*J. of Ramanujan Society of Mathematics and Mathematical Sciences*
*Vol. 10, No. 1 (2022), pp. 47-78*

DOI: 10.56827/JRSMMS.2022.1001.5 **ISSN (Online): 2582-5461**

**ISSN (Print): 2319-1023**

# SOME RESULTS IN CONNECTION WITH SUM AND PRODUCT THEOREMS RELATED TO GENERALIZED RELATIVE ORDER $(\alpha, \beta)$ AND GENERALIZED RELATIVE TYPE $(\alpha, \beta)$ OF ENTIRE FUNCTIONS IN THE UNIT DISC

**Tanmay Biswas and Chinmay Biswas***

Rajbari, Rabindrapally, R. N. Tagore Road,
Krishnagar, Nadia - 741101, West Bengal, INDIA

E-mail : tanmaybiswas_math@rediffmail.com

*Department of Mathematics,
Nabadwip Vidyasagar College,
Nabadwip, Nadia - 741302, West Bengal, INDIA

E-mail : chinmay.shib@gmail.com

(**Received:** *Jul. 23, 2022* **Accepted:** *Oct. 21, 2022* **Published:** *Dec. 30, 2022)*

**Abstract:** Orders and types of entire functions have been actively investigated by many authors. In this paper, we investigate some basic properties in connection with sum and product of generalized relative order $(\alpha, \beta)$, generalized relative type $(\alpha, \beta)$ and generalized relative weak type $(\alpha, \beta)$ of entire functions in the unit disc $D$ with respect to another entire function where $\alpha$, $\beta$ are continuous non-negative functions on $(-\infty, +\infty)$.

**Keywords and Phrases:** Entire function, growth, composition, generalized relative order $(\alpha, \beta)$, generalized relative type $(\alpha, \beta)$, generalized relative weak type $(\alpha, \beta)$.

**2020 Mathematics Subject Classification:** 30D35, 30D30, 30D20.

## 1. Introduction and Definitions

Let $h(z) = \sum\limits_{n=0}^{\infty} c_n z^n$ be analytic in the unit disc $U = \{z : |z| < 1\}$ and $M_h(r)$ be the maximum of $|h(z)|$ on $|z| = r$. In [12], Sons defined the order $\varrho(h)$ and the lower order $\lambda(h)$ as

$$\varrho(h) = \limsup_{r\to 1} \frac{\log^{[2]} M_h(r)}{-\log(1-r)} \text{ and } \lambda(h) = \liminf_{r\to 1} \frac{\log^{[2]} M_h(r)}{-\log(1-r)}.$$

However during the last several years many authors have investigated different properties of analytic function in the unit disc $U$ and derived so many great results e.g. [7, 8, 9, 10, 11]. The notion of relative order was first introduced by Bernal [2, 3].

An entire function $h$ is said to have Property (D), if for any $\delta > 1$, $\gamma > 0$ and for all $r$ $(0 < r < 1)$ sufficiently close to 1,

$$\left(M_h\left(\beta\left(\frac{1}{1-r}\right)^{\gamma}\right)\right)^2 \le M_h\left(\left(\beta\left(\frac{1}{1-r}\right)^{\gamma}\right)^{\delta}\right).$$

Now let $L$ be a class of continuous non-negative functions $\alpha$ defined on $(-\infty,\infty)$ such that $\alpha(x) = \alpha(x_0) \ge 0$ for $x \le x_0$ with $\alpha(x) \uparrow \infty$ as $x \to \infty$. Further we assume that throughout the present paper $\alpha$, $\beta \in L$. Now considering this, Biswas et al. [4] have introduced the definitions of the generalized order $(\alpha,\beta)$ and generalized lower order $(\alpha,\beta)$ of an entire function $h$ in the unit disc $U$ which are as follows:

**Definition 1.** [4] *The generalized order $(\alpha,\beta)$ denoted by $\varrho_{(\alpha,\beta)}[h]$ and generalized lower order $(\alpha,\beta)$ denoted by $\lambda_{(\alpha,\beta)}[h]$ of an entire function $h$ in the unit disc $U$ are defined as:*

$$\varrho_{(\alpha,\beta)}[h] = \limsup_{r\to 1} \frac{\alpha(M_h(r))}{\beta((1-r)^{-1})} \text{ and } \lambda_{(\alpha,\beta)}[h] = \liminf_{r\to 1} \frac{\alpha(M_h(r))}{\beta((1-r)^{-1})}.$$

Now for making some progresses about the works of relative order, one can introduce the definitions of generalized relative order $(\alpha,\beta)$ and generalized relative lower order $(\alpha,\beta)$ of an entire functions in the unit disc $U$ with respect to another entire function in the following way:

**Definition 2.** *Let $h$ and $k$ be entire functions defined in the unit disc $U$, the quantities $\varrho_{(\alpha,\beta)}[h]_k$ and $\lambda_{(\alpha,\beta)}[h]_k$ respectively called generalized relative order $(\alpha,\beta)$ and generalized relative lower order $(\alpha,\beta)$ of $h$ with respect to $k$, are defined as:*

$$\varrho_{(\alpha,\beta)}[h]_k = \limsup_{r\to 1} \frac{\alpha(M_k^{-1}(M_h(r)))}{\beta((1-r)^{-1})} \text{ and } \lambda_{(\alpha,\beta)}[h]_k = \liminf_{r\to 1} \frac{\alpha(M_k^{-1}(M_h(r)))}{\beta((1-r)^{-1})}.$$

Further if $\varrho_{(\alpha,\beta)}[h]_k$ and $\lambda_{(\alpha,\beta)}[h]_k$ are the same, then we call $h$ as a function of regular generalized relative growth $(\alpha,\beta)$ with respect to $k$. Otherwise, we call $h$ as a function of irregular generalized relative growth $(\alpha,\beta)$ with respect to $k$.

Now in order to refine the growth scale namely the generalized relative order $(\alpha, \beta)$, we introduce the definitions of another growth indicators, called generalized relative type $(\alpha, \beta)$ and generalized relative lower type $(\alpha, \beta)$ respectively of an entire function $h$ with respect to an entire function $k$ in the unit disc $U$ which are as follows:

**Definition 3.** *Let $h$ and $k$ be entire functions defined in the unit disc $U$ with $h$ have finite positive generalized relative order $(\alpha, \beta)$ with respect to $k$ (i.e., $0 < \varrho_{(\alpha,\beta)}[h]_k < \infty$), then the quantities $\sigma_{(\alpha,\beta)}[h]_k$ and $\overline{\sigma}_{(\alpha,\beta)}[h]_k$ respectively called generalized relative type $(\alpha, \beta)$ and generalized relative lower type $(\alpha, \beta)$ of $h$ with respect to $k$, are defined as:*

$$\begin{aligned}\sigma_{(\alpha,\beta)}[h]_k &= \limsup_{r\to 1} \frac{\exp(\alpha(M_k^{-1}(M_h(r))))}{(\beta((1-r)^{-1}))^{\varrho_{(\alpha,\beta)}[h]_k}} \text{ and} \\ \overline{\sigma}_{(\alpha,\beta)}[h]_k &= \liminf_{r\to 1} \frac{\exp(\alpha(M_k^{-1}(M_h(r))))}{(\beta((1-r)^{-1}))^{\varrho_{(\alpha,\beta)}[h]_k}}.\end{aligned}$$

It is obvious that $0 \le \overline{\sigma}_{(\alpha,\beta)}[h]_k \le \sigma_{(\alpha,\beta)}[h]_k \le \infty$.

Analogously, to determine the relative growth of two entire functions in the unit disc $U$ having same non zero finite generalized relative lower order $(\alpha, \beta)$, one can introduce the definitions of generalized relative weak type $(\alpha, \beta)$ denoted by $\tau_{(\alpha,\beta)}[h]_k$ and generalized relative upper weak type $(\alpha, \beta)$ denoted by $\overline{\tau}_{(\alpha,\beta)}[h]_k$ of an entire function $h$ with respect to entire function $k$ in the unit disc $U$ in the following way:

**Definition 4.** *Let $h$ and $k$ be entire functions defined in the unit disc $U$ with $h$ have finite positive generalized relative lower order $(\alpha, \beta)$ (i.e., $0 < \lambda_{(\alpha,\beta)}[h]_k < \infty$), then the quantities $\tau_{(\alpha,\beta)}[h]_k$ and $\overline{\tau}_{(\alpha,\beta)}[h]_k$ respectively called generalized relative weak type $(\alpha, \beta)$ and generalized relative upper weak type $(\alpha, \beta)$ of $h$ with respect to $k$, are defined as:*

$$\begin{aligned}\tau_{(\alpha,\beta)}[h]_k &= \liminf_{r\to 1} \frac{\exp(\alpha(M_k^{-1}(M_h(r))))}{(\beta((1-r)^{-1}))^{\lambda_{(\alpha,\beta)}[h]_k}} \text{ and} \\ \overline{\tau}_{(\alpha,\beta)}[h]_k &= \limsup_{r\to 1} \frac{\exp(\alpha(M_k^{-1}(M_h(r))))}{(\beta((1-r)^{-1}))^{\lambda_{(\alpha,\beta)}[h]_k}}.\end{aligned}$$

It is obvious that $0 \le \tau_{(\alpha,\beta)}[h]_k \le \overline{\tau}_{(\alpha,\beta)}[h]_k \le \infty$.

We finally remind the following definition which is needed in the sequel.

**Definition 5.** *Let $h$ and $k$ be entire functions defined in the unit disc $U$. Then* *they are said to have mutually Property (X) in $U$ if for all $r$, $0 < r < 1$, sufficiently close to* 1,

$$M_{h\cdot k}(r) > M_h(r) \quad \textit{and} \quad M_{h\cdot k}(r) > M_k(r)$$

*hold simultaneously.*

Here, in this paper, our aim is to investigate some basic properties of entire functions in the unit disc $U$ connected to generalized relative order $(\alpha, \beta)$, generalized relative type $(\alpha, \beta)$ and generalized relative weak type $(\alpha, \beta)$ with respect to another entire function under somewhat different conditions. In this paper, we suppose that all the growth indicators are nonzero finite.We do not explain the standard definitions and notations in the theory of entire functions as those are available in [1], [5], [6], [13] and [14].

## 2. Main Results

In this section, we present the main results of the paper.

