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Certain Explicit Evaluation of Ramanujan's Theta Functions

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Abstract: In this paper, using certain known modular equations, we have established certain modular identities which have been made use of, to evaluate certain theta functions. We shall attempt to evaluate Ramanujan's cubic continued fraction and also certain Ramanujan-Weber class invariants with the help of some of our results. The results established herein may prove useful in further investigations in the subject.
Keywords and Phrases: Modular identities, modular equation, Ramanujan's theta function, cubic continued fraction, Ramanujan-Weber class invariants.
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1. Introduction, Notations and Definitions : In what follows, for real or complex α and $q(|q| < 1)$, let

$$[\alpha; q]_\infty = \prod_{r=0}^{\infty}(1 - \alpha q^r). \tag{1.1}$$

Ramanujan's theta functions are defined by

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = [q^2; q^2]_\infty[-q; q^2]_\infty^2 \tag{1.2}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{[q^2; q^2]_\infty}{[q; q^2]_\infty} \tag{1.3}$$

$$f(-q) = \sum_{n=-\infty}^{\infty} (-)^n q^{n(3n-1)/2} = [q; q]_\infty \tag{1.4}$$

and

$$\chi(-q) = [q; q^2]_\infty \tag{1.5}$$

We shall use the following modular equations also;
If β and the multipliers m are of degree three over α, then

$$\sqrt[8]{\frac{\beta^3}{\alpha}} = \frac{m-1}{2} \tag{1.6}$$

$$\sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}} = \frac{3-m}{2m} \tag{1.7}$$

$$\sqrt[8]{\frac{\alpha^3}{\beta}} = \frac{3+m}{2m} \tag{1.8}$$

$$\sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} = \frac{m+1}{2} \tag{1.9}$$

Berndt [2; (5.1), p. 232]

The following results due to Ramanujan will also be used in our study,

$$\phi(q) = \sqrt{z} \tag{1.10}$$

$$\phi(-q) = \sqrt{z}\sqrt[4]{1-x} \tag{1.11}$$

$$\phi(-q^2) = \sqrt{z}\sqrt[8]{1-x} \tag{1.12}$$

$$\psi(q) = \sqrt{z/2}\sqrt[8]{x/q} \tag{1.13}$$

$$\psi(-q) = \sqrt{z/2}\sqrt[8]{x(1-x)/q} \tag{1.14}$$

$$\psi(q^2) = \frac{\sqrt{z}}{2}\sqrt[4]{x/q} \tag{1.15}$$

$$f(q) = \frac{\sqrt{z}}{\sqrt[6]{2}}\sqrt[24]{x(1-x)/q} \tag{1.16}$$

$$f(-q) = \frac{\sqrt{z}}{\sqrt[6]{2}}\sqrt[6]{1-x}\,\sqrt[24]{x/q} \tag{1.17}$$

$$f(-q^2) = \frac{\sqrt{z}}{\sqrt[3]{2}}\sqrt[12]{x(1-x)/q} \tag{1.18}$$

$$f(-q^4) = \frac{\sqrt{z}}{\sqrt[3]{4}}\sqrt[24]{1-x}\,\sqrt[6]{x/q} \tag{1.19}$$

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{x(1-x)/q}} \tag{1.20}$$

$$\chi(-q) = \frac{\sqrt[6]{2}\ \sqrt[12]{1-x}}{\sqrt[24]{x/q}} \tag{1.21}$$

$$\chi(-q^2) = \frac{\sqrt[3]{2}\ \sqrt[24]{1-x}}{\sqrt[12]{x/q}} \tag{1.22}$$

Ramanujan [4; Chapter 17, entries 10, 11, 12]

$$\frac{\phi(e^{-\pi})}{\phi(e^{-3\pi})} = \sqrt[4]{6\sqrt{3}-9} \tag{1.23}$$

Berndt [3; Chapter 35, entry 4, p. 327]

Ramanujan-Weber class invariants G_n and g_n are defined as,

$$2^{1/4}e^{-\pi\sqrt{n}/24}G_n = \chi(e^{-\pi\sqrt{n}}) \tag{1.24}$$

and

$$2^{1/4}e^{-\pi\sqrt{n}/24}g_n = \chi(-e^{-\pi\sqrt{n}}) \tag{1.25}$$

Berndt [3; Chapter 34, (1.3) 4, p. 183]

