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SOME RESULTS ON ATOMI GRAPH OF THE LATTICES

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Abstract: This paper deals with an atomi graph of the finite lattices. Let L be a finite lattice with one atom denoted by L_a and $A(L_a) = \{x|$ there exist $y \in L_a$ such that $x \wedge y = a$, and $x, y \neq a$, a is an atom of the lattice$\}$. We defined a relation $x \wedge y = a$, and $x, y \neq a$ as the atomi of the lattice L_a. The atomi graph of the lattice L_a, is denoted by $\gamma(L_a)$, is a graph with the vertex set $A(L_a)$ and two distinct vertices $x, y \in A(L_a)$ are adjacent if and only if they are atomi. We study some properties of atomi graph of the lattices.

Keywords and Phrases: Diameter, connected graph, complete graph, regular graph, complete bipartite graph.

2020 Mathematics Subject Classification: 05E30, 05C76, 05C99.

1. Introduction

The study of algebraic structures by using the properties of the associated graphs has gained considerable attention in the last few years, so the study of the graphs of the lattices has emerged as a growing field in graph theory. In literature, we find zero divisor, incomparability graphs, comparability graphs which are associated with this kind of study.

Duffus and Rival [6] considered the covering graphs of posets. This graph has vertices which are the elements of P and edges are those pairs $\{a, b\}$, $a, b \in P$, satisfying a covers b or b covers a. Allan and Laskar [3] have studied the domination and independent domination numbers of a graph. A graph $G = (V, E)$ considered as finite, undirected, with no multiple edges and with no loops.

Filipov [8] discussed the comparability graph of a partially ordered set by defining the adjacency between two elements a, b of a poset P, by using the comparability relation that is a, b are adjacent if either $a \leq b$ or $b \leq a$. The study of graphs associated with rings was initiated by Beck [4]. Two elements x, y in a commutative ring R are called adjacent if and only if $xy = 0$. He studied coloring of such graphs.

Akbari and Mohammadian [2] discussed zero-divisor graph of a commutative ring. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a graph with vertex set $Z(R)^*$ in which two vertices x and y are adjacent if and only if $x \neq y$ and $xy = 0$. It is shown that for any finite commutative ring R, the edge chromatic number of $\Gamma(R)$ is equal to the maximum degree of $\Gamma(R)$, unless $\Gamma(R)$ is a complete graph of odd order. Bresar, et. al. [5] introduce the cover-incomparability graph of a poset and called these graph as $C - I$ graph of P. The cover-incomparability graph of a poset P is the edge-union of the covering and the incomparability graph of P.

Nimbhorkar et. al. [11] defined the zero-divisor graphs of a lattices L with 0, by defining the vertex set as the set of all elements in L and two vertices $x, y \in L$ are adjacent if and only if $x \wedge y = 0$. For a finite bounded lattice L, Estaji and Khashyarmanesh [7] associate a zero-divisor graph $G(L)$ which is a natural generalization of the concept of zero-divisor graph for a Boolean algebra. Also they study the interplay of lattice-theoretic properties of L with graph-theoretic properties of $G(L)$.

Foxa and Pach [9] defined incomparability graph with vertex set P, in which two elements of P are adjacent if and only if they are incomparable. They studied applications to extremal problems for string graphs and edge intersection patterns in topological graphs. Wasadikar and Survase [12] studied the incomparability graph of lattices. For $a, b \in P$, they define a, b are incomparable if neither $a \leq b$ nor $b \leq a$ denoted by $a \parallel b$. For a finite lattice L and $W(L)= \{x \mid$ there exist

$y \in L$ such that $x \parallel y\}$. The incomparability graph, is a graph with the vertex set $W(L)$ and two distinct vertices $a, b \in W(L)$ are adjacent if and only if they are incomparable.

Another graph associated with a lattice was discussed by Afkhami et.al. [1]. They associate a simple graph to a lattice L in which the vertex set is being the set of all elements of L and two distinct vertices x and y are adjacent if $x \vee y \in S$, when S is a multiplicatively closed subset of L. Golovach et. al. [10] studied enumeration algorithms and lower and upper bounds for the maximum number of minimal dominating sets in interval graphs and trees. They have shown that every interval graph on n vertices has at most $3^{\frac{n}{3}}$ minimal dominating sets. Also they discussed the upper bound on the number of minimal dominating sets of trees. Wasadikar and Dabhole [13] discussed the minimum dominating set and order of incomparability graph of the lattices L_n and $L_n^{1^2}$. Also they express the cardinality of neighbourhood of an atom by a binomial expression.