**Theorem 1.** *Let $h_1$, $h_2$ and $k_1$ be entire functions defined in the unit disc $U$ such that at least $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$. Then*

$$\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} \leq \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}.$$

*The equality holds when $\lambda_{(\alpha,\beta)}[h_i]_{k_1} > \lambda_{(\alpha,\beta)}[h_j]_{k_1}$ with at least $h_j$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$ where $i, j = 1, 2$ and $i \neq j$.*

**Proof.** If $\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} = 0$ then theorem is trivially true. So we take $\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} > 0$. Clearly $\lambda_{(\alpha,\beta)}[h_k]_{k_1}$ is finite for $k = 1, 2$. Also let $\max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\} = \Delta$ and $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$. Now for any arbitrarily chosen $\eta > 0$ from the definition of $\lambda_{(\alpha,\beta)}[h_1]_{k_1}$, we get for a sequence of $r$ tending to 1 that

$$M_{h_1}(r) \leq M_{k_1}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[h_1]_{k_1} + \eta)\beta((1-r)^{-1})])$$

$$i.e.,\ M_{h_1}(r) \leq M_{k_1}(\alpha^{-1}[(\Delta + \eta)\beta((1-r)^{-1})]). \tag{2.1}$$

Also for any arbitrarily chosen $\eta > 0$ and from the definition of $\varrho_{(\alpha,\beta)}[h_2]_{k_1}(= \lambda_{(\alpha,\beta)}[h_2]_{k_1})$, we obtain for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$M_{h_2}(r) \leq M_{k_1}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[h_2]_{k_1} + \eta)\beta((1-r)^{-1})])$$

$$i.e.,\ M_{h_2}(r) \leq M_{k_1}(\alpha^{-1}[(\Delta + \eta)\beta((1-r)^{-1})]). \tag{2.2}$$

So from (2.1) and (2.2), we have for a sequence of $r$ tending to 1 that

$$M_{h_1 \pm h_2}(r) < 2M_{k_1}(\alpha^{-1}[(\Delta + \eta)\beta((1-r)^{-1})]).$$

$$M_{h_1 \pm h_2}(r) < 2M_{k_1}(\alpha^{-1}[\log(\exp \beta((1-r)^{-1}))^{(\Delta+\eta)}]).$$
$$i.e., M_{h_1 \pm h_2}(r) < M_{k_1}(\alpha^{-1}[\log(\exp \beta((1-r)^{-1}))^{(\Delta+2\varepsilon)}])$$
$$i.e., \; \frac{\alpha(M_{k_1}^{-1}(M_{h_1 \pm h_2}(r)))}{\beta((1-r)^{-1})} < (\Delta + 2\varepsilon).$$

Hence

$$\begin{aligned} \liminf_{r \to 1} \frac{\alpha(M_{k_1}^{-1}(M_{h_1 \pm h_2}(r)))}{\beta((1-r)^{-1})} &\leq \Delta + 2\varepsilon. \\ i.e., \; \lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} &\leq \Delta + 2\varepsilon. \end{aligned}$$

Since $\eta > 0$ is arbitrary, we get above

$$\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} \leq \Delta = \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}.$$

Similarly, if we take $h_1$ as a function of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$ or both $h_1$ and $h_2$ are of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$, then we can verify that

$$\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} \leq \Delta = \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}. \tag{2.3}$$

Moreover without loss of any generality, let $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ and $h = h_1 \pm h_2$. Then in view of (2.3) we get that $\lambda_{(\alpha,\beta)}[h]_{k_1} \leq \lambda_{(\alpha,\beta)}[h_2]_{k_1}$. As, $h_2 = \pm(h - h_1)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[h_2]_{k_1} \leq \max\{\lambda_{(\alpha,\beta)}[h]_{k_1}, \lambda_{(\alpha,\beta)}[h_1]_{k_1}\}$ . As we assume that $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_2]_{k_1}$, therefore we have $\lambda_{(\alpha,\beta)}[h_2]_{k_1} \leq \lambda_{(\alpha,\beta)}[h]_{k_1}$ and
hence $\lambda_{(\alpha,\beta)}[h]_{k_1} = \lambda_{(\alpha,\beta)}[h_2]_{k_1} = \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}$. Therefore, $\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} = \lambda_{(\alpha,\beta)}[h_i]_{k_1} \mid i = 1, 2$ provided $\lambda_{(\alpha,\beta)}[h_1]_{k_1} \neq \lambda_{(\alpha,\beta)}[h_2]_{k_1}$. Thus the theorem follows.

**Theorem 2.** *Let $h_1$, $h_2$, $k_1$ be all entire functions defined in the unit disc $U$ such that $\varrho_{(\alpha,\beta)}[h_1]_{k_1}$ and $\varrho_{(\alpha,\beta)}[h_2]_{k_1}$ exist. Then*

$$\varrho_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} \leq \max\{\varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_2]_{k_1}\}.$$

*The equality holds when $\varrho_{(\alpha,\beta)}[h_1]_{k_1} \neq, \varrho_{(\alpha,\beta)}[h_2]_{k_1}$.*

We omit the proof of Theorem 2 as easily it can be derived in view of Theorem 1.

**Theorem 3.** *Let $h_1$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$ such that $\lambda_{(\alpha,\beta)}[h_1]_{k_1}$ and $\lambda_{(\alpha,\beta)}[h_1]_{k_2}$ exist. Then*

$$\lambda_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} \geq \min\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_1]_{k_2}\}.$$

*The equality holds when* $\lambda_{(\alpha,\beta)}[h_1]_{k_1} \neq \lambda_{(\alpha,\beta)}[h_1]_{k_2}$.

**Proof.** If $\lambda_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \infty$ then the theorem is trivially true. So we suppose that $\lambda_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} < \infty$. We can clearly assume that $\lambda_{(\alpha,\beta)}[h_1]_{k_k}$ is finite for $k = 1, 2$. Also let $\Psi = \min\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_1]_{k_2}\}$. Now for any arbitrary $\eta > 0$ from the definition of $\lambda_{(\alpha,\beta)}[h_1]_{k_k}$ where $k = 1, 2$, we have for all $r$, $0 < r < 1$, sufficiently close to 1

$$M_{k_k}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[h_1]_{k_k} - \eta)\beta((1-r)^{-1})]) \leq M_{h_1}(r)$$

$$i.e,\ M_{k_k}(\alpha^{-1}[(\Psi - \eta)\beta((1-r)^{-1})]) \leq M_{h_1}(r)$$

Hence, we obtain from above for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$\begin{aligned} &M_{k_1 \pm k_2}(\alpha^{-1}[(\Psi - \eta)\beta((1-r)^{-1})]) \\ &< M_{k_1}(\alpha^{-1}[(\Psi - \eta)\beta((1-r)^{-1})]) + M_{k_2}(\alpha^{-1}[(\Psi - \eta)\beta((1-r)^{-1})]) \end{aligned}$$

$$i.e.,\ M_{k_1 \pm k_2}(\alpha^{-1}[(\Psi - \eta)\beta((1-r)^{-1})]) < 2M_{h_1}(r)$$

$$i.e.,\ M_{k_1 \pm k_2}(\alpha^{-1}[\log(\exp\beta((1-r)^{-1}))^{(\Psi-\eta)}]) < 2M_{h_1}(r)$$

$$i.e.,\ \frac{1}{2}M_{k_1 \pm k_2}(\alpha^{-1}[\log(\exp\beta((1-r)^{-1}))^{(\Psi-\eta)}]) < M_{h_1}(r)$$

$$i.e.,\ M_{k_1 \pm k_2}(\alpha^{-1}[\log(\exp\beta((1-r)^{-1}))^{(\Psi-2\varepsilon)}]) < M_{h_1}(r)$$

$$i.e.,\ \frac{\alpha(M^{-1}_{k_1 \pm k_2}(M_{h_1}(r)))}{\beta((1-r)^{-1})} > \Psi - 2\varepsilon.$$

Since $\eta > 0$ is arbitrary, we get from above that

$$\lambda_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} \geq \Psi = \min\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_1]_{k_2}\}. \tag{2.4}$$

Now without loss of any generality, we can take that $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$ and $k = k_1 \pm k_2$. Then in view of (2.4) we get that $\lambda_{(\alpha,\beta)}[h_1]_k \geq \lambda_{(\alpha,\beta)}[h_1]_{k_1}$. Further, $k_1 = (k \pm k_2)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[h_1]_{k_1} \geq \min\{\lambda_{(\alpha,\beta)}[h_1]_k, \lambda_{(\alpha,\beta)}[h_1]_{k_2}\}$ . As we assume that $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, therefore we have $\lambda_{(\alpha,\beta)}[h_1]_{k_1} \geq \lambda_{(\alpha,\beta)}[h_1]_k$ and hence $\lambda_{(\alpha,\beta)}[h_1]_k = \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \min\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_1]_{k_2}\}$. Therefore, $\lambda_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \lambda_{(\alpha,\beta)}[h_1]_{k_i} \mid i = 1, 2$ provided $\lambda_{(\alpha,\beta)}[h_1]_{k_1} \neq \lambda_{(\alpha,\beta)}[h_1]_{k_2}$. Thus the theorem follows.

**Theorem 4.** *Let $h_1$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$ such that $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to at least any one of $k_1$ or $k_2$. Then*

$$\varrho_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} \geq \min\{\varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_1]_{k_2}\}.$$

*The equality holds when $\varrho_{(\alpha,\beta)}[h_1]_{k_i} < \varrho_{(\alpha,\beta)}[h_1]_{k_j}$ with at least $h_1$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_j$ where $i = j = 1, 2$ and $i \neq j$.*

We omit the proof of Theorem 4 as it can be easily derived in view of Theorem 3.

**Theorem 5.** *Let $h_1$, $h_2$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$, then*

$$\begin{aligned} & \varrho_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} \\ \leq \;& \max[\min\{\varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_1]_{k_2}\}, \min\{\varrho_{(\alpha,\beta)}[h_2]_{k_1}, \varrho_{(\alpha,\beta)}[h_2]_{k_2}\}] \end{aligned}$$

*when the following two conditions holds:*
*(i) $\varrho_{(\alpha,\beta)}[h_1]_{k_i} < \varrho_{(\alpha,\beta)}[h_1]_{k_j}$ with at least $h_1$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_j$ for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and*
*(ii) $\varrho_{(\alpha,\beta)}[h_2]_{k_i} < \varrho_{(\alpha,\beta)}[h_2]_{k_j}$ with at least $h_2$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_j$ for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.*
*The equality holds when both $\varrho_{(\alpha,\beta)}[h_i]_{k_1} < \varrho_{(\alpha,\beta)}[h_j]_{k_1}$ and $\varrho_{(\alpha,\beta)}[h_i]_{k_2} < \varrho_{(\alpha,\beta)}[h_j]_{k_2}$ hold for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.*

**Proof.** Let both the conditions $(i)$ and $(ii)$ hold. Then from Theorem 2 and Theorem 4, we get

$$\begin{aligned} & \max[\min\{\varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_1]_{k_2}\}, \min\{\varrho_{(\alpha,\beta)}[h_2]_{k_1}, \varrho_{(\alpha,\beta)}[h_2]_{k_2}\}] \\ & = \max[\varrho_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2}, \varrho_{(\alpha,\beta)}[h_2]_{k_1 \pm k_2}] \\ & \geq \varrho_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} \; . \end{aligned} \tag{2.5}$$

As $\varrho_{(\alpha,\beta)}[h_i]_{k_1} < \varrho_{(\alpha,\beta)}[h_j]_{k_1}$ and $\varrho_{(\alpha,\beta)}[h_i]_{k_2} < \varrho_{(\alpha,\beta)}[h_j]_{k_2}$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we obtain that

$$\text{either } \min\{\varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_1]_{k_2}\} > \min\{\varrho_{(\alpha,\beta)}[h_2]_{k_1}, \varrho_{(\alpha,\beta)}[h_2]_{k_2}\} \text{ or}$$

$$\min\{\varrho_{(\alpha,\beta)}[h_2]_{k_1}, \varrho_{(\alpha,\beta)}[h_2]_{k_2}\} > \min\{\varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_1]_{k_2}\} \text{ holds.}$$

Hence from the conditions $(i)$ and $(ii)$, we have from above that

$$\text{either } \varrho_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} > \varrho_{(\alpha,\beta)}[h_2]_{k_1 \pm k_2} \text{ or } \varrho_{(\alpha,\beta)}[h_2]_{k_1 \pm k_2} > \varrho_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2}$$

which is the condition for holding equality in (2.5). Hence the theorem follows.