If $a = \sqrt[4]{\pi}/\Gamma(3/4)$, then

$$\phi(e^{-\pi}) = a \tag{1.26}$$

Berndt [3; Chapter 35, entry (1.1)(1.3), p. 325]

Ramanujan's cubic continued fraction is given by,

$$v = \frac{q^{1/3}}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+...} \tag{1.27}$$

where,

$$\frac{1}{v} = \left\{\frac{\psi^4(q)}{q\psi^4(q^3)} - 1\right\}^{1/3} \tag{1.28}$$

and,

$$2v = \left\{1 - \frac{\phi^4(-q)}{\phi^4(-q^3)}\right\}^{1/3} \tag{1.29}$$

Andrews and Berndt [1; entry (3.3.1), p. 94]

2. Results to be established

In this section we shall establish the following results,

$$\frac{\phi^3(-e^{-\pi})}{\phi(-e^{-3\pi})} = a^2\sqrt[4]{6\sqrt{3}-9}\frac{\{3-\sqrt{6\sqrt{3}-9}\}^2}{4(6\sqrt{3}-9)} \tag{2.1}$$

$$\frac{\phi^3(-e^{-3\pi})}{\phi(-e^{-\pi})} = a^2\sqrt[4]{6\sqrt{3}-9}\frac{\{1+\sqrt{6\sqrt{3}-9}\}^2}{4(6\sqrt{3}-9)} \tag{2.2}$$

$$\phi(-e^{-\pi})\phi(-e^{-3\pi}) = \frac{a^2\sqrt[4]{6\sqrt{3}-9}\{3-\sqrt{6\sqrt{3}-9}\}}{4(6\sqrt{3}-9)}\{1+\sqrt{6\sqrt{3}-9}\} \tag{2.3}$$

$$\frac{\phi(-e^{-\pi})}{\phi(-e^{-3\pi})} = \sqrt{\frac{3-\sqrt{6\sqrt{3}-9}}{1+\sqrt{6\sqrt{3}-9}}} \tag{2.4}$$

$$\frac{\phi^3(-e^{-2\pi})}{\phi(-e^{-6\pi})} = \frac{a^2\sqrt[4]{6\sqrt{3}-9}\{3-\sqrt{6\sqrt{3}-9}\}}{(2\sqrt{6\sqrt{3}-9})} \tag{2.5}$$

$$\frac{\phi^3(-e^{-6\pi})}{\phi(-e^{-2\pi})} = \frac{a^2\sqrt[4]{6\sqrt{3}-9}(1+\sqrt{6\sqrt{3}-9})}{(2\sqrt{6\sqrt{3}-9})} \tag{2.6}$$

$$\phi(-e^{-2\pi})\phi(-e^{-6\pi}) = a^2\frac{\sqrt{(3-\sqrt{6\sqrt{3}-9})(1+\sqrt{6\sqrt{3}-9})}}{(2\sqrt{6\sqrt{3}-9})} \tag{2.7}$$

$$\frac{\phi(-e^{-2\pi})}{\phi(-e^{-6\pi})} = \sqrt[8]{6\sqrt{3}-9}\times\sqrt[4]{\frac{3-\sqrt{6\sqrt{3}-9}}{1+\sqrt{6\sqrt{3}-9}}} \tag{2.8}$$

$$\frac{\psi^3(e^{-\pi})}{\psi(e^{-3\pi})} = a^2\sqrt[4]{6\sqrt{3}-9}\frac{3+\sqrt{6\sqrt{3}-9}}{4\sqrt{6\sqrt{3}-9}} \tag{2.9}$$

$$\frac{e^{-\pi}\psi^3(e^{-3\pi})}{\psi(e^{-\pi})} = a^2\sqrt[4]{6\sqrt{3}-9}\frac{\sqrt{6\sqrt{3}-9}-1}{4(6\sqrt{3}-9)} \tag{2.10}$$

$$e^{-\pi/2}\psi(e^{-\pi})\psi(e^{-3\pi}) = \frac{a^2\sqrt{(3+\sqrt{6\sqrt{3}-9})(\sqrt{6\sqrt{3}-9}-1)}}{4\sqrt{6\sqrt{3}-9}} \tag{2.11}$$