In the present research article, let L be a lattice with one atom a, denoted by L_a. We associate a graph with L_a and denote it by $\gamma(L_a)$. Let $A(L_a) = \{x|$ there exist $y \in L_a$ such that $x \wedge y = a$, and $x, y \neq a$, a is an atom of the lattice$\}$. Define a simple graph to the lattice L_a as, a) the vertices of $\gamma(L_a)$ are the elements of $A(L_a)$; b) two elements $x, y \in L_a$ are adjacent if and only if $x \wedge y = a$, and $x, y \neq a$ in L_a. We called this graph as the atomi graph of the lattice L_a. Further, the distance $d(x, y)$ between two vertices x and y is the length of a shortest path joining x and y. Some properties of atomi graph of the lattice L_a is studied. We proved that $\gamma(L_a)$ is always connected and its diameter is ≤ 3. Let L_a have chains from a to 1, where 1 is greatest element of the lattice L_a. We have discussed, the chains of L_a which gives $\gamma(L_a)$ are a complete, regular and a bipartite graph. Further, we have studied the atomi graph of linear sum of two lattices.

Theorem 1.1. *The atomi graph of a chain is the empty graph.*
Proof. Let the lattice L_a is a chain. Consider any two elements $x, y \in L_a$. As $x \wedge y = a$ exists only if $x \wedge a = a$, so for any x there does not exists y such that $x \wedge y = a$, and $x, y \neq a$.

Hence, the atomi graph of a chain is the empty graph i.e. $A(L_a) = \Phi$.

Corollary 1.2. *In the lattice L_a, if $a \prec x$ and there is no $y \in L_a$ such that $a \prec y$ then atomi graph of L_a is the empty graph.*

Example 1.3. The atomi graph of the lattice shown in the *Figure* 1.1 is the empty graph.

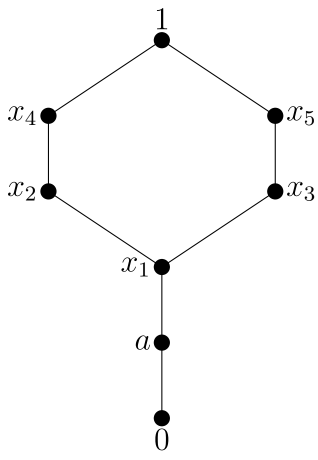

Figure 1.1

Proposition 1.4. *The non-isomorphic lattices may have isomorphic atomi graph* $\gamma(L_a)$.

Example 1.5. The lattices in the $Figure$ 1.2(a) and $Figure$ 1.2(b) are non-isomorphic but the atomi graph of the lattices of $Figure$ 1.2(a) and $Figure$ 1.2(b) shown in the $Figure$ 1.2(c) is isomorphic i.e. $L_{a_1} \ncong L_{a_2}$ but $\gamma(L_{a_1}) \cong \gamma(L_{a_2})$.

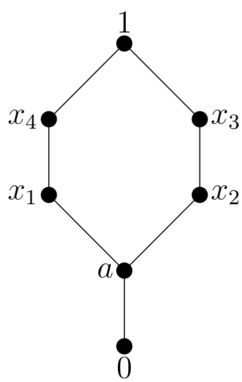

Figure 1.2(a)

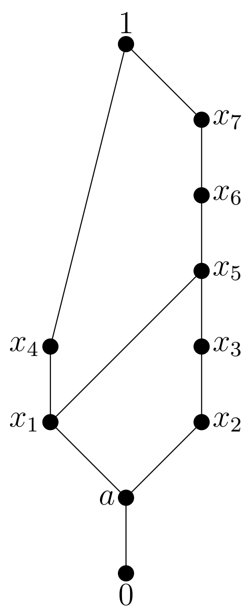

Figure 1.2(b)

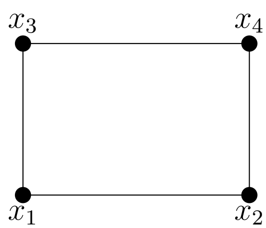

Figure 1.2(c)

2. Connectedness of Atomi Graph and its Diameter

In this section we have proved the connectedness of atomi graph of the lattice L_a and calculated the distance between any two vertices in $\gamma(L_a)$. Further we have calculated the diameter of $\gamma(L_a)$.