**Theorem 6.** *Let $h_1$, $h_2$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$, then*

$$\begin{aligned} & \lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} \\ \geq \;& \min[\max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}, \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_2}, \lambda_{(\alpha,\beta)}[h_2]_{k_2}\}] \end{aligned}$$

*when the following two conditions hold:*
*(i)* $\lambda_{(\alpha,\beta)}[h_i]_{k_1} > \lambda_{(\alpha,\beta)}[h_j]_{k_1}$ *with at least* $h_j$ *is of regular generalized relative growth* $(\alpha, \beta)$ *with respect to* $k_1$ *for* $i = 1, 2, j = 1, 2$ *and* $i \neq j$*; and*
*(ii)* $\lambda_{(\alpha,\beta)}[h_i]_{k_2} > \lambda_{(\alpha,\beta)}[h_j]_{k_2}$ *with at least* $h_j$ *is of regular generalized relative growth* $(\alpha, \beta)$ *with respect to* $k_2$ *for* $i = 1, 2, j = 1, 2$ *and* $i \neq j$*.*
*The sign of equality holds when both the conditions* $\lambda_{(\alpha,\beta)}[h_1]_{k_i} < \lambda_{(\alpha,\beta)}[h_1]_{k_j}$ *and* $\lambda_{(\alpha,\beta)}[h_2]_{k_i} < \lambda_{(\alpha,\beta)}[h_2]_{k_j}$ *hold for* $i = 1, 2$*;* $j = 1, 2$ *and* $i \neq j$*.*

**Proof.** Let both the conditions $(i)$ and $(ii)$ hold. Then from Theorem 1 and Theorem 3, we get that

$$\begin{aligned}
&\min[\max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}, \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_2}, \lambda_{(\alpha,\beta)}[h_2]_{k_2}\}] \\
&= \min[\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1}, \lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_2}] \\
&\leq \lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} \ . \qquad (2.6)
\end{aligned}$$

Since $\lambda_{(\alpha,\beta)}[h_1]_{k_i} < \lambda_{(\alpha,\beta)}[h_1]_{k_j}$ and $\lambda_{(\alpha,\beta)}[h_2]_{k_i} < \lambda_{(\alpha,\beta)}[h_2]_{k_j}$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we get that

$$\text{either } \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\} < \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_2}, \lambda_{(\alpha,\beta)}[h_2]_{k_2}\} \text{ or}$$

$$\max\{\lambda_{(\alpha,\beta)}[h_1]_{k_2}, \lambda_{(\alpha,\beta)}[h_2]_{k_2}\} < \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\} \text{ holds.}$$

Since conditions $(i)$ and $(ii)$ hold, we get from above that

$$\text{either } \lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} < \lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_2} \text{ or } \lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_2} < \lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1}$$

which is the condition for holding the equality in (2.6).

This completes the theorem.

**Theorem 7.** *Let* $h_1$*,* $h_2$*,* $k_1$ *be all entire functions defined in the unit disc* $U$ *such that at least* $h_1$ *or* $h_2$ *is of regular generalized relative growth* $(\alpha, \beta)$ *with respect to* $k_1$ *where* $k_1$ *satisfy the Property (D) and* $h_1$*,* $h_2$ *satisfy the Property (X), then*

$$\lambda_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} = \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}.$$

**Proof.** Suppose that $\lambda_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} > 0$. Otherwise if $\lambda_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} = 0$ then the theorem is trivially true. Let us consider that $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$. Also let $\max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}$ $= \Delta$. We can clearly assume that $\lambda_{(\alpha,\beta)}[h_k]_{k_1}$ is finite for $k = 1, 2$. Now for any arbitrarily chosen $\frac{\eta}{2} > 0$, it follows from the definition of $\lambda_{(\alpha,\beta)}[h_1]_{k_1}$, for a sequence of $r$ tending to 1 that

$$M_{h_1}(r) \leq M_{k_1}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[h_1]_{k_1} + \frac{\eta}{2})\beta((1-r)^{-1})])$$

$$i.e.,\ M_{h_1}(r) \leq M_{k_1}(\alpha^{-1}[(\Delta + \frac{\eta}{2})\beta((1-r)^{-1})]). \tag{2.7}$$

Also for any arbitrarily chosen $\frac{\eta}{2} > 0$, we obtain from the definition of $\varrho_{(\alpha,\beta)}[h_2]_{k_1}$ $(= \lambda_{(\alpha,\beta)}[h_2]_{k_1})$, for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$M_{h_2}(r) \leq M_{k_1}(\alpha^{-1}[(\varrho_{(\alpha,\beta)}[h_2]_{k_1} + \frac{\eta}{2})\beta((1-r)^{-1})])$$

$$i.e.,\ M_{h_2}(r) \leq M_{k_1}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[h_2]_{k_1} + \frac{\eta}{2})\beta((1-r)^{-1})])$$

$$i.e.,\ M_{h_2}(r) \leq M_{k_1}(\alpha^{-1}[(\Delta + \frac{\eta}{2})\beta((1-r)^{-1})]). \tag{2.8}$$

Observe that

$$\frac{\Delta + \eta}{\Delta + \frac{\eta}{2}} > 1.$$

Therefore we consider the expression $\frac{\log[\alpha^{-1}[(\Delta+\eta)\beta((1-r)^{-1})]]}{\log[\alpha^{-1}[(\Delta+\frac{\eta}{2})\beta((1-r)^{-1})]]}$ for all $r$, $0 < r < 1$, sufficiently close to 1. Thus for any $\delta > 1$, it follows from the above that there is $r_0$ such that, $0 < r_0 < 1$, for which

$$\frac{\log[\alpha^{-1}[(\Delta + \eta)\beta(r_0)]]}{\log[\alpha^{-1}[(\Delta + \frac{\eta}{2})\beta(r_0)]]} = \delta. \tag{2.9}$$

Hence from (2.7) and (2.8), we have for a sequence of $r$ tending to 1 that

$$M_{h_1 \cdot h_2}(r) < [M_{k_1}(\alpha^{-1}[(\Delta + \frac{\eta}{2})\beta((1-r)^{-1})])]^2.$$

Now we obtain from above for a sequence of $r$ tending to 1 that

$$M_{h_1 \cdot h_2}(r) < M_{k_1}((\alpha^{-1}[(\Delta + \frac{\eta}{2})\beta((1-r)^{-1})])^{\delta}),$$

since $k_1$ has the Property (D) and $\delta > 1$. Therefore from (2.9), we get from above for a sequence of $r$ tending to 1 that

$$M_{h_1 \cdot h_2}(r) < M_{k_1}(\alpha^{-1}[(\Delta + \eta)\beta((1-r)^{-1})]).$$

Since $\eta > 0$ is arbitrary, we get from above that

$$\lambda_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} \leq \Delta = \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}.$$

Similarly, if we take $h_1$ as a function of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$ or both $h_1$ and $h_2$ as functions of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$, then we can easily show that

$$\lambda_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} \leq \Delta = \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}.$$

Let us now show that $\lambda_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} \geq \Delta$. Since $h_1$, $h_2$ satisfy the Property (X), we have $M_{h_1 \cdot h_2}(r) > M_{h_1}(r)$ for all $r$, $0 < r < 1$, sufficiently close to 1 and therefore

$$\frac{\alpha(M_{k_1}^{-1}(M_{h_1 \cdot h_2}(r)))}{\beta((1-r)^{-1})} > \frac{\alpha(M_{k_1}^{-1}(M_{h_1}(r)))}{\beta((1-r)^{-1})}$$

since $M_{k_1}^{-1}(r)$ is an increasing function of $r$. So $\lambda_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} \geq \lambda_{(\alpha,\beta)}[h_1]_{k_1}$ and similarly, $\lambda_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} \geq \lambda_{(\alpha,\beta)}[h_2]_{k_1}$.

This completes the proof.

Now we state the next theorem which can be easily followed in view of Theorem 7 and so its proof is omitted.

**Theorem 8.** *Let $h_1$, $h_2$, $k_1$ be all entire functions defined in the unit disc $U$ such that $\varrho_{(\alpha,\beta)}[h_1]_{k_1}$, $\varrho_{(\alpha,\beta)}[h_1]_{k_2}$ exist where $k_1$ satisfies the Property (D) and $h_1$ , $h_2$ satisfy the Property (X), then*

$$\varrho_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} = \max\{\varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_2]_{k_1}\}.$$

**Theorem 9.** *Let $h_1$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$ such that $\lambda_{(\alpha,\beta)}[h_1]_{k_1}$, $\lambda_{(\alpha,\beta)}[h_1]_{k_2}$ exist where $k_1 \cdot k_2$ satisfies the Property (D) and $k_1$, $k_2$ satisfy the Property (X), then*

$$\lambda_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2} = \min\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_1]_{k_2}\}.$$

**Proof.** Suppose that $\lambda_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2} < \infty$. Otherwise if $\lambda_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2} = \infty$ then the theorem is trivially true. Also let $\min\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_1]_{k_2}\} = \Psi$. We can clearly assume that $\lambda_{(\alpha,\beta)}[h_1]_{k_k}$ is finite for $k = 1, 2$. Now for any arbitrary $\eta > 0$, with $\eta < \Psi$, we obtain for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$M_{k_k}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[h_1]_{k_k} - \frac{\eta}{2})\beta((1-r)^{-1})]) \leq M_{h_1}(r),$$

$$i.e., \;\; M_{k_k}(\alpha^{-1}[(\Psi - \frac{\eta}{2})\beta((1-r)^{-1})]) \leq M_{h_1}(r).$$

Hence we have

$$M_{k_1\cdot k_2}(\alpha^{-1}[(\Psi-\frac{\eta}{2})\beta((1-r)^{-1})])<[M_{h_1}(r)]^2,$$

$$i.e.,\ [M_{k_1\cdot k_2}(\alpha^{-1}[(\Psi-\frac{\eta}{2})\beta((1-r)^{-1})])]^{\frac{1}{2}}<M_{h_1}(r),$$

$$i.e.,\ [M_{k_1\cdot k_2}(\alpha^{-1}[\log(\exp\beta((1-r)^{-1}))^{(\Psi-\frac{\eta}{2})}])]^{\frac{1}{2}}<M_{h_1}(r).$$

We have from above for all $r$, $0<r<1$, sufficiently close to 1 that

$$M_{k_1\cdot k_2}(\alpha^{-1}[\log(\exp\beta((1-r)^{-1}))^{(\Psi-\frac{\eta}{2})}])^{\frac{1}{\delta}}<M_{h_1}(r)$$

since $k_1\cdot\ k_2$ has the Property (D) and $\delta>1$.