$$\frac{\psi(e^{-\pi})}{e^{-\pi/4}\psi(e^{-3\pi})} = \sqrt[8]{6\sqrt{3}-9}\sqrt[4]{\frac{3+\sqrt{6\sqrt{3}-9}}{\sqrt{6\sqrt{3}-9}-1}} \tag{2.12}$$

$$\frac{\psi^3(-e^{-\pi})}{\psi(-e^{-3\pi})} = \frac{a^2\sqrt[4]{6\sqrt{3}-9}}{2(\sqrt{3}-1)} \tag{2.13}$$

$$\frac{e^{-\pi}\psi^3(-e^{-3\pi})}{\psi(-e^{-\pi})} = \frac{a^2\sqrt[4]{6\sqrt{3}-9}}{6\sqrt{3}(\sqrt{3}+1)} \tag{2.14}$$

$$e^{-\pi/2}\psi(-e^{-\pi})\psi(-e^{-3\pi}) = \frac{a^2\sqrt[4]{2\sqrt{3}-3}}{2\sqrt{6}} \tag{2.15}$$

$$\frac{\psi(-e^{-\pi})}{e^{-\pi/4}\psi(-e^{-3\pi})} = \sqrt[4]{6\sqrt{3}-9} \tag{2.16}$$

$$\frac{\psi^3(e^{-2\pi})}{\psi(e^{-6\pi})} = a^2(6\sqrt{3}-9)^{1/4}\frac{\{3+\sqrt{6\sqrt{3}-9}\}^2}{16(6\sqrt{3}-9)} \tag{2.17}$$

$$\frac{e^{-2\pi}\psi^3(e^{-6\pi})}{\psi(e^{-2\pi})} = a^2(6\sqrt{3}-9)^{1/4}\frac{\{\sqrt{6\sqrt{3}-9}-1\}^2}{16(6\sqrt{3}-9)} \tag{2.18}$$

$$e^{-\pi}\psi(e^{-2\pi})\psi(e^{-6\pi}) = \frac{a^2(6\sqrt{3}-9)^{1/4}\{\sqrt{6(\sqrt{3}-9)}+3\}\{\sqrt{6(\sqrt{3}-9)}-1\}}{16(6\sqrt{3}-9)} \tag{2.19}$$

$$\frac{\psi(e^{-2\pi})}{e^{-\pi/2}\psi(e^{-6\pi})} = \left\{\frac{\sqrt{6(\sqrt{3}-9)}+3}{\sqrt{6(\sqrt{3}-9)}-1}\right\}^{1/2} \tag{2.20}$$

$$\frac{f^3(e^{-\pi})}{f(e^{-3\pi})} = \frac{a^2(6\sqrt{3}-9)^{1/4}}{\{2(\sqrt{3}-1)\}^{1/3}} \tag{2.21}$$

$$\frac{e^{-\pi/3}f^3(e^{-3\pi})}{f(e^{-\pi})} = \frac{a^2(6\sqrt{3}-9)^{1/4}\{(2-\sqrt{3})(\sqrt{3}-1)/4\}^{1/3}}{6\sqrt{3}-9} \tag{2.22}$$

$$e^{-\pi/6}f(e^{-\pi})f(e^{-3\pi}) = \frac{a^2}{(6\sqrt{3}-9)^{1/4}}\{(2-\sqrt{3})/8\}^{1/6} \tag{2.23}$$

$$\frac{f(e^{-\pi})}{e^{-\pi/12}f(e^{-3\pi})} = (3\sqrt{3})^{1/4}(2-\sqrt{3})^{1/12} \tag{2.24}$$

$$\frac{f^3(-e^{-\pi})}{f(-e^{-3\pi})} = \frac{a^2(6\sqrt{3}-9)^{1/4}}{\{2(\sqrt{3}-1)\}^{1/3}} \times \frac{3-\sqrt{(6\sqrt{3}-9)}}{2\sqrt{(6\sqrt{3}-9)}} \tag{2.25}$$

