $(\mu\times_P\vartheta)(\text{w})= \sigma(\text{u}_1)\cap\sigma(\text{u}_2)\cap\tau(\text{v}_1)$ and $v_1 = v_2$.
So $(\mu\times_P\vartheta)(\text{w})= \sigma(\text{u}_1)\cap\sigma(\text{u}_2)\cap\tau(\text{v}_1) = \sigma(\text{u}_1)\cap\sigma(\text{u}_2)\cap\tau(\text{v}_2)\cap\tau(\text{v}_1)$
$= (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$

$= (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.

**Case v:** Suppose that $w\in E_5$.

Then $(\mu\times_P\vartheta)(w) = \mu(u_1, u_2)\cap\tau(v_1)\cap\tau(v_2)$ and $(u_1,u_2)\in E_U$.

Since $(\mu\times_P\vartheta)(w) > 0$ we have $\mu(u_1, u_2) > 0$.

Which implies $\mu(u_1, u_2) = \sigma(u_1)\cap\sigma(u_2)$ as G is strong.

So $(\mu\times_P\vartheta)(w) = \mu(u_1, u_2)\cap\tau(v_2)\cap\tau(v_1)$

$=\sigma(u_1)\cap\sigma(u_2)\cap\tau(v_2)\cap\tau(v_1) = (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$

$= (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.

**Case vi:** Suppose that $w\in E_6$.

Then $(\mu\times_P\vartheta)(w) = \sigma(u_1)\cap\sigma(u_2)\cap\vartheta(v_1,v_2)$ and $(v_1,v_2)\in E_V$. Since $(\mu\times_P\vartheta)(w) > 0$ we have $\vartheta(v_1,v_2) > 0$.

Which implies $\vartheta(v_1,v_2) = \tau(v_2)\cap\tau(v_1)$ as H is strong.

So $(\mu\times_P\vartheta)(w) = \sigma(u_1)\cap\sigma(u_2)\cap\vartheta(v_1,v_2) = \sigma(u_1)\cap\sigma(u_2)\cap\tau(v_1)\cap\tau(v_2)$

$= (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2)) = (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.

**Case vii:** Suppose that $w\in E_7$.

Then $(\mu\times_P\vartheta)(w) = \mu(u_1, u_2)\cap\vartheta(v_1,v_2)$ and $(u_1,u_2)\in E_U$, $(v_1,v_2)\in E_V$.

Since $(\mu\times_P\vartheta)(w) > 0$ we have $\vartheta(v_1,v_2) > 0$ and $\mu(u_1, u_2) > 0$.

Which implies $\vartheta(v_1,v_2) = \tau(v_2)\cap\tau(v_1)$ and $\mu(u_1, u_2) = \sigma(u_1)\cap\sigma(u_2)$, as G and H are strong fuzzy graphs.

So $(\mu\times_P\vartheta)(w) = \mu(u_1, u_2)\cap\vartheta(v_1,v_2) = (\sigma(u_1)\cap\sigma(u_2))\cap(\tau(v_1)\cap\tau(v_2))$

$=(\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$

$= (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.

**Case viii:** Suppose that $w\in E_8$.

Then $(\mu\times_P\vartheta)(w) = \sigma(u_1)\cap\sigma(u_2)\cap\tau(v_1)\cap\tau(v_2)$

$= (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2)) = (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.

In all the above cases we have shown that

$(\mu\times_P\vartheta)((u_1,v_1),(u_2,v_2)) = (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$

Hence $G\times_P H$ is a strong fuzzy graph.

**Theorem 3.2.** *If $G:(\sigma,\mu)$ and $H:(\tau,\vartheta)$ are any two complete fuzzy graphs then their complete product $G\times_P H$ is also a complete fuzzy graph.*

**Proof.** Given $G:(\sigma,\mu)$ and $H:(\tau,\vartheta)$ are two complete fuzzy graphs.

So $\mu(u,v) = \sigma(u)\cap\sigma(v)\forall u,v \in \sigma^*$ and $\vartheta(a,b) = \tau(a)\cap\tau(a)\forall a,b \in \tau^*$

we prove this theorem by using the case division method.

Let $w = ((u_1,v_1),(u_2,v_2))\in U\times V$ such that $(\sigma\times_P\tau)(u_1,v_1) > 0$ and $(\sigma\times_P\tau)(u_2,v_2) > 0$.

Which implies $\sigma(u_1)\cap\tau(v_1) > 0$ and $\sigma(u_2)\cap\tau(v_2) > 0$.

$\Rightarrow \sigma(u_1) > 0, \tau(v_1) > 0, \sigma(u_2) > 0, \tau(v_2) > 0$. ...........(*)

**Case i:** Suppose that $w\in E_1$.

Then $(\mu\times_P\vartheta)(w) = \sigma(u_1)\cap\vartheta(v_1,v_2)$ where $u_1 = u_2$ and $(v_1,v_2)\in E_V$.