Therefore taking $\delta\to 1+$, we have

$$M_{k_1\cdot k_2}(\alpha^{-1}[(\Psi-\frac{\eta}{2})\beta((1-r)^{-1})])<M_{h_1}(r).$$

It follows from above for all $r$, $0<r<1$, sufficiently close to 1 that

$$\frac{\alpha(M^{-1}_{k_1\cdot k_2}(M_{h_1}(r)))}{\beta((1-r)^{-1})}>\Psi-\frac{\eta}{2}.$$

Since $\eta>0$ is arbitrary, from above we get that

$$\lambda_{(\alpha,\beta)}[h_1]_{k_1\cdot k_2}\geq\Psi=\min\{\lambda_{(\alpha,\beta)}[h_1]_{k_1},\lambda_{(\alpha,\beta)}[h_1]_{k_2}\}.$$

Let us now show that $\lambda_{(\alpha,\beta)}[h_1]_{k_1\cdot k_2}\leq\Psi$. Since $k_1$, $k_2$ satisfy the Property (X), we have $M_{k_1\cdot k_2}(r)>M_{k_1}(r)$ for all $r$, $0<r<1$, sufficiently close to 1 and therefore $M^{-1}_{k_1\cdot k_2}(r)<M^{-1}_{k_1}(r)$. Hence

$$\frac{\alpha(M^{-1}_{k_1\cdot k_2}(M_{h_1}(r)))}{\beta((1-r)^{-1})}<\frac{\alpha(M^{-1}_{k_1}(M_{h_1}(r)))}{\beta((1-r)^{-1})}.$$

So $\lambda_{(\alpha,\beta)}[h_1]_{k_1\cdot k_2}\leq\lambda_{(\alpha,\beta)}[h_1]_{k_1}$ and similarly, $\lambda_{(\alpha,\beta)}[h_1]_{k_1\cdot k_2}\leq\lambda_{(\alpha,\beta)}[h_1]_{k_2}$.

Hence the theorem follows.

**Theorem 10.** *Let $h_1$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$ such that $h_1$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to at least any one of $k_1$ or $k_2$ and $k_1\cdot k_2$ satisfy the Property (D) and $k_1$, $k_2$ satisfy the Property (X), then*

$$\varrho_{(\alpha,\beta)}[h_1]_{k_1\cdot k_2}=\min\{\varrho_{(\alpha,\beta)}[h_1]_{k_1},\varrho_{(\alpha,\beta)}[h_1]_{k_2}\}.$$

We omit the proof of Theorem 10 as it can easily be followed from Theorem 9.

Now we state the next two theorems without their proofs as one can easily derived their proofs from Theorem 5 and Theorem 6 respectively.

**Theorem 11.** *Let $h_1$, $h_2$, $k_1$, $k$ be all entire functions defined in the unit disc $U$ such that $k_1 \cdot k_2$ be satisfies the Property (D), $h_1$, $h_2$ satisfy the Property (X) and $k_1$, $k_2$ satisfy the Property (X), then,*

$$\begin{aligned} & \varrho_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1 \cdot k_2} \\ = \; & \max[\min\{\varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_1]_{k_2}\}, \min\{\varrho_{(\alpha,\beta)}[h_2]_{k_1}, \varrho_{(\alpha,\beta)}[h_2]_{k_2}\}], \end{aligned}$$

*when the following two conditions hold:*
*(i) $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to at least any one of $k_1$ or $k_2$; and*
*(ii) $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to at least any one of $k_1$ or $k_2$.*

**Theorem 12.** *Let $h_1$, $h_2$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$ such that $k_1 \cdot k_2$, $k_1$, $k_2$ be satisfy the Property (D), $h_1$, $h_2$ satisfy the Property (X) and $k_1$, $k_2$ satisfy the Property (X), then,*

$$\begin{aligned} & \lambda_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1 \cdot k_2} \\ = \; & \min[\max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}, \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_2}, \lambda_{(\alpha,\beta)}[h_2]_{k_2}\}] \end{aligned}$$

*when the following two conditions hold:*
*(i) At least $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$; and*
*(ii) At least $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_2$.*

Next we find out the sum and product related theorems with generalized relative type $(\alpha, \beta)$ ( respectively generalized relative lower type $(\alpha, \beta)$) and generalized relative weak type $(\alpha, \beta)$ of an entire functions in the unit disc $U$ with respect to an entire function taking into consideration of the above theorems.

**Theorem 13.** *Let $h_1$, $h_2$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$ such that $\varrho_{(\alpha,\beta)}[h_1]_{k_1}$, $\varrho_{(\alpha,\beta)}[h_2]_{k_1}$, $\varrho_{(\alpha,\beta)}[h_1]_{k_2}$ and $\varrho_{(\alpha,\beta)}[h_2]_{k_2}$ are all non-zero and finite.*
***(A)*** *If $\varrho_{(\alpha,\beta)}[h_i]_{k_1} > \varrho_{(\alpha,\beta)}[h_j]_{k_1}$ for $i$, $j = 1, 2$ and $i \neq j$, then*

$$\sigma_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} = \sigma_{(\alpha,\beta)}[h_i]_{k_1} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} = \overline{\sigma}_{(\alpha,\beta)}[h_i]_{k_1}.$$

***(B)*** *If* $\varrho_{(\alpha,\beta)}[h_1]_{k_i} < \varrho_{(\alpha,\beta)}[h_1]_{k_j}$ *with at least* $h_1$ *is of regular generalized relative growth* $(\alpha,\beta)$ *with respect to* $k_j$ *for* $i,\ j = 1,2$ *and* $i \neq j$, *then*

$$\sigma_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \sigma_{(\alpha,\beta)}[h_1]_{k_i} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_i}.$$

***(C)*** *Assume the functions* $h_1, h_2, k_1$ *and* $k_2$ *satisfy the following conditions:*
*(i)* $\varrho_{(\alpha,\beta)}[h_1]_{k_i} < \varrho_{(\alpha,\beta)}[h_1]_{k_j}$ *with at least* $h_1$ *is of regular generalized relative growth* $(\alpha,\beta)$ *with respect to* $k_j$ *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$;
*(ii)* $\varrho_{(\alpha,\beta)}[h_2]_{k_i} < \varrho_{(\alpha,\beta)}[h_2]_{k_j}$ *with at least* $h_2$ *is of regular generalized relative growth* $(\alpha,\beta)$ *with respect to* $k_j$ *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$;
*(iii)* $\varrho_{(\alpha,\beta)}[h_i]_{k_1} < \varrho_{(\alpha,\beta)}[h_j]_{k_1}$ *and* $\varrho_{(\alpha,\beta)}[h_i]_{k_2} < \varrho_{(\alpha,\beta)}[h_j]_{k_2}$ *hold simultaneously for* $i = 1,2;\ j = 1,2$ *and* $i \neq j$;
*(iv)* $\varrho_{(\alpha,\beta)}[h_l]_{k_m} =$
$\max[\min\{\varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_1]_{k_2}\}, \min\{\varrho_{(\alpha,\beta)}[h_2]_{k_1}, \varrho_{(\alpha,\beta)}[h_2]_{k_2}\}] \mid l = m = 1,2;$
*then we have*

$$\sigma_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} = \sigma_{(\alpha,\beta)}[h_l]_{k_m} \mid l,m = 1,2$$

*and*

$$\overline{\sigma}_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} = \overline{\sigma}_{(\alpha,\beta)}[h_l]_{k_m} \mid l,m = 1,2.$$

**Proof.** From the definitions of generalized relative type $(\alpha,\beta)$ and generalized relative lower type $(\alpha,\beta)$, we get for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$M_{h_k}(r) \leq M_{k_l}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_k]_{k_l} + \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_k]_{k_l}}\}), \tag{2.10}$$

$$M_{h_k}(r) \geq M_{k_l}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_k]_{k_l} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_k]_{k_l}}\}] \tag{2.11}$$

$$\text{i.e., } M_{k_l}(r) \leq M_{h_k}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[h_k]_{k_l} - \eta)})^{\frac{1}{\varrho_{(\alpha,\beta)}[h_k]_{k_l}}}))), \tag{2.12}$$

and for a sequence of values of $r$ tending to 1, we obtain that

$$M_{h_k}(r) \geq M_{k_l}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_k]_{k_l} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_k]_{k_l}}\}) \tag{2.13}$$

$$\text{i.e., } M_{k_l}(r) \leq M_{h_k}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[h_k]_{k_l} - \eta)})^{\frac{1}{\varrho_{(\alpha,\beta)}[h_k]_{k_l}}}))), \tag{2.14}$$

and

$$M_{h_k}(r) \leq M_{k_l}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_k]_{k_l} + \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_k]_{k_l}}\}), \tag{2.15}$$

where $\eta > 0$ is any arbitrary positive number, $k = 1,\ 2$ and $l = 1,2$.
**Case I.** Suppose that $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$ holds. Also let $\eta (> 0)$ be arbitrary. Now in view of (2.10), we get for all $r$ , $0 < r < 1$, sufficiently close to 1 that

$$M_{h_1 \pm h_2}(r) \leq M_{k_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} + \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \cdot (1 + A), \tag{2.16}$$

where $A = \frac{M_{k_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_2]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_2]_{k_1}}\})}{M_{k_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})}$, and in view of $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$, and for all $r$, $0 < r < 1$, sufficiently close to 1, we can make the term $A$ sufficiently small.

Hence for any $\xi = 1+\eta_1$, where $\eta_1 = A$, it follows from (2.16) for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$M_{h_1\pm h_2}(r) \leq M_{k_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \cdot (1+\eta_1)$$
$$i.e.,\ M_{h_1\pm h_2}(r) \leq M_{k_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}] \cdot \xi.$$

Hence making $\xi \to 1+$, we get in view of Theorem 2 , $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$ and above for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$\limsup_{r\to 1} \frac{\exp(\alpha(M_{k_1}^{-1}(M_{h_1\pm h_2}(r))))}{[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1\pm h_2]_{k_1}}} \leq \sigma_{(\alpha,\beta)}[h_1]_{k_1}$$

$$i.e.,\ \sigma_{(\alpha,\beta)}[h_1\pm h_2]_{k_1} \leq \sigma_{(\alpha,\beta)}[h_1]_{k_1}. \tag{2.17}$$

Now we may consider that $h = h_1 \pm h_2$. Since $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$ holds, then $\sigma_{(\alpha,\beta)}[h]_{k_1} = \sigma_{(\alpha,\beta)}[h_1\pm h_2]_{k_1} \leq \sigma_{(\alpha,\beta)}[h_1]_{k_1}$. Further, let $h_1 = (h \pm h_2)$. Therefore in view of Theorem 2 and $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$, we obtain that $\varrho_{(\alpha,\beta)}[h]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$ holds. Hence in view of (2.17) $\sigma_{(\alpha,\beta)}[h_1]_{k_1} \leq \sigma_{(\alpha,\beta)}[h]_{k_1} = \sigma_{(\alpha,\beta)}[h_1\pm h_2]_{k_1}$. Therefore $\sigma_{(\alpha,\beta)}[h]_{k_1} = \sigma_{(\alpha,\beta)}[h_1]_{k_1} \Rightarrow \sigma_{(\alpha,\beta)}[h_1\pm h_2]_{k_1} = \sigma_{(\alpha,\beta)}[h_1]_{k_1}$.

Similarly, if we consider $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_2]_{k_1}$, then one can easily verify that $\sigma_{(\alpha,\beta)}[h_1\pm h_2]_{k_1} = \sigma_{(\alpha,\beta)}[h_2]_{k_1}$.

**Case II.** Let us consider that $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$ holds. Also let $\eta(>0)$ be arbitrary. By (2.10) and (2.15), we have for a sequence of values of $r$ tending to 1 that

$$M_{h_1\pm h_2}(r) \leq M_{k_1}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \cdot (1+B), \tag{2.18}$$

where $B = \frac{M_{k_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_2]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_2]_{k_1}}\})}{M_{k_1}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})}$, and in view of $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$, we can make the term $B$ sufficiently small by taking $r$ ( where $0 < r < 1$) sufficiently close to 1 and therefore by the same technique of the proof of Case I, we have from (2.18) that $\overline{\sigma}_{(\alpha,\beta)}[h_1\pm h_2]_{k_1} = \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1}$ when $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$ holds.