$$\frac{e^{-\pi/3}f^3(-e^{-3\pi})}{f(-e^{-\pi})}=\frac{a^2(6\sqrt{3}-9)^{1/4}}{6\sqrt{3}-9}\times\frac{1+\sqrt{(6\sqrt{3}-9)}}{2}\{(2-\sqrt{3})(\sqrt{3}-1)/4\}^{1/3} \tag{2.26}$$

$$e^{-\pi/6}f(-e^{-3\pi})f(-e^{-\pi})=\frac{a^2}{2^{4/3}\sqrt{(6\sqrt{3}-9)}}\left\{\frac{2-\sqrt{3}}{2}\right\}^{1/6}\times$$
$$\times[\{3-\sqrt{(6\sqrt{3}-9)}\}\{1+\sqrt{(6\sqrt{3}-9)}\}]^{1/2} \tag{2.27}$$

$$\frac{f(-e^{-\pi})}{e^{-\pi/12}f(-e^{-3\pi})}=\frac{(6\sqrt{3}-9)^{1/8}}{(2-\sqrt{3})^{1/6}}\times\left\{\frac{3-\sqrt{(6\sqrt{3}-9)}}{1+\sqrt{(6\sqrt{3}-9)}}\right\}^{1/4} \tag{2.28}$$

$$\frac{f^3(-e^{-2\pi})}{f(-e^{-6\pi})}=\frac{a^2(6\sqrt{3}-9)^{1/4}}{\{2(\sqrt{3}-1)\}^{2/3}} \tag{2.29}$$

$$\frac{e^{-2\pi/3}f^3(-e^{-6\pi})}{f(-e^{-2\pi})}=\frac{a^2(6\sqrt{3}-9)^{1/4}}{6\sqrt{3}} \tag{2.30}$$

$$e^{-\pi/3}f(-e^{-2\pi})f(-e^{-6\pi})=\frac{a^2(6\sqrt{3}-9)^{1/4}}{2^{5/6}\sqrt{(3\sqrt{3})}(\sqrt{3}-1)^{1/3}} \tag{2.31}$$

$$\frac{f(-e^{-2\pi})}{e^{-\pi/6}f(-e^{-6\pi})}=\frac{2^{1/12}3^{3/8}}{(\sqrt{3}-1)^{1/6}} \tag{2.32}$$

$$\frac{f^3(-e^{-4\pi})}{f(-e^{-12\pi})}=\frac{a^2(6\sqrt{3}-9)^{1/4}\{3+\sqrt{(6\sqrt{3}-9)}\}}{2^{7/3}\sqrt{(6\sqrt{3}-9)}(\sqrt{3}-1)^{1/3}} \tag{2.33}$$

$$\frac{e^{-4\pi/3}f^3(-e^{-12\pi})}{f(-e^{-4\pi})}=\frac{a^2(6\sqrt{3}-9)^{1/4}}{2^{4/3}(6\sqrt{3}-9)}\times$$
$$\{(2-\sqrt{3})(\sqrt{3}-1)/2\}^{1/3}\{\sqrt{(6\sqrt{3}-9)}-1\}/2 \tag{2.34}$$

$$e^{-2\pi/3}f(-e^{-4\pi})f(-e^{-12\pi})=\frac{a^2}{4\sqrt{(6\sqrt{3}-9)}}\{(2-\sqrt{3})/8\}^{1/6}\times$$
$$\times\{\sqrt{(6\sqrt{3}-9)}+3\}\{\sqrt{(6\sqrt{3}-9)}-1\} \tag{2.35}$$

$$\frac{f(-e^{-4\pi})}{e^{-\pi/3}f(-e^{-12\pi})}=\frac{(6\sqrt{3}-9)^{1/8}}{(2-\sqrt{3})^{1/6}}\times\left\{\frac{3+\sqrt{(6\sqrt{3}-9)}}{\sqrt{(6\sqrt{3}-9)}-1}\right\}^{1/4} \tag{2.36}$$

$$\frac{\chi^3(e^{-\pi})}{\chi(e^{-3\pi})}=\{2(\sqrt{3}-1)\}^{1/3} \tag{2.37}$$

$$\frac{\chi^3(e^{-3\pi})}{e^{-\pi/3}\chi(e^{-\pi})}=\{4/(2-\sqrt{3})(\sqrt{3}-1)\}^{1/3} \tag{2.38}$$