$(\mu\times_P\vartheta)$(w) $= \sigma(u_1\ )\cap\ \vartheta(v_1,v_2\ ) = \ \sigma(u_1\ )\cap\tau(v_1)\cap\tau(v_2)$, as H is a complete fuzzy graph.
$= \sigma(u_1\ )\cap\sigma(u_2)\cap\tau(v_1)\cap\tau(v_2) = (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$
$= (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.
**Case ii:** Suppose that w$\in E_2$.
Then $(\mu\times_P\vartheta)$(w)= $\sigma(u_1\ )\cap\tau(v_1)\cap\tau(v_2)$ and $u_1 = u_2$.
So $(\mu\times_P\vartheta)$(w)= $\sigma(u_1\ )\cap\tau(v_1)\cap\tau(v_2)$
$= \sigma(u_1)\cap\sigma(u_2)\cap\tau(v_1)\cap\tau(v_2) = (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$
$= (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.
**Case iii:** Suppose that w$\in E_3$.
Then $(\mu\times_P\vartheta)$(w)= $\mu(u_1,u_2)\cap\tau(v_1)$ and $v_1 = v_2$ and $(u_1,u_2)\in E_U$.
so $(\mu\times_P\vartheta)$(w)= $\mu(u_1,u_2)\cap\tau(v_1) = \sigma(u_1)\cap\sigma(u_2)\cap\tau(v_1)$, as G is a complete fuzzy graph.
$= \sigma(u_1)\cap\sigma(u_2)\cap\tau(v_1)\cap\tau(v_2) = (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$
$= (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.
Case-iv: Suppose that w$\in E_4$ .
Then $(\mu\times_P\vartheta)$(w)= $\sigma(u_1\ )\cap\sigma(u_2)\cap\tau(v_1)$ and $v_1 = v_2$.
So $(\mu\times_P\vartheta)$(w)= $\sigma(u_1\ )\cap\sigma(u_2)\cap\tau(v_1) = \sigma(u_1\ )\cap\sigma(u_2\ )\cap\tau(v_2)\cap\tau(v_1)$
$= (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$
$= (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.
**Case v:** Suppose that w$\in E_5$.
Then $(\mu\times_P\vartheta)(w) = \mu(u_1, u_2)\cap\tau(v_1)\cap\tau(v_2)$ and $(u_1,u_2)\in E_U$.
So $(\mu\times_P\vartheta)(w) = \mu(u_1, u_2)\cap\tau(v_2)\cap\tau(v_1)$
$=(\sigma(u_1)\cap\sigma(u_2))\cap\tau(v_2)\cap\tau(v_1)$, as G is a complete fuzzy graph.
$= (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$
$= (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.
**Case vi:** Suppose that w$\in E_6$.
Then $(\mu\times_P\vartheta)(w) = \sigma(u_1\ )\cap\sigma(u_2)\cap\vartheta(v_1,v_2)$ and $(v_1,v_2)\in E_V$. So $(\mu\times_P\vartheta)(w) = \sigma(u_1\ )\cap\sigma(u_2)\cap\vartheta(v_1,v_2) = \sigma(u_1\ )\cap\sigma(u_2)\cap(\tau(v_1)\cap\tau(v_2))$, as H is a complete fuzzy graph.
$= (\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2)) = (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.
**Case vii:** Suppose that w$\in E_7$.
Then $(\mu\times_P\vartheta)(w) = \mu(u_1, u_2)\cap\vartheta(v_1,v_2)$ and $(u_1,u_2)\in E_U$ , $(v_1,v_2)\in E_V$.
So $(\mu\times_P\vartheta)(w) = \mu(u_1, u_2)\cap\vartheta(v_1,v_2)$
$= (\sigma(u_1\ )\cap\sigma(u_2))\cap(\tau(v_1)\cap\tau(v_2))$,as G and H are complete fuzzy graphs.
$=(\sigma(u_1)\cap\tau(v_1))\cap(\sigma(u_2)\cap\tau(v_2))$
$= (\sigma\times_P\tau)(u_1,v_1)\cap(\sigma\times_P\tau)(u_2,v_2)$.
**Case viii:** Suppose that w$\in E_8$.
Then $(\mu\times_P\vartheta)$(w)= $\sigma(u_1\ )\cap\sigma(u_2)\cap\tau(v_1)\cap\tau(v_2)$

$= (\sigma(u_1) \cap \tau(v_1)) \cap (\sigma(u_2) \cap \tau(v_2)) = (\sigma \times_P \tau)(u_1, v_1) \cap (\sigma \times_P \tau)(u_2, v_2)$.
In all the above cases we have shown that
$(\mu \times_P \vartheta)((u_1, v_1), (u_2, v_2)) = (\sigma \times_P \tau)(u_1, v_1) \cap (\sigma \times_P \tau)(u_2, v_2), \forall (u_1, v_1), (u_2, v_2) \in (\sigma \times_P \tau)^*$.
Hence $G \times_P H$ is a complete fuzzy graph.