Likewise, if we consider $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_2]_{k_1}$, then one can easily verify that $\overline{\sigma}_{(\alpha,\beta)}[h_1\pm h_2]_{k_1} = \overline{\sigma}_{(\alpha,\beta)}[h_2]_{k_1}$.

Hence from Case I and Case II, we get the first part of the theorem.

**Case III.** Let us consider that $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ with at least $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_2$. Hence from (2.11) and (2.13), we obtain for a sequence of values of $r$ tending to 1 that

$$\begin{aligned}
&M_{k_1\pm k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}-\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&\le M_{k_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}-\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&+ M_{k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}-\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&i.e., M_{k_1\pm k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}-\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \le M_{h_1}(r)(1+C)
\end{aligned} \tag{2.19}$$

where $C = \frac{M_{k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}-\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})}{M_{k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_2}-\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_2}}\})}$, and since $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$, we can make the term $C$ sufficiently small for a sequence of values of $r$ sufficiently close to 1. Hence for any $\xi = 1 + \eta_1$, where $\eta_1 = C$, we get from (2.19) and Theorem 4, for a sequence of values of $r$ tending to 1 that

$$M_{k_1\pm k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}-\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \le M_{h_1}(r)(1+\eta_1)$$

$$i.e., M_{k_1\pm k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}-\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \le M_{h_1}(r)\xi.$$

Hence, making $\xi \to 1+$, we obtain from above for a sequence of values of $r$ tending to 1 that

$$(\sigma_{(\alpha,\beta)}[h_1]_{k_1}-\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1\pm k_2}} < \exp(\alpha(M^{-1}_{k_1\pm k_2}(M_{h_1}(r)))).$$

Since $\eta > 0$ is arbitrary, we find that

$$\sigma_{(\alpha,\beta)}[h_1]_{k_1\pm k_2} \ge \sigma_{(\alpha,\beta)}[h_1]_{k_1}. \tag{2.20}$$

Now we may consider that $k = k_1 \pm k_2$. Also $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ and at least $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_2$. Then $\sigma_{(\alpha,\beta)}[h_1]_k = \sigma_{(\alpha,\beta)}[h_1]_{k_1\pm k_2} \ge \sigma_{(\alpha,\beta)}[h_1]_{k_1}$. Further let $k_1 = (k \pm k_2)$. Therefore in view of Theorem 4 and $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$, we obtain that $\varrho_{(\alpha,\beta)}[h_1]_k < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ as at least $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_2$. Hence in view of (2.20), $\sigma_{(\alpha,\beta)}[h_1]_{k_1} \ge \sigma_{(\alpha,\beta)}[h_1]_k = \sigma_{(\alpha,\beta)}[h_1]_{k_1\pm k_2}$. Therefore $\sigma_{(\alpha,\beta)}[h_1]_k = \sigma_{(\alpha,\beta)}[h_1]_{k_1}$
$\Rightarrow \sigma_{(\alpha,\beta)}[h_1]_{k_1\pm k_2} = \sigma_{(\alpha,\beta)}[h_1]_{k_1}$.

Similarly if we consider $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ with at least $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$, then $\sigma_{(\alpha,\beta)}[h_1]_{k_1\pm k_2} = \sigma_{(\alpha,\beta)}[h_1]_{k_2}$.

**Case IV.** In this case suppose that $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ with at least $h_1$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_2$. Therefore in view of (2.11), we have for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$\begin{aligned}
&M_{k_1 \pm k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&\leq M_{k_1}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&+ M_{k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&M_{k_1 \pm k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \leq M_{h_1}(r)(1+D),
\end{aligned} \tag{2.21}$$

where $D = \frac{M_{k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})}{M_{k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_2} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_2}}\})}$ and in view of $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$, we can make the term $D$ sufficiently small by taking $r$ sufficiently close to 1 and hence by the similar way of the proof of Case III we have from (2.21) that $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1}$ where $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ and at least $h_1$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_2$.

Similarly if we take $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ with at least $h_1$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_1$, then $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_2}$.

Thus from Case III and Case IV, we get the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem 5 and the first part and second part of the theorem. So its proof is omitted.

**Theorem 14.** *Let $h_1$, $h_2$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$ such that $\lambda_{(\alpha,\beta)}[h_1]_{k_1}$, $\lambda_{(\alpha,\beta)}[h_2]_{k_1}$, $\lambda_{(\alpha,\beta)}[h_1]_{k_2}$ and $\lambda_{(\alpha,\beta)}[h_2]_{k_2}$ are all non-zero and finite.*

**(A)** *If $\lambda_{(\alpha,\beta)}[h_i]_{k_1} > \lambda_{(\alpha,\beta)}[h_j]_{k_1}$ with at least $h_j$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_1$ for $i$, $j = 1,2$ and $i \neq j$, then*

$$\tau_{(\alpha,\beta)}[h_1 + h_2]_{k_1} = \tau_{(\alpha,\beta)}[h_i]_{k_1} \text{ and } \overline{\tau}_{(\alpha,\beta)}[h_1 + h_2]_{k_1} = \overline{\tau}_{(\alpha,\beta)}[h_i]_{k_1}.$$

**(B)** *If $\lambda_{(\alpha,\beta)}[h_1]_{k_i} < \lambda_{(\alpha,\beta)}[h_1]_{k_j}$ for $i = j = 1,2$ and $i \neq j$, then*

$$\tau_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \tau_{(\alpha,\beta)}[h_1]_{k_i} \text{ and } \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_i}.$$

**(C)** *Assume the functions $h_1, h_2, k_1$ and $k_2$ satisfy the following conditions:*
*(i)$\lambda_{(\alpha,\beta)}[h_i]_{k_1} > \lambda_{(\alpha,\beta)}[h_j]_{k_1}$ with at least $h_j$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_1$ for i,j $= 1,2$ and $i \neq j$;*
*(ii) $\lambda_{(\alpha,\beta)}[h_i]_{k_2} > \lambda_{(\alpha,\beta)}[h_j]_{k_2}$ with at least $h_j$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_2$ for $i$, $j = 1,2$ and $i \neq j$;*
*(iii) Both of $\lambda_{(\alpha,\beta)}[h_1]_{k_i} < \lambda_{(\alpha,\beta)}[h_1]_{k_j}$ and $\lambda_{(\alpha,\beta)}[h_2]_{k_i} < \lambda_{(\alpha,\beta)}[h_2]_{k_j}$ hold for $i$,*

*$j = 1, 2$ and $i \neq j$;*
*(iv)$\lambda_{(\alpha,\beta)}[h_l]_{k_m} =$*
$\min[\max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}, \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_2}, \lambda_{(\alpha,\beta)}[h_2]_{k_2}\}] \mid l = m = 1, 2$;
*then we have*

$$\tau_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} = \tau_{(\alpha,\beta)}[h_l]_{k_m} \mid l, m = 1, 2$$

*and*

$$\overline{\tau}_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} = \overline{\tau}_{(\alpha,\beta)}[h_l]_{k_m} \mid l, m = 1, 2.$$

**Proof.** We obtain for any $\eta(> 0)$ and for all $r$ with $0 < r < 1$, sufficiently close to 1 that

$$M_{h_k}(r) \leq M_{k_l}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_k]_{k_l} + \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_k]_{k_l}}\}), \quad (2.22)$$

$$M_{h_k}(r) \geq M_{k_l}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_k]_{k_l} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_k]_{k_l}}\}) \quad (2.23)$$

$$i.e., \ M_{k_l}(r) \leq M_{h_k}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[h_k]_{k_l} - \eta)})^{\frac{1}{\lambda_{(\alpha,\beta)}[h_k]_{k_l}}})), \quad (2.24)$$

and for a sequence of values of $r \to 1$, we get that

$$M_{h_k}(r) \geq M_{k_l}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_k]_{k_l} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_k]_{k_l}}\}) \quad (2.25)$$

$$i.e., \ M_{k_l}(r) \leq M_{h_k}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\tau}_{(\alpha,\beta)}[h_k]_{k_l} - \eta)})^{\frac{1}{\lambda_{(\alpha,\beta)}[h_k]_{k_l}}})), \quad (2.26)$$

and

$$M_{h_k}(r) \leq M_{k_l}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_k]_{k_l} + \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_k]_{k_l}}\}), \quad (2.27)$$

where $k = 1, 2$ and $l = 1, 2$.

**Case I.** Let $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ with at least $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$. Also let $\eta(> 0)$ be arbitrary. Now we obtain from (2.22) and (2.27), for a sequence of values of $r$ tending to 1 that

$$M_{h_1 \pm h_2}(r)(r) \leq M_{k_1}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} + \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \cdot (1 + E). \quad (2.28)$$

where $E = \frac{M_{k_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_2]_{k_1}}\})}{M_{k_1}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})}$ and in view of $\lambda_{(\alpha,\beta)}[h_1]_{k_1}$ $> \lambda_{(\alpha,\beta)}[h_2]_{k_1}$, we can make the term $E$ sufficiently small by taking $r$ sufficiently close to 1. Hence with the help of Theorem 1 and using the same technique of Case I of Theorem 13, we have from (2.28) that

$$\tau_{(\alpha,\beta)}[h_1 + h_2]_{k_1} \leq \tau_{(\alpha,\beta)}[h_1]_{k_1}. \quad (2.29)$$

Further, we may consider that $h = h_1 \pm h_2$. Also suppose that $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ and at least $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$. Then $\tau_{(\alpha,\beta)}[h]_{k_1} = \tau_{(\alpha,\beta)}[h_1+h_2]_{k_1} \leq \tau_{(\alpha,\beta)}[h_1]_{k_1}$. Now let $h_1 = (h \pm h_2)$. Therefore in view of Theorem 1,
$\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ and at least $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$, we obtain that $\lambda_{(\alpha,\beta)}[h]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ holds. Hence in view of (2.29), $\tau_{(\alpha,\beta)}[h_1]_{k_1} \leq \tau_{(\alpha,\beta)}[h]_{k_1} = \tau_{(\alpha,\beta)}[h_1 + h_2]_{k_1}$. Therefore $\tau_{(\alpha,\beta)}[h]_{k_1} = \tau_{(\alpha,\beta)}[h_1]_{k_1}$ i.e., $\tau_{(\alpha,\beta)}[h_1 + h_2]_{k_1} = \tau_{(\alpha,\beta)}[h_1]_{k_1}$.

Similarly, if we consider $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ with at least $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$ then we can easily verify that $\tau_{(\alpha,\beta)}[h_1 + h_2]_{k_1} = \tau_{(\alpha,\beta)}[h_2]_{k_1}$.

**Case II.** Let us consider that $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ with at least $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$. Also let $\eta(> 0)$ be arbitrary. Hence we get from (2.22) for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$M_{h_1 \pm h_2}(r)(r) \leq M_{k_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \cdot (1+K)\,. \tag{2.30}$$

where $K = \frac{M_{k_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_2]_{k_1}}\})}{M_{k_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})}$, and in view of $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$, we can make the term $K$ sufficiently small by taking $r$ sufficiently close to 1 and therefore for similar reasoning of Case I we get from (2.30) that $\overline{\tau}_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} = \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1}$ when $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ and at least $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$.

Similarly, if we take $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ with at least $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$ then we can easily verify that $\overline{\tau}_{(\alpha,\beta)}[h_1 + h_2]_{k_1} = \overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1}$

Thus from Case I and Case II, we get the first part of the theorem.