$$e^{\pi/6}\chi(e^{-\pi})\chi(e^{-3\pi}) = \{8/(\sqrt{3}-1)\}^{1/6} \tag{2.39}$$

$$\frac{e^{-\pi/12}\chi(e^{-\pi})}{\chi(e^{-3\pi})} = (2-\sqrt{3})^{1/6} \tag{2.40}$$

$$\frac{\chi^3(-e^{-\pi})}{\chi(-e^{-3\pi})} = \frac{\{3-\sqrt{(6\sqrt{3}-9)}\}}{\sqrt{(6\sqrt{3})}(2-\sqrt{3})^{1/3}} \tag{2.41}$$

$$\frac{\chi^3(-e^{-3\pi})}{e^{-\pi/3}\chi(-e^{-\pi})} = \frac{\sqrt{(6\sqrt{3}-9)}+1}{\{2(2-\sqrt{3})(\sqrt{3}-1)\}^{1/3}} \tag{2.42}$$

$$e^{\pi/6}\chi(-e^{-\pi})\chi(-e^{-3\pi}) = \frac{[\{3-\sqrt{(6\sqrt{3}-9)}\}\{\sqrt{(6\sqrt{3}-9)}+1\}]^{1/2}}{2^{5/12}(2-\sqrt{3})^{1/3}3^{3/8}(\sqrt{3}-1)^{1/6}} \tag{2.43}$$

$$\frac{\chi(-e^{-\pi})}{e^{\pi/12}\chi(-e^{-3\pi})} = \frac{(\sqrt{3}-1)^{1/12}}{3^{3/16}2^{1/24}}\left\{\frac{3-\sqrt{(6\sqrt{3}-9)}}{\sqrt{(6\sqrt{3}-9)}+1}\right\}^{1/4} \tag{2.44}$$

$$\frac{\chi(-e^{-2\pi})}{\chi(-e^{-6\pi})} = \frac{2^{5/6}\sqrt{(6\sqrt{3}-9)}}{\{3+\sqrt{(6\sqrt{3}-9)}\}(\sqrt{3}-1)^{1/3}} \tag{2.45}$$

$$\frac{\chi^3(-e^{-6\pi})}{e^{-2\pi/3}\chi(-e^{-2\pi})} = \frac{2^{4/3}\{(2-\sqrt{3})(\sqrt{3}-1)\}^{1/3}}{\sqrt{(6\sqrt{3}-9)}-1} \tag{2.46}$$

$$e^{\pi/3}\chi(-e^{-2\pi})\chi(-e^{-6\pi}) = \frac{2^{3/2}(2-\sqrt{3})^{5/12}3^{3/8}}{[\{3+\sqrt{(6\sqrt{3}-9)}\}\{\sqrt{(6\sqrt{3}-9)}-1\}]^{1/2}} \tag{2.47}$$

$$\frac{\chi(-e^{-2\pi})}{e^{\pi/6}\chi(-e^{-6\pi})} = \frac{3^{3/16}}{(2-\sqrt{3})^{1/24}} \times \left\{\frac{\sqrt{(6\sqrt{3}-9)}-1}{\sqrt{(6\sqrt{3}-9)}+3}\right\}^{1/4} \tag{2.48}$$

3. Proof of (2.1) - (2.48):

In this section we shall discuss the proof of our results listed in section 2. We know that in modular equations (1.6)-(1.10), β and the multiplier m are of degree 3. To proceed further let us define the following functions with the help of (1.10) and (1.22) say,

$$P = \frac{\phi(q)}{\phi(q^3)} = \sqrt{m} \tag{3.1}$$

$$Q = \frac{\phi^3(-q)}{\phi(-q^3)} = \sqrt{(z_1^3/z_3)} \times \left\{\frac{(1-\alpha)^3}{(1-\beta)}\right\}^{1/4} \tag{3.2}$$

$$Q_1 = \frac{\phi^3(-q^3)}{\phi(-q)} = \sqrt{(z_3^3/z_1)} \times \left\{\frac{(1-\beta)^3}{(1-\alpha)}\right\}^{1/4} \tag{3.3}$$

$$Q_2 = \frac{\phi^3(-q^2)}{\phi(-q^6)} = \sqrt{(z_1^3/z_3)} \times \left\{\frac{(1-\alpha)^3}{(1-\beta)}\right\}^{1/8} \tag{3.4}$$