Theorem 2.1. *The atomi graph of the lattice L_a is always connected and $diam(\gamma(L_a)) \leq 3$.*

Proof. Let $x, y \in \gamma(L_a)$ be distinct.

a) If $x \wedge y = a$, and $x, y \neq a$ in L_a then x is adjacent to y. So $d(x, y) = 1$.

b) Suppose that $x \wedge y \neq a$, and $x, y \neq a$ then there exists elements $z_1, z_2 \in A(L) - \{x, y\}$ such that $z_1 \wedge x = a, z_2 \wedge y = a$, and $x, y, z_1, z_2 \neq a$.

i) If $z_1 = z_2$ then $x - z_1 - y$ is a path of length 2; thus $d(x, y) = 2$.

ii) Let us assume that $z_1 \neq z_2$. If $z_1 \wedge z_2 = a$, and $z_1, z_2 \neq a$ then we have $z_1 \wedge x = a$ and $z_2 \wedge y = a$, and $x, y, z_1, z_2 \neq a$ i.e. $x - z_1 - z_2 - y$ is a path of length 3. Hence $d(x, y) = 3$.

If $z_1 \wedge z_2 \neq a$ then for some $z_1 \wedge z_2 = q_1$, we have $x - q_1 - y$ is a path of length 2. Thus $d(x, y) = 2$.

Hence, $diam(\gamma(L_a)) \leq 3$.

Now there exists a path between any two distinct elements in $\gamma(L_a)$, hence $\gamma(L_a)$ is always a connected graph.

3. Some Properties of $\gamma(L_a)$ when the Lattice L_a Contains Chains

Let L_a be a lattice consisting of chains C_i^k between a and 1 such that $C_i^k \cap C_j^k = \{a, 1\}$, $(C_i \neq \{a, 1\})$ for every $i, j = 1, 2, ..., n, i \neq j$. In C_i^k, i denote the number of a chain and k denote the number of elements in the i^{th} chain excluding $\{a, 1\}$.

Theorem 3.1. *The atomi graph $\gamma(L_a)$ is a complete graph, if in the lattice L_a, each chain $C_i^{k=1}$, $i = 1, 2, ..., n$, $C_i^{k=1} \cap C_j^{k=1} = \{a, 1\}, i \geq 2$ contain one element.*

Proof. Let L_a be a finite lattice and $\gamma(L_a)$ is atomi graph of the lattice L_a. Consider each chain $C_i, i = 1, 2, ..., n$ contain one element other than $a, 1$ denoted by $C_i^{k=1}$,.

If $x_i \in C_i^{k=1}$, we have $x_{j_1} \wedge x_{j_2} = a$, $x_{j_1}, x_{j_2} \neq a$, $x_{j_1}, x_{j_2} = 1, 2, ..., n$, x_{j_1}, x_{j_2}, are distinct in the lattice L_a, since $C_i^{k=1} \cap C_j^z = \{a, 1\}$. So each x_{j_1} is adjacent to every x_{j_2} in $\gamma(L_a)$ i.e. every two distinct vertices in $\gamma(L_a)$ have an edge.

Therefore $\gamma(L_a)$ is a complete graph.

Corollary 3.2. *The order of complete atomi graph is n and the degree of each vertex is $n - 1$.*

Theorem 3.3. *If the lattice L_a contains $C_i^k, i \geq 3$, k is any positive integer $k \geq 2$ $i = 1, 2, ..., n$, then $\gamma(L_a)$ is a regular graph.*

Proof. Consider the lattice L_a contains $C_i, i = 1, 2, ..., n, i \geq 3,\ i = 1, 2, ..., n$, $C_i \cap C_j = \{a, 1\}$ chains and each chain C_i containing k elements denoted $C_i^k, k \geq 2$.

Let $x_p, x_q \in C_1^k$, then $x_p \wedge x_q \neq a, x_p, x_q \neq a$, since no two elements in C_1^k have $g.l.b$ as an atom a. But for $x_p \in C_1^k, x_z \in C_i^k, i \neq 1,\ x_p, x_q, x_z$ are the k elements in each chain, we have $x_p \wedge x_z = a$, i.e. each x_p is adjacent to every element of $C_i^k, i \neq 1$ chains.

Similarly, let $x_p, x_q \in C_2^k$, then $x_p \wedge x_q \neq a, x_p, x_q \neq a$, since no two elements in C_2^k have $g.l.b$ as an atom a. But for $x_p \in C_2^k, x_z \in C_i^k, i \neq 2,\ p, z = 1, 2, ..., n$, we have $x_p \wedge x_z = a$, i.e. each x_p is adjacent to every element of $C_i^k, i \neq 2$ chains.

Continuing this for the i^{th} chain of the lattice L_a. Let $x_p, x_q \in C_i^k$, then $x_p \wedge x_q \neq a, x_p, x_q \neq a$, since no two elements in C_i^k have $g.l.b$ as an atom a. But for $x_p \in C_i^k, x_z \in C_j^k, j = 1, 2, ..., (n-1)$, we have $x_p \wedge x_z = a$, i.e. each x_p is adjacent to every element of $C_j^k, j = 1, 2, ..., n$ chains.