**Case III.** Let us consider that $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$. Hence we get from (2.23) for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$\begin{aligned}
&M_{k_1 \pm k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&\leq M_{k_1}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&+ M_{k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&i.e., M_{k_1 \pm k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&\leq M_{h_1}(r)(1+L)
\end{aligned} \tag{2.31}$$

where $L = \frac{M_{k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1}-\eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})}{M_{k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_2}-\eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_2}}\})}$, and since $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, we can make the term $L$ sufficiently small by taking $r$

sufficiently closed to 1. Therefore observing Theorem 3 and by the same way of Case III of Theorem 13, we obtain from (2.31) that

$$\tau_{(\alpha,\beta)}[h_1]_{k_1\pm k_2} \geq \tau_{(\alpha,\beta)}[h_1]_{k_1}. \tag{2.32}$$

Further, we may consider that $k = k_1 \pm k_2$. As $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, so $\tau_{(\alpha,\beta)}[h_1]_k = \tau_{(\alpha,\beta)}[h_1]_{k_1\pm k_2} \geq \tau_{(\alpha,\beta)}[h_1]_{k_1}$. Further let $k_1 = (k \pm k_2)$. Therefore in view of Theorem 3 and $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$ we obtain that $\lambda_{(\alpha,\beta)}[h_1]_k < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$ holds. Hence in view of (2.32) $\tau_{(\alpha,\beta)}[h_1]_{k_1} \geq \tau_{(\alpha,\beta)}[h_1]_k = \tau_{(\alpha,\beta)}[h_1]_{k_1\pm k_2}$. Therefore $\tau_{(\alpha,\beta)}[h_1]_k = \tau_{(\alpha,\beta)}[h_1]_{k_1}$
i.e., $\tau_{(\alpha,\beta)}[h_1]_{k_1\pm k_2} = \tau_{(\alpha,\beta)}[h_1]_{k_1}$.

Likewise, if we consider that $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, then one can easily verify that $\tau_{(\alpha,\beta)}[h_1]_{k_1\pm k_2} = \tau_{(\alpha,\beta)}[h_1]_{k_2}$.

**Case IV.** In this case further we consider $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$.Therefore we obtain from (2.23) and (2.25), for a sequence of $r$ tending to 1, that

$$\begin{aligned}
&M_{k_1\pm k_2}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})\\
&\leq M_{k_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})\\
&+ M_{k_2}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})\\
&M_{k_1\pm k_2}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})\\
&\leq M_{h_1}(r)(1+H),
\end{aligned} \tag{2.33}$$

where $H = \frac{M_{k_2}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})}{M_{k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_2} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_2}}\})}$. Now in view of $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, we can make the term $H$ sufficiently small by taking $r$ sufficiently close to 1 and therefore using the similar technique for as executed in the proof of Case IV of Theorem 13, we get from (2.33) that $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1\pm k_2} = \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1}$ when $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$.

Similarly, if we consider that $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1\pm k_2} = \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_2}$.

Thus from Case III and Case IV, we get the second part of the theorem.

The proof of the third part of the Theorem is omitted as it can be followed from Theorem 6 and the above cases.

In the following two theorems we retake the equalities in Theorem 1 to Theorem 4 under somewhat different conditions.

**Theorem 15.** *Let $h_1$, $h_2$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$.*
**(A)** *The following condition is assumed to be satisfied:*

*(i) If either* $\sigma_{(\alpha,\beta)}[h_1]_{k_1} \neq \sigma_{(\alpha,\beta)}[h_2]_{k_1}$ *or* $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\sigma}_{(\alpha,\beta)}[h_2]_{k_1}$ *holds, then*

$$\varrho_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} = \varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_2]_{k_1}.$$

**(B)** *The following conditions are assumed to be satisfied:*
*(i) Either* $\sigma_{(\alpha,\beta)}[h_1]_{k_1} \neq \sigma_{(\alpha,\beta)}[h_1]_{k_2}$ *or* $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_2}$ *holds;*
*(ii) If* $h_1$ *is of regular generalized relative growth* $(\alpha, \beta)$ *with respect to at least any one of* $k_1$ *or* $k_2$*, then*

$$\varrho_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_1]_{k_2}.$$

**Proof. Case I.** Suppose that $\varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_2]_{k_1}$ $(0 < \varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_2]_{k_1} < \infty)$. Now in view of Theorem 2 it is easy to see that $\varrho_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} \leq \varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_2]_{k_1}$. If possible let

$$\varrho_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_2]_{k_1} \ . \tag{2.34}$$

Let $\sigma_{(\alpha,\beta)}[h_1]_{k_1} \neq \sigma_{(\alpha,\beta)}[h_2]_{k_1}$. Then from the first part of Theorem 13 and (2.34) we obtain that $\sigma_{(\alpha,\beta)}[h_1]_{k_1} = \sigma_{(\alpha,\beta)}[h_1 \pm h_2 \mp h_2]_{k_1} = \sigma_{(\alpha,\beta)}[h_2]_{k_1}$ which is a contradiction. Hence $\varrho_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} = \varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_2]_{k_1}$ . Similarly by the first part of Theorem 13, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\sigma}_{(\alpha,\beta)}[h_2]_{k_1}$. This completes the proof of the first part of the theorem.

**Case II.** Let us assume that $\varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ $(0 < \varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_1]_{k_2} < \infty)$ and $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to at least any one of $k_1$ or $k_2$ and $(k_1 \pm k_2)$. Therefore in view of Theorem 4, it follows that $\varrho_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} \geq \varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ and if possible let

$$\varrho_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} > \varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_1]_{k_2}. \tag{2.35}$$

Let us take $\sigma_{(\alpha,\beta)}[h_1]_{k_1} \neq \sigma_{(\alpha,\beta)}[h_1]_{k_2}$. Then from the proof of second part of Theorem 13 and (2.35) we have $\sigma_{(\alpha,\beta)}[h_1]_{k_1} = \sigma_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2 \mp k_2} = \sigma_{(\alpha,\beta)}[h_1]_{k_2}$ which is a contradiction. Hence $\varrho_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ . Also from the proof of second part of Theorem 13 we can get the same conclusion under the condition $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_2}$ and hence the second part of the theorem is established.

**Theorem 16.** *Let* $h_1, h_2, k_1, k_2$ *be all entire functions defined in the unit disc* $U$*.*
**(A)** *The following conditions are assumed to be satisfied:*
*(i)* $(h_1 \pm h_2)$ *is of regular generalized relative growth* $(\alpha, \beta)$ *with respect to at least any one of* $k_1$ *or* $k_2$*;*

*(ii) Either* $\sigma_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} \neq \sigma_{(\alpha,\beta)}[h_1 \pm h_2]_{k_2}$ *or* $\overline{\sigma}_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} \neq \overline{\sigma}_{(\alpha,\beta)}[h_1 \pm h_2]_{k_2}$*;*
*(iii) Either* $\sigma_{(\alpha,\beta)}[h_1]_{k_1} \neq \sigma_{(\alpha,\beta)}[h_2]_{k_1}$ *or* $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\sigma}_{(\alpha,\beta)}[h_2]_{k_1}$*;*
*(iv) Either* $\sigma_{(\alpha,\beta)}[h_1]_{k_2} \neq \sigma_{(\alpha,\beta)}[h_2]_{k_2}$ *or* $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_2} \neq \overline{\sigma}_{(\alpha,\beta)}[h_2]_{k_2}$*; then*

$$\varrho_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} = \varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_2]_{k_1} = \varrho_{(\alpha,\beta)}[h_1]_{k_2} = \varrho_{(\alpha,\beta)}[h_2]_{k_2}.$$

**(B)** *The following conditions are assumed to be satisfied:*
*(i)* $h_1$ *and* $h_2$ *are of regular generalized relative growth* $(\alpha, \beta)$ *with respect to at least any one of* $k_1$ *or* $k_2$*;*
*(ii) Either* $\sigma_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} \neq \sigma_{(\alpha,\beta)}[h_2]_{k_1 \pm k_2}$ *or* $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} \neq \overline{\sigma}_{(\alpha,\beta)}[h_2]_{k_1 \pm k_2}$*;*
*(iii) Either* $\sigma_{(\alpha,\beta)}[h_1]_{k_1} \neq \sigma_{(\alpha,\beta)}[h_1]_{k_2}$ *or* $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_2}$*;*
*(iv) Either* $\sigma_{(\alpha,\beta)}[h_2]_{k_1} \neq \sigma_{(\alpha,\beta)}[h_2]_{k_2}$ *or* $\overline{\sigma}_{(\alpha,\beta)}[h_2]_{k_1} \neq \overline{\sigma}_{(\alpha,\beta)}[h_2]_{k_2}$*; then*

$$\varrho_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} = \varrho_{(\alpha,\beta)}[h_1]_{k_1} = \varrho_{(\alpha,\beta)}[h_2]_{k_1} = \varrho_{(\alpha,\beta)}[h_1]_{k_2} = \varrho_{(\alpha,\beta)}[h_2]_{k_2}.$$

The proof of Theorem 16 is similar to Theorem 15, so we neglect it.

**Theorem 17.** *Let* $h_1$*,* $h_2$*,* $k_1$*,* $k_2$ *be all entire functions defined in the unit disc* $U$*.*
**(A)** *The following conditions are assumed to be satisfied:*
*(i) At least any one of* $h_1$ *or* $h_2$ *is of regular generalized relative growth* $(\alpha, \beta)$ *with respect to* $k_1$*;*
*(ii) Either* $\tau_{(\alpha,\beta)}[h_1]_{k_1} \neq \tau_{(\alpha,\beta)}[h_2]_{k_1}$ *or* $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1}$ *holds, then*

$$\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} = \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_2]_{k_1}.$$

**(B)** *The following conditions are assumed to be satisfied:*
*(i)* $h_1$*,* $k_1$ *and* $k_2$ *be any three entire functions such that* $\lambda_{(\alpha,\beta)}[h_1]_{k_1}$ *and* $\lambda_{(\alpha,\beta)}[h_1]_{k_2}$ *exists;*
*(ii) Either* $\tau_{(\alpha,\beta)}[h_1]_{k_1} \neq \tau_{(\alpha,\beta)}[h_1]_{k_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_2}$ *holds, then*

$$\lambda_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_1]_{k_2}.$$

**Proof. Case I.** Let $\lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ $(0 < \lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1} < \infty)$ and at least $h_1$ or $h_2$ and $(h_1 \pm h_2)$ are of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$. Now, from seeing Theorem 1, it is easy to say that $\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} \leq \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_2]_{k_1}$. If possible let

$$\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_2]_{k_1}. \tag{2.36}$$

Let $\tau_{(\alpha,\beta)}[h_1]_{k_1} \neq \tau_{(\alpha,\beta)}[h_2]_{k_1}$. Then from the proof of the first part of Theorem 14 and (2.36) we have $\tau_{(\alpha,\beta)}[h_1]_{k_1} = \tau_{(\alpha,\beta)}[h_1 \pm h_2 \mp h_2]_{k_1} = \tau_{(\alpha,\beta)}[h_2]_{k_1}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1} = \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ . Similarly from the proof of the first part of Theorem 14, we can get the same conclusion under the hypothesis $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1}$. This completes the proof of the first part of the theorem.