$$Q_3 = \frac{\phi^3(-q^3)}{\phi(-q^2)} = \sqrt{(z_3^3/z_1)} \times \left\{\frac{(1-\beta)^3}{(1-\alpha)}\right\}^{1/8} \tag{3.5}$$

$$Q_4 = \frac{\psi^3(q)}{\psi(q^3)} = \frac{1}{2}\sqrt{(z_1^3/z_3)} \times \left\{\frac{\alpha^3}{\beta}\right\}^{1/8} \tag{3.6}$$

$$Q_5 = \frac{q\psi^3(q^3)}{\psi(q)} = \frac{1}{2}\sqrt{(z_3^3/z_1)} \times \left\{\frac{\beta^3}{\alpha}\right\}^{1/8} \tag{3.7}$$

$$Q_6 = \frac{\psi^3(-q)}{\psi(-q^3)} = \frac{1}{2}\sqrt{(z_1^3/z_3)} \times \left\{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right\}^{1/8} \tag{3.8}$$

$$Q_7 = \frac{q\psi^3(-q^3)}{\psi(-q)} = \frac{1}{2}\sqrt{(z_3^3/z_1)} \times \left\{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right\}^{1/8} \tag{3.9}$$

$$Q_8 = \frac{\psi^3(q^2)}{\psi(q^6)} = \frac{1}{4}\sqrt{(z_1^3/z_3)} \times \left\{\frac{\alpha^3}{\beta}\right\}^{1/4} \tag{3.10}$$

$$Q_9 = \frac{q^2\psi^3(q^6)}{\psi(q^2)} = \frac{1}{4}\sqrt{(z_3^3/z_1)} \times \left\{\frac{\beta^3}{\alpha}\right\}^{1/4} \tag{3.11}$$

$$Q_{10} = \frac{f^3(q)}{f(q^3)} = \frac{1}{2^{1/3}}\sqrt{(z_1^3/z_3)} \times \left\{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right\}^{1/24} \tag{3.12}$$

$$Q_{11} = \frac{q^{1/3}f^3(q^3)}{f(q)} = \frac{1}{2^{1/3}}\sqrt{(z_3^3/z_1)} \times \left\{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right\}^{1/24} \tag{3.13}$$

$$Q_{12} = \frac{f^3(-q)}{f(-q^3)} = 2^{1/3}\sqrt{(z_1^3/z_3)}\{(1-\alpha)^3/(1-\beta)\}^{1/6}(\alpha^3/\beta)^{1/24} \tag{3.14}$$

$$Q_{13} = \frac{q^{1/3}f^3(-q^3)}{f(-q)} = 2^{-1/3}\sqrt{(z_3^3/z_1)}\{(1-\beta)^3/(1-\alpha)\}^{1/6}(\beta^3/\alpha)^{1/24} \tag{3.15}$$

$$Q_{14} = \frac{f^3(-q^2)}{f(-q^6)} = 2^{2/3}\sqrt{(z_1^3/z_3)}\{\alpha^3(1-\alpha)^3/\beta(1-\beta)\}^{1/12} \tag{3.16}$$

$$Q_{15} = \frac{q^{2/3}f^3(-q^6)}{f(-q^2)} = 2^{-2/3}\sqrt{(z_3^3/z_1)}\{\beta^3(1-\beta)^3/\alpha(1-\alpha)\}^{1/12} \tag{3.17}$$

$$Q_{16} = \frac{f^3(-q^4)}{f(-q^{12})} = 4^{2/3}\sqrt{(z_1^3/z_3)}\left\{\frac{(1-\alpha)^3}{(1-\beta)}\right\}^{1/24}\left\{\frac{\alpha^3}{\beta}\right\}^{1/6} \tag{3.18}$$

$$Q_{17} = \frac{q^{4/3} f^3(-q^{12})}{f(-q^4)} = 4^{2/3} \sqrt{(z_3^3/z_1)} \left\{ \frac{(1-\beta)^3}{(1-\alpha)} \right\}^{1/24} \left\{ \frac{\beta^3}{\alpha} \right\}^{1/6} \tag{3.19}$$