**Theorem 3.3.** *If $G \times_P H$ is a Regular fuzzy graph and if $G:(\sigma, \mu)$ and $H: (\tau, \vartheta)$ are two complete fuzzy graphs such that $\sigma < \tau$ then $G:(\sigma, \mu)$ is a Regular fuzzy graph.*
**Proof.** Given $G:(\sigma, \mu)$ and $H: (\tau, \vartheta)$ are two complete fuzzy graphs such that $\sigma < \tau$. So we have $(\mu \times_P \vartheta)(w) = \sigma(u_1) \cap \sigma(u_2) \cap \tau(v_1) \cap \tau(v_2)$ $\forall$ w $\in$E where $w = ((u_1, v_1), (u_2, v_2))$.
Given $G \times_P H$ is a Regular fuzzy graph.So $d_L((u,v)) = k$ $\forall (u,v) \in U \times V$ for some positive real k, where $L = G \times_P H$.
Implies $\sum (\mu \times_P \vartheta)((u,v),(u',v')) = k$ $\forall$ $(u',v') \in U \times V$ such that $(u',v') \neq (u,v)$.
Implies $\sum (\sigma(u) \cap \sigma(u') \cap \tau(v) \cap \tau(v')) = k$ $\forall$ $(u',v') \in U \times V$ such that $(u',v') \neq (u,v)$.
Implies $\sum (\sigma(u) \cap \sigma(u')) = k$ $\forall$ $u' \in U$ such that $u' \neq u$, as $\sigma < \tau$.
Implies $\sum \mu(u,u') = k$ $\forall$ $u' \in U$ such that $u' \neq u$, as G is a complete fuzzy graph.
Implies $d_G(u) = k$ $\forall$ $u \in U$.
Hence $G:(\sigma, \mu)$ is a Regular fuzzy graph.

**Theorem 3.4.** *If $G:(\sigma, \mu)$ and $H: (\tau, \vartheta)$ are two complete fuzzy graphs such that $\sigma < \tau$ and If $G:(\sigma, \mu)$ is a regular fuzzy graph then $G \times_P H$ is a Regular fuzzy graph.*
**Proof.** Given $G:(\sigma, \mu)$ and $H: (\tau, \vartheta)$ are two complete fuzzy graphs.
So we have $(\mu \times_P \vartheta)(w) = \sigma(u_1) \cap \sigma(u_2) \cap \tau(v_1) \cap \tau(v_2)$ $\forall$ w $\in$E
where $w = ((u_1, v_1), (u_2, v_2))$.
Given $G:(\sigma, \mu)$ a is a regular fuzzy graph. So $d_G(u) = k$ $\forall$ u $\in E_U$. Let $L = G \times_P H$.
$d_L((u,v)) = \sum (\mu \times_P \vartheta)((u,v),(u',v'))$ , $\forall$ $(u',v') \in U \times V$ such that $(u',v') \neq (u,v)$.
Implies $d_L((u,v)) = \sum (\sigma(u) \cap \sigma(u') \cap \tau(v) \cap \tau(v'))$,$\forall$ $(u',v') \in U \times V$ such that $(u',v') \neq (u,v)$.
Implies $d_L((u,v)) = \sum (\sigma(u) \cap \sigma(u'))$ where $u' \neq u$, as $\sigma < \tau$.
Implies $d_L((u,v)) = d_G(u) = k$ . Hence $G \times_P H$ is a Regular fuzzy graph.

## 4. Conclusion

The application of fuzzy relations is widespread and important in the field of clustering analysis, neural networks, computer networks, pattern recognition, decision making, and expert systems. In each of these, the basic mathematical structure is that of a fuzzy graph. Fuzzy-graph theory has a large number of applications in modeling various real-time systems where the level of information inherent in the system varies with different levels of precision. In this article, we dealt with a new framework called the complete product of fuzzy graphs in which we pool two fuzzy graphs and produce a new comprehensive fuzzy graph to handle fuzzy information in a more elegant way. We proved that the complete product of two strong fuzzy graphs is again a strong fuzzy graph and the complete product of two complete fuzzy graphs is again a complete fuzzy graph. Also, we discussed the conditions under which the property of regularity will be mutually transmitted between the complete product of two fuzzy graphs and one of its factor fuzzy graphs. In the future, we will extend our fuzzification to the complete product of compliments of a pair of fuzzy graphs and the complement of the complete product of a pair of fuzzy graphs.

**Acknowledgements**

The authors are very thankful to the editor and referees for their valuable comments and suggestions for improving the paper.

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