**Case II.** Let us consider that $\lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_1]_{k_2}$ $(0 < \lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_1]_{k_2} < \infty)$. Therefore from Theorem 3, we get that $\lambda_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} \geq \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_1]_{k_2}$ and if possible let

$$\lambda_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} > \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_1]_{k_2}. \tag{2.37}$$

Suppose $\tau_{(\alpha,\beta)}[h_1]_{k_1} \neq \tau_{(\alpha,\beta)}[h_1]_{k_2}$. Then from the second part of Theorem 14 and (2.37), we have $\tau_{(\alpha,\beta)}[h_1]_{k_1} = \tau_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2 \mp k_2} = \tau_{(\alpha,\beta)}[h_1]_{k_2}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} = \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_1]_{k_2}$ . Similarly with the help of the second part of Theorem 14, we can get the same conclusion under the condition $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_2}$ and therefore the second part of the theorem is established.

**Theorem 18.** *Let $h_1$, $h_2$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$.*
**(A)** *The following conditions are assumed to be satisfied:*
*(i) At least any one of $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1$ and $k_2$;*
*(ii) Either $\tau_{(\alpha,\beta)}[h_1 + h_2]_{k_1} \neq \tau_{(\alpha,\beta)}[h_1 \pm h_2]_{k_2}$ or $\overline{\tau}_{(\alpha,\beta)}[h_1 + h_2]_{k_1} \neq \overline{\tau}_{(\alpha,\beta)}[h_1 \pm h_2]_{k_2}$;*
*(iii) Either $\tau_{(\alpha,\beta)}[h_1]_{k_1} \neq \tau_{(\alpha,\beta)}[h_2]_{k_1}$ or $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1}$;*
*(iv) Either $\tau_{(\alpha,\beta)}[h_1]_{k_2} \neq \tau_{(\alpha,\beta)}[h_2]_{k_2}$ or $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_2} \neq \overline{\tau}_{(\alpha,\beta)}[h_2]_{k_2}$; then*

$$\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} = \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_2]_{k_1} = \lambda_{(\alpha,\beta)}[h_1]_{k_2} = \lambda_{(\alpha,\beta)}[h_2]_{k_2}.$$

**(B)** *The following conditions are assumed to be satisfied:*
*(i) At least any one of $h_1$ or $h_2$ are of regular generalized relative growth $(\alpha, \beta)$ with respect to $k_1 \pm k_2$;*
*(ii) Either $\tau_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} \neq \tau_{(\alpha,\beta)}[h_2]_{k_1 \pm k_2}$ or $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1 \pm k_2} \neq \overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1 \pm k_2}$ holds;*
*(iii) Either $\tau_{(\alpha,\beta)}[h_1]_{k_1} \neq \tau_{(\alpha,\beta)}[h_1]_{k_2}$ or $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} \neq \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_2}$ holds;*
*(iv) Either $\tau_{(\alpha,\beta)}[h_2]_{k_1} \neq \tau_{(\alpha,\beta)}[h_2]_{k_2}$ or $\overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1} \neq \overline{\tau}_{(\alpha,\beta)}[h_2]_{k_2}$ holds, then*

$$\lambda_{(\alpha,\beta)}[h_1 \pm h_2]_{k_1 \pm k_2} = \lambda_{(\alpha,\beta)}[h_1]_{k_1} = \lambda_{(\alpha,\beta)}[h_2]_{k_1} = \lambda_{(\alpha,\beta)}[h_1]_{k_2} = \lambda_{(\alpha,\beta)}[h_2]_{k_2}.$$

The proof of Theorem 18 is similar as of Theorem 17, so we neglect it.

**Theorem 19.** *Let $h_1$, $h_2$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$ such that $\varrho_{(\alpha,\beta)}[h_1]_{k_1}$, $\varrho_{(\alpha,\beta)}[h_2]_{k_1}$, $\varrho_{(\alpha,\beta)}[h_1]_{k_2}$ and $\varrho_{(\alpha,\beta)}[h_2]_{k_2}$ are all non-zero.*
**(A)** *The following conditions are assumed to be satisfied:*

*$(i)$ $k_1$ satisfies the Property (D);*
*$(ii)$ $h_1$, $h_2$ satisfy the Property (X), then*

$$\sigma_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} = \sigma_{(\alpha,\beta)}[h_i]_{k_1} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} = \overline{\sigma}_{(\alpha,\beta)}[h_i]_{k_1}.$$

**(B)** *The following conditions are assumed to be satisfied:*
*$(i)$ $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to at least any one of $k_1$ or $k_2$ ;*
*$(ii)$ $k_1 \cdot k_2$ satisfies the Property (D);*
*$(iii)$ $k_1$, $k_2$ satisfy the Property (X), then,*

$$\sigma_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2} = \sigma_{(\alpha,\beta)}[h_1]_{k_i} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2} = \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_i}.$$

**(C)** *The following conditions are assumed to be satisfied:*
*$(i)$ $k_1 \cdot k_2$, $k_1$ and $k_2$ satisfy the Property (D);*
*$(ii)$ $h_1$, $h_2$ satisfy the Property (X) and $k_1$, $k_2$ satisfy the Property (X);*
*$(iii)$ $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to at least any one of $k_1$ or $k_2$;*
*$(iv)$ $h_2$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to at least any one of $k_1$ or $k_2$;*
*$(v)$ $\varrho_{(\alpha,\beta)}[h_l]_{k_m} =$
$\max[\min\{\varrho_{(\alpha,\beta)}[h_1]_{k_1}, \varrho_{(\alpha,\beta)}[h_1]_{k_2}\}, \min\{\varrho_{(\alpha,\beta)}[h_2]_{k_1}, \varrho_{(\alpha,\beta)}[h_2]_{k_2}\}] \mid l, m = 1, 2$; then*

$$\sigma_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1 \cdot k_2} = \sigma_{(\alpha,\beta)}[h_l]_{k_m} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1 \cdot k_2} = \overline{\sigma}_{(\alpha,\beta)}[h_l]_{k_m}.$$

**Proof. Case I.** Suppose that $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$. Also let $k_1$ be satisfy the Property (D). Now from (2.10), we have for any $\eta > 0$ and for all $r$ with $0 < r < 1$ and sufficiently close to 1 that

$$M_{h_1 \cdot h_2}(r) \leq M_{k_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} + \frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}))$$
$$\times M_{k_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_2]_{k_1} + \frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_2]_{k_1}}\})). \quad (2.38)$$

Since $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$, we get that

$$\lim_{r \to +\infty} \frac{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} + \frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}}{(\sigma_{(\alpha,\beta)}[h_2]_{k_1} + \frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_2]_{k_1}}} = \infty.$$

Therefore we get from (2.38) for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$M_{h_1 \cdot h_2}(r) < [M_{k_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} + \frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})]^2. \quad (2.39)$$

Let us observe that

$$\frac{\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\eta}{\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\frac{\eta}{2}}>1$$

$$\Rightarrow \frac{\log(\alpha^{-1}(\log(\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\eta)))[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}}{\log(\alpha^{-1}(\log(\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\frac{\eta}{2})))[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}}=\delta(\text{say})>1. \tag{2.40}$$

Since $k_1$ satisfies the Property (D), we get from (2.40) and (2.39) for all $r$ with $0<r<1$ and sufficiently close to 1 that

$$\begin{aligned} M_{h_1\cdot h_2}(r) &< M_{k_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})]^{\delta} \\ i.e.,\ M_{h_1\cdot h_2}(r) &< M_{k_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})]. \\ \text{for }\delta &\to 1+ \end{aligned}$$

Now in view of Theorem 8, we get from above for all $r$ with $0<r<1$ and sufficiently close to 1 that

$$M_{h_1\cdot h_2}(r)<M_{k_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}}\})].$$

$$i.e.,\ \frac{\exp(\alpha(M_{k_1}^{-1}(M_{h_1\cdot h_2}(r))))}{[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}}}<(\sigma_{(\alpha,\beta)}[h_1]_{k_1}+\eta)$$

$$i.e.,\ \sigma_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}\le\sigma_{(\alpha,\beta)}[h_1]_{k_1}. \tag{2.41}$$

Now we establish the equality of (2.41). Since $h_1$, $h_2$ satisfy the Property (X), we have $M_{h_1\cdot h_2}(r)>M_{h_1}$ for all $r$ with $0<r<1$ and sufficiently close to 1 and therefore

$$\frac{\exp(\alpha(M_{k_1}^{-1}(M_{h_1\cdot h_2}(r))))}{[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}}}>\frac{\exp(\alpha(M_{k_1}^{-1}(M_{h_1}(r))))}{[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}}$$

as $M_{k_1}^{-1}(r)$ is an increasing function of $r$. So $\sigma_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}\ge\sigma_{(\alpha,\beta)}[h_1]_{k_1}$. Hence $\sigma_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}\le\sigma_{(\alpha,\beta)}[h_1]_{k_1}$.

Similarly, if we consider $\varrho_{(\alpha,\beta)}[h_1]_{k_1}<\varrho_{(\alpha,\beta)}[h_2]_{k_1}$, then one can verify that $\sigma_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}=\sigma_{(\alpha,\beta)}[h_2]_{k_1}$.

**Case II.** Let $\varrho_{(\alpha,\beta)}[h_1]_{k_1}>\varrho_{(\alpha,\beta)}[h_2]_{k_1}$ and $k_1$ be satisfy the Property (D). Now we get from (2.10) and (2.15) for any $\eta>0$ and for a sequence of values of $r$ tending to infinity that

$$\begin{aligned} M_{h_1\cdot h_2}(r)\le{}& M_{k_1}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})] \\ &\times M_{k_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_2]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_2]_{k_1}}\})]. \end{aligned} \tag{2.42}$$

Now in view of $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_2]_{k_1}$, we get that

$$\lim_{r\to 1} \frac{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} + \frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}}{(\sigma_{(\alpha,\beta)}[h_2]_{k_1} + \frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_2]_{k_1}}} = \infty.$$

Hence we get from (2.42) for a sequence of values of $r \to 1$ that

$$M_{h_1\cdot h_2}(r) < [M_{k_1}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} + \frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})]]^2.$$

Now by the same technique of the proof of Case I, we can easily show for a sequence of values of $r \to 1$ that $\overline{\sigma}_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} = \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1}$ under the conditions specified in the theorem.

In the same way, assuming $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_2]_{k_1}$ we can verify that $\overline{\sigma}_{(\alpha,\beta)}[h_1 \cdot h_2]_{k_1} = \overline{\sigma}_{(\alpha,\beta)}[h_2]_{k_1}$.

Hence the first part of theorem follows from Case I and Case II.