$$Q_{18} = \frac{\chi^3(q)}{\chi(q^3)} = 2^{1/3} \left\{ \frac{\beta(1-\beta)}{\alpha^3(1-\alpha)^3} \right\}^{1/24} \tag{3.20}$$

$$Q_{19} = \frac{\chi^3(q^3)}{q^{1/3}\chi(q)} = 2^{1/3} \left\{ \frac{\alpha(1-\alpha)}{\beta^3(1-\beta)^3} \right\}^{1/24} \tag{3.21}$$

$$Q_{20} = \frac{\chi^3(-q)}{\chi(-q^3)} = 2^{1/3} \left\{ \frac{(1-\alpha)^3}{(1-\beta)} \right\}^{1/12} \left\{ \frac{\beta}{\alpha^3} \right\}^{1/24} \tag{3.22}$$

$$Q_{21} = \frac{\chi^3(-q^3)}{q^{1/3}\chi(-q)} = 2^{1/3} \left\{ \frac{(1-\beta)^3}{(1-\alpha)} \right\}^{1/12} \left\{ \frac{\alpha}{\beta^3} \right\}^{1/24} \tag{3.23}$$

$$Q_{22} = \frac{\chi^3(-q^2)}{\chi(-q^6)} = 2^{2/3} \left\{ \frac{(1-\alpha)^3}{(1-\beta)} \right\}^{1/24} \left\{ \frac{\beta}{\alpha^3} \right\}^{1/12} \tag{3.24}$$

$$Q_{23} = \frac{\chi^3(-q^6)}{q^{1/3}\chi(-q^2)} = 2^{2/3} \left\{ \frac{(1-\beta)^3}{(1-\alpha)} \right\}^{1/24} \left\{ \frac{\alpha}{\beta^3} \right\}^{1/12} \tag{3.25}$$

We shall also need the following results,
Putting $q = e^{-\pi}$ in (3.1) and using (1.23), we get

$$P = \sqrt{m} = \frac{\phi(e^{-\pi})}{\phi(e^{-3\pi})} = (6\sqrt{3} - 9)^{1/4} \tag{3.26}$$

Substituting for m from (3.26) in (1.6)-(1.9), we have

$$\left\{ \frac{\beta^3}{\alpha} \right\}^{1/8} = \frac{\sqrt{(6\sqrt{3}-9)} - 1}{2} \tag{3.27}$$

$$\left\{ \frac{(1-\beta)^3}{1-\alpha} \right\}^{1/8} = \frac{\sqrt{(6\sqrt{3}-9)} + 1}{2} \tag{3.28}$$

$$\left\{ \frac{(1-\alpha)^3}{1-\beta} \right\}^{1/8} = \frac{3 - \sqrt{(6\sqrt{3}-9)}}{2\sqrt{(6\sqrt{3}-9)}} \tag{3.29}$$

$$\left\{ \frac{\alpha^3}{\beta} \right\}^{1/8} = \frac{3 + \sqrt{(6\sqrt{3}-9)}}{2\sqrt{(6\sqrt{3}-9)}} \tag{3.30}$$

Now, from (1.10) and (1.26), we get

$$\sqrt{z_1} = \phi(e^{-\pi}) = a \tag{3.31}$$

Thus, using (3.26) and (3.31), we have

$$\sqrt{z_3} = \phi(e^{-3\pi}) = \frac{a}{(6\sqrt{3}+9)^{1/4}} \tag{3.32}$$

Thus, we have

$$\sqrt{(z_1^3/z_3)} = a^2(6\sqrt{3}-9)^{1/4} \tag{3.33}$$

and

$$\sqrt{(z_3^3/z_1)} = \frac{a^2(6\sqrt{3}-9)^{1/4}}{6\sqrt{3}-9} \tag{3.34}$$

Now, applying (3.27)-(3.34) we get our results (3.2)-(3.25) for $q = e^{-\pi}$. For example, (3.2), (3.29) and (3.33) yield (2.1); (3.3), (3.38) and (3.34) yield (2.2). Similarly, other results of section 2 could also be proved. Specifically (2.5) and (2.6) lead to (2.7) and (2.8); (2.9) and (2.10) yield (2.11) and (2.12); (2.13) and (2.14) lead to (2.15) and (2.16); (2.17) and (2.18) lead to (2.19) and (2.20).