Hence the degree of each vertex in $\gamma(L_a)$ is same and therefore $\gamma(L_a)$ is a regular graph.

Theorem 3.4. *The atomi graph of the lattice L_a is a complete bipartite graph $K_{m,n}$ if L_a contains two chains C_1^m, C_2^n, $m, n \geq 2$, $m, n = 1, 2, ..., n$, $C_1^m \cap C_2^n = \{a, 1\}$.*

Proof. Let the lattice L_a having two chains C_1^m and C_2^n, $m, n \geq 2$, $m, n = 1, 2, ..., n$. A chain C_1^m containing $m \geq 2$ elements and C_2^n contains $n \geq 2$ elements, $C_1^m \cap C_2^n = \{a, 1\}$.

Let $p_1 = \{x_i | x_i \in C_1, i = 1, 2, ..., n, i \geq 2\}$. Since no two elements of p_1 are adjacent in $\gamma(L_a)$, hence assume that p_1 as one partite set and similarly consider p_2 as another partite set, $p_12 = \{x_j | x_j \in C_2, j = 1, 2, ..., n, j \geq 2\}$. We claim that x is not adjacent to y if $x, y \in p_1$ or $x, y \in p_2$.

Assume that x is adjacent to y for some $x, y \in p_1$ then we have $x \wedge y = a, x, y \neq a$. But this contradicts to assumption of atomi relation in the chain C_1, for any $x \in p_1$ there does not exists $y \in p_1$ such that $x \wedge y = a, x, y \neq a$. Hence x is not adjacent to y. Similarly $x, y \in p_2$, we have x is not adjacent to y.

Now $C_i \cap C - j = \{a, 1\}$ we have for each $x_i \in C_1$ there exist $x_j \in C_2$ such that $x_i \wedge x_j = a, x_i, x_j \neq a$. Hence each $x_i \in C_i$ is adjacent to every $x_j \in C_2$ in $\gamma(L_a)$.

Therefore, $\gamma(L_a)$ is a complete bipartite graph $K_{m,n}$.

4. Discussion of Atomi Graph the Lattice L^*

In this section we have studies the atomi graph of the lattice L^*, where L^* is the linear sum of lattices L_{a_1} and L_{a_2} that is $L^* = L_{a_1} \bigoplus L_{a_2}$.

Definition 4.1. *Let L_1 and L_2 be two lattices, the linear sum of L_1 and L_2 denoted by $L_1 \bigoplus L_2$ is obtained by placing the diagram of L_1 directly below the diagram of L_2 and adding a line segment from the maximum element of L_1 to the minimum* *element of L_2.*

Theorem 4.2. *If $L^* = L_{a_1} \bigoplus L_{a_2}$ is the linear sum of the lattices L_{a_1} and L_{a_2} then the atomi graph of L^* is the atomi graph of L_{a_1}.*

Proof. Let $L^* = L_{a_1} \bigoplus L_{a_2}$ is the linear sum of the lattices L_{a_1} and L_{a_2}. In the lattice L_*, the lattices L_{a_1} and L_{a_2} are joined by an edge $x - y$, where x is the maximum element of L_{a_1} and y is the minimum element of L_{a_2} and a_1, a_2 are atoms of the lattices L_{a_1}, L_{a_2} respectively. The lattice L^* contains a chain $x - y - a_2$ which is formed by linear sum of L_{a_1} and L_{a_2}.

Consider $z_1, z_2 \in L_{a_2}$ then there does not exists $z_1 \wedge z_2 = a_1, z_1, z_2 \neq a_1$. Here a_1 is an atom of the lattice L_{a_1} which is also an atom of L^*. Now $z_1 \wedge z_2 = a_1, z_1, z_2 \neq a_1$ exist only if $z_1, z_2 \in L_{a_1}$ i.e. adjacency of L^* is derived from adjacency of L_{a_1}. Hence no elements of L_{a_2} is in the atomi graph of L^*.

Therefore, the atomi graph of L^* is the atomi graph of L_{a_1}.

5. Conclusion.

In this paper we have considered finite lattice with one atom L_a. Let L be a finite lattice with one atom denoted by L_a and $A(L_a) = \{x|$ there exist $y \in L_a$ such that $x \wedge y = a, x, y \neq a$, a is an atom of the lattice$\}$, where a relation $x \wedge y = a, x, y \neq a$ as the atomi of the lattice L_a. Here $\gamma(L_a)$ denotes the atomi graph of the lattice L_a which is a graph of vertex set $A(L_a)$. Two distict vertices are adjacent if and only if they are atomi. We have studied the atomi graph of linear sum of two lattices.

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