**Case III.** Let $k_1 \cdot k_2$ be satisfy the Property (D) and $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ with $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to at least any one of $k_1$ or $k_2$. So by (2.11) and (2.13), we get for a sequence of values of $r \to 1$, that

$$\begin{aligned}
&M_{k_1\cdot k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&\le M_{k_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&\times M_{k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}). \qquad (2.43)
\end{aligned}$$

Now in view of $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$, we obtain that

$$\lim_{r\to 1} \frac{M_{k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})}{M_{k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_2} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_2}}\})} = \infty.$$

Now from (2.43) we have for a sequence of values of $r \to 1$, that

$$\begin{aligned}
&M_{k_1\cdot k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\
&\le M_{h_1}(r) \times M_{h_2}(r) \\
&i.e., [M_{k_1\cdot k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})]^{\frac{1}{2}} \le M_{h_1}(r) \qquad (2.44)
\end{aligned}$$

Since $k_1 \cdot k_2$ satisfies the Property (D), we get from (2.44) for a sequence of values of $r \to 1$, that

$$i.e., [M_{k_1\cdot k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})^{\frac{1}{\delta}}] \le M_{h_1}(r)$$

Now letting $\delta \to 1+$ we have from above and Theorem 10 for a sequence of values of $r \to 1$, that

$$M_{k_1 \cdot k_2}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2}}\}) \leq M_{h_1}(r)$$

$$\frac{\exp(\alpha(M^{-1}_{k_1 \cdot k_2}(M_h(r))))}{(\exp\beta((1-r)^{-1}))^{\varrho_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2}}} > \sigma_{(\alpha,\beta)}[h_1]_{k_1} - \eta$$

Since $\eta > 0$ is arbitrary, it follows from above that

$$\sigma_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2} \geq \sigma_{(\alpha,\beta)}[h_1]_{k_1} \ . \tag{2.45}$$

Now we establish the equality of (2.45). Since $k_1$, $k_2$ satisfy the Property (X), we have $M_{k_1 \cdot k_2}(r) > M_{k_1}(r)$ for all $r$, $0 < r < 1$, sufficiently close to 1 and therefore $M^{-1}_{k_1 \cdot k_2}(r) < M^{-1}_{k_1}(r)$. Hence

$$\frac{\exp(\alpha(M^{-1}_{k_1 \cdot k_2}(M_{h_1}(r))))}{[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2}}} < \frac{\exp(\alpha(M^{-1}_{k_1}(M_{h_1}(r)))}{[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}}$$

as $M_{h_1}(r)$ is an increasing function of $r$. So $\sigma_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2} = \sigma_{(\alpha,\beta)}[h_1]_{k_1}$.

**Case IV.** Suppose $k_1 \cdot k_2$ be satisfy the Property (D). Also let $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ where $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to at least any one of $k_1$ or $k_2$. Therefore in view of (2.11), we obtain for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$\begin{aligned} &M_{k_1 \cdot k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\ &\leq M_{k_1}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\ &\times M_{k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}). \end{aligned} \tag{2.46}$$

Now in view of $\varrho_{(\alpha,\beta)}[h_1]_{k_1} < \varrho_{(\alpha,\beta)}[h_1]_{k_2}$, we obtain that

$$\lim_{r\to 1} \frac{M_{k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\})}{M_{k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_2} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_2}}\})} = \infty.$$

Therefore it follows from (2.46) for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$M_{k_1 \cdot k_2}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\varrho_{(\alpha,\beta)}[h_1]_{k_1}}\}) \leq [M_{h_1}(r)]^2.$$

Now by the similar technique of the proof of Case III, we can show, for all $r$ with $0 < r < 1$ and sufficiently close to 1, that $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2} = \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1}$ under the given conditions.

Similarly, if we take $\varrho_{(\alpha,\beta)}[h_1]_{k_1} > \varrho_{(\alpha,\beta)}[h_1]_{k_2}$ where at least $h_1$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_1$, then we can show that $\overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_1\cdot k_2} = \overline{\sigma}_{(\alpha,\beta)}[h_1]_{k_2}$.

Hence Case III and Case IV completes the second part of theorem.

The proof of the third part can be easily carried out from Theorem 11 and the above cases.

**Theorem 20.** *Let $h_1$, $h_2$, $k_1$, $k_2$ be all entire functions defined in the unit disc $U$ such that $\lambda_{(\alpha,\beta)}[h_1]_{k_1}$, $\lambda_{(\alpha,\beta)}[h_2]_{k_1}$, $\lambda_{(\alpha,\beta)}[h_1]_{k_2}$ and $\lambda_{(\alpha,\beta)}[h_2]_{k_2}$ are all non-zero and finite.*

**(A)** *The following conditions are assumed to be satisfied:*

*(i) At least $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_1$ for $i$, $j = 1,2$ and $i \neq j$;*

*(ii) $k_1$ satisfies the Property (D) and $h_1$, $h_2$ satisfy the Property (X), then*

$$\tau_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1} = \tau_{(\alpha,\beta)}[h_i]_{k_1} \text{ and } \overline{\tau}_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1} = \overline{\tau}_{(\alpha,\beta)}[h_i]_{k_1}.$$

**(B)** *The following condition is assumed to be satisfied:*

*(i) $k_1\cdot k_2$ satisfies the Property (D) and $k_1$, $k_2$ satisfy the Property (X),*

$$\tau_{(\alpha,\beta)}[h_1]_{k_1\cdot k_2} = \tau_{(\alpha,\beta)}[h_1]_{k_i} \text{ and } \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1\cdot k_2} = \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_i}.$$

**(C)** *The following conditions are assumed to be satisfied:*

*(i) $k_1\cdot k_2$, $k_1$ and $k_2$ be satisfy the Property (D);*

*(ii) $h_1$, $h_2$ satisfy the Property (X) and $k_1$, $k_2$ satisfy the Property (X);*

*(iii) At least $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_1$ for $i = 1$, $2$, $j = 1,2$ and $i \neq j$;*

*(iv) At least $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_2$ for $i = 1$, $2$, $j = 1,2$ and $i \neq j$;*

*(v) $\lambda_{(\alpha,\beta)}[h_l]_{k_m} =$ $\min[\max\{\lambda_{(\alpha,\beta)}[h_1]_{k_1}, \lambda_{(\alpha,\beta)}[h_2]_{k_1}\}, \max\{\lambda_{(\alpha,\beta)}[h_1]_{k_2}, \lambda_{(\alpha,\beta)}[h_2]_{k_2}\}] \mid l, m = 1,2$ ; then*

$$\tau_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1\cdot k_2} = \tau_{(\alpha,\beta)}[h_l]_{k_m} \text{ and } \overline{\tau}_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1\cdot k_2} = \overline{\tau}_{(\alpha,\beta)}[h_l]_{k_m}.$$

**Proof. Case I.** Suppose $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ where at least $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_1$ and $k_1$ satisfies the Property (D). Now we get from (2.22) and (2.25) for any $\eta > 0$, for a sequence of $r \to 1$ that

$$\begin{aligned} M_{h_1\cdot h_2}(r) \leq{}& M_{k_1}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} + \frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})) \\ &\times M_{k_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1} + \frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_2]_{k_1}}\})). \quad (2.47) \end{aligned}$$

Now in view of $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$, we get that

$$\lim_{r\to 1}\frac{(\tau_{(\alpha,\beta)}[h_1]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}}{(\overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_2]_{k_1}}}=\infty.$$

As $M_{k_1}(r)$ increases with $r$, so we obtain from (2.47) for a sequence of values of $r\to 1$ that

$$M_{h_1\cdot h_2}(r)<[M_{k_1}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}))]^2. \quad (2.48)$$

Now by similar proof of Case I of Theorem 19 we have from (2.48) that

$$\tau_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}=\tau_{(\alpha,\beta)}[h_1]_{k_1}.$$

Similarly, if we consider $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ with at least $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_1$, then we can show that $\tau_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}=\tau_{(\alpha,\beta)}[h_2]_{k_1}$.

**Case II.** Let $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ where at least $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_1$ and $k_1$ which satisfy the Property (D). Now we get from (2.22) for any $\eta>0$ and for all $r$ with $0<r<1$, sufficiently close to 1 that

$$\begin{aligned}M_{h_1\cdot h_2}(r)\le{}& M_{k_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}))\\ &\times M_{k_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_2]_{k_1}}\})). \quad (2.49)\end{aligned}$$

Now in view of $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_2]_{k_1}$, we get that

$$\lim_{r\to 1}\frac{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}}{(\overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_2]_{k_1}}}=\infty.$$

As $M_{k_1}(r)$ increases with $r$, so we obtain from (2.49) for all $r$, $0<r<1$, sufficiently close to 1 that

$$M_{h_1\cdot h_2}(r)<[M_{k_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1}+\frac{\eta}{2})[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})]^2. \quad (2.50)$$

Now by similar argument of the proof of Case I of Theorem 20 we get from (2.50) that $\overline{\tau}_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}=\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1}$ under the conditions specified in the theorem.

Likewise, if we take $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_2]_{k_1}$ where at least $h_1$ or $h_2$ is of regular generalized relative growth $(\alpha,\beta)$ with respect to $k_1$, then we can show that $\overline{\tau}_{(\alpha,\beta)}[h_1\cdot h_2]_{k_1}=\overline{\tau}_{(\alpha,\beta)}[h_2]_{k_1}$.

Therefore from Case I and Case II, the first part of theorem follows.

**Case III.** Let $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$ and $k_1 \cdot k_2$ be satisfy the Property (D). We get for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$\begin{aligned} &M_{k_1 \cdot k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\ &\le M_{k_1}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\ &\times M_{k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}). \end{aligned} \tag{2.51}$$

Now in view of $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, we get that

$$\lim_{r\to 1} \frac{M_{k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})}{M_{k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_2} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_2}}\})} = \infty.$$

Hence it follows from (2.51) and (2.23) for all $r$, $0 < r < 1$, sufficiently close to 1 that

$$\begin{aligned} &M_{k_1 \cdot k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\ &\le M_{h_1}(r) \times M_{h_2}(r) \\ &i.e., [M_{k_1 \cdot k_2}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})]^{\frac{1}{2}} \le M_{h_1}(r) \end{aligned} \tag{2.52}$$

Now by the similar technique of the proof of Case III of Theorem 19 we get from (2.52) that $\tau_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2} = \tau_{(\alpha,\beta)}[h_1]_{k_1}$. If $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, then one can easily verify that $\tau_{(\alpha,\beta)}[h_1]_{k_1 \cdot k_2} = \tau_{(\alpha,\beta)}[h_1]_{k_2}$ .

**Case IV.** Suppose $k_1 \cdot k_2$ be satisfy the Property (D) and $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, where $h_1$ is of regular generalized relative growth $(\alpha, \beta)$ with respect to at least any one of $k_1$ or $k_2$. Now we obtain for a sequence of values of $r$ tending to 1, that

$$\begin{aligned} &M_{k_1 \cdot k_2}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\ &\le M_{k_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \\ &\times M_{k_2}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \end{aligned} \tag{2.53}$$

Now in view of $\lambda_{(\alpha,\beta)}[h_1]_{k_1} < \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, we get that

$$\lim_{r\to 1} \frac{M_{k_2}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\})}{M_{k_2}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_2} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_2}}\})} = \infty.$$

Hence it follows from (2.53), (2.23) and (2.25), for a sequence of values of $r$ tending to 1, that

$$M_{k_1 \cdot k_2}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1} - \eta)[\exp(\beta((1-r)^{-1}))]^{\lambda_{(\alpha,\beta)}[h_1]_{k_1}}\}) \le [M_{h_1}(r)]^2 \tag{2.54}$$

Now by the similar argument of the proof of Case III of Theorem 20, we get from (2.54) that $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1\cdot k_2} = \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1}$. Similarly if we take $\lambda_{(\alpha,\beta)}[h_1]_{k_1} > \lambda_{(\alpha,\beta)}[h_1]_{k_2}$, then we can easily verify that $\overline{\tau}_{(\alpha,\beta)}[h_1]_{k_1\cdot k_2} = \overline{\tau}_{(\alpha,\beta)}[h_1]_{k_2}$. Hence from Case III and Case IV, the second part of the theorem follows.

Proof of the third part of the Theorem can be easily followed from Theorem 12 and the above cases.

**Acknowledgement**

The authors are grateful to the reviewer for his/her valuable suggestions and constructive comments for the improvement of the paper.

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