Also applying (2.21) and (2.22) we get (2.23) and (2.24); (2.25) and (2.26) provide the proof of (2.27) and (2.28). Similarly (2.29) and (2.30) yield (2.31) and (2.32); (2.33) and (2.34) yield (2.35) and (2.36); (2.37) and (2.38) lead to (2.39) and (2.40); (2.41) and (2.42) provide the proof of (2.43) and (2.44) and finally (2.45) and (2.46) lead to then proof of (2.47) and (2.48). The results discussed here are useful in the evaluation of functions involving theta functions, cubic continued fractions and Ramanujan-Weber class invariants.

4. Application

In this section we shall discuss some very interesting applications of a few of our results. We can easily show that

$$\frac{q^{1/3}}{1+}\,\frac{q+q^2}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^3+q^6}{1+\ldots}$$

$$= \frac{\{(6\sqrt{3}-9)-1\}^{1/3}}{\{\sqrt{(6\sqrt{3}-9)}+1\}^{2/3}} \qquad (\text{for } \; q = e^{-\pi}) \tag{4.1}$$

(with the help of (1.27), (1.29) and (2.4)).

$$= \frac{\{(6\sqrt{3}-9)-1\}^{2/3}}{2\{\sqrt{(6\sqrt{3}-9)}+1\}^{1/3}} \qquad (\text{for } \; q = e^{-2\pi}) \tag{4.2}$$

(with the help of (1.27), (1.29) and (2.8)).

$$= \frac{1}{(6\sqrt{3}+8)^{1/3}} \qquad (\text{for } \; q = -e^{-\pi}) \tag{4.3}$$

(with the help of (1.27), (1.28) and (2.16)).

Next, we discuss the evaluation of Ramanujan-Webers constants. If we make use of (1.24) with n=1 and n=9, respectively and applying (2.40), we get

$$\frac{G_1}{G_9} = \left\{\frac{\sqrt{2}}{1+\sqrt{3}}\right\}^{1/3} \tag{4.4}$$

Since $G_1 = 1$ (cf. Berndt [3; page 189]) we get

$$G_9 = \left\{\frac{1+\sqrt{3}}{\sqrt{2}}\right\}^{1/3} \tag{4.5}$$

This is a known result (cf. Berndt [3; page 189]).

Again, if we put n=1 and n=4, respectively in (1.25) and make use of a known result (cf. Berndt [3; Chapter 35, entries 2(V) and 2(VI), p. 326]), we get

$$g_1 = 2^{-1/8} \tag{4.6}$$

and

$$g_4 = 2^{1/8} \tag{4.7}$$

Also, (1.25) and (2.44) lead to

$$\frac{g_1}{g_9} = \frac{(2-\sqrt{3})^{1/24}}{3^{3/16}} \times \left\{\frac{3-\sqrt{(6\sqrt{3}-9)}}{1+\sqrt{(6\sqrt{3}-9)}}\right\}^{1/4} \tag{4.8}$$

Again, (1.25) and (2.48) lead to

$$\frac{g_4}{g_{36}} = \frac{3^{3/16}}{(2-\sqrt{3})^{1/24}} \times \left\{\frac{\sqrt{(6\sqrt{3}-9)}-1}{\sqrt{(6\sqrt{3}-9)}+3}\right\}^{1/4} \tag{4.9}$$

From (4.6) and (4.8), we get

$$g_9 = \frac{3^{3/16}}{2^{1/8}(2-\sqrt{3})^{1/24}} \times \left\{\frac{1+\sqrt{(6\sqrt{3}-9)}}{3-\sqrt{(6\sqrt{3}-9)}}\right\}^{1/4} \tag{4.10}$$

Similarly, (4.7) and (4.9) yield

$$g_{36} = \frac{2^{1/8}(2-\sqrt{3})^{1/24}}{3^{3/16}} \times \left\{\frac{\sqrt{(6\sqrt{3}-9)}+3}{\sqrt{(6\sqrt{3}-9)}-1}\right\}^{1/4} \tag{4.11}$$

g_1, g_4, g_9 and g_{36} are new addition to the list of Ramanujan-Weber constants. Similarly, several other values of the invariants can also be calculated. The results established herein may be useful in the study of theta functions and related results.

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