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δ^c-CLOSURE OPERATOR IN FUZZY SETTING

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Abstract: This paper deals with fuzzy regular open set [1]. Here a new type of fuzzy closure operator is introduced which is not an idempotent operator. First we characterize this operator by fuzzy open set. It is shown that this operator is distributed over union but not on intersection. Next we establish the mutual relationship of this operator with the operators defined in [2, 3, 4, 6, 7, 8, 9, 11]. Lastly, we show that in fuzzy almost regular space [14], this newly defined closure operator will be idempotent.

Keywords and Phrases: Fuzzy regular open set, fuzzy semiopen set, fuzzy β-open set, fuzzy preopen set, fuzzy δ^c-closed set, fuzzy γ-open set, fuzzy almost regular space, δ^c-convergence of a fuzzy net.

2020 Mathematics Subject Classification: 54A40, 03E72.

1. Introduction

After introducing fuzzy closure operator by Chang [9], several types of fuzzy closure-like operators are introduced and studied by many mathematicians. In this context we have to mention fuzzy β^*-closure operator [2], fuzzy s^*-closure operator [3], fuzzy p^*-closure operator [4], fuzzy γ^*-closure operator [6], fuzzy s^c-closure operator [7], fuzzy β^c-closure operator [8]. Fuzzy δ-closure operator is introduced in [11]. Here we introduce fuzzy δ^c-closure operator which is coarser than the fuzzy δ-closure operator and for a fuzzy open set, these two operators coincide. Again this newly defined operator is not an idempotent operator, in general. Also it is shown that this newly defined operator is coarser than the operators defined in [2, 3, 4, 6, 7, 8, 9].

2. Preliminaries

Throughout the paper, by (X, τ) or simply by X we mean a fuzzy topological space (fts, for short) in the sense of Chang [9]. A fuzzy set A is a function from a non-empty set X into a closed interval $I = [0, 1]$, i.e., $A \in I^X$ [15]. The support of a fuzzy set A in X will be denoted by $suppA$ [15] and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [13] with the singleton support $x \in X$ and the value t $(0 < t \leq 1)$ at x will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 in X respectively. The complement of a fuzzy set A in X will be denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for all $x \in X$ [15]. For two fuzzy sets A and B in X, we write $A \leq B$ if and only if $A(x) \leq B(x)$, for each $x \in X$, and AqB means A is quasi-coincident (q-coincident, for short) with B if $A(x) + B(x) > 1$, for some $x \in X$ [13]. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. clA and $intA$ of a fuzzy set A in X respectively stand for the fuzzy closure [9] and fuzzy interior [9] of A in X. A fuzzy set A in X is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy preopen [12], fuzzy β-open [10], fuzzy γ-open [5]) if $A = intclA$ (resp., $A \leq clintA$, $A \leq intclA$, $A \leq clintclA$, $A \leq (intclA) \bigcup (clintA)$). The complement of a fuzzy regular open (resp., fuzzy semiopen, fuzzy preopen, fuzzy β-open, fuzzy γ-open) set is called a fuzzy regular closed [1] (resp., fuzzy semiclosed [1], fuzzy preclosed [12], fuzzy β-closed [10], fuzzy γ-closed [5]) set. The smallest fuzzy semiclosed (resp., fuzzy preclosed, fuzzy β-closed, fuzzy γ-closed) set containing a fuzzy set A is called fuzzy semiclosure [1] (resp., fuzzy preclosure [12], fuzzy β-closure [10], fuzzy γ-closure [5]) of A and is denoted by $sclA$ (resp., $pclA$, βclA, γclA). A fuzzy point x_t is called a fuzzy δ-cluster point [11] of a fuzzy set A in an fts X if UqA for every fuzzy regular open set U in X q-coincident with x_t. The union of all fuzzy δ-cluster points of a fuzzy set A in an fts X is called fuzzy δ-closure of A, denoted by A_δ [11]. A fuzzy set A is called fuzzy δ-closed if $A = [A]_\delta$ [11]. The collection of all fuzzy regular open (resp., fuzzy semiopen, fuzzy preopen, fuzzy β-open, fuzzy γ-open) sets in an fts X is denoted by $FRO(X)$ (resp., $FSO(X)$, $FPO(X)$, $F\beta O(X)$, $F\gamma O(X)$) and that of fuzzy regular closed (resp., fuzzy semiclosed, fuzzy preclosed, fuzzy β-closed, fuzzy γ-closed) sets is denoted by $FRC(X)$ (resp., $FSC(X)$, $FPC(X)$, $F\beta C(X)$, $F\gamma C(X)$). A function $S : (D, \gg) \to J$ where $(D, \gg)$ is a directed set and J is the collection of all fuzzy points in an ordinary set X, is called a fuzzy net in X. It is denoted by $\{S_n : n \in D\}$. In this paper fuzzy regular open sets are required for making $(D, \gg)$ as a directed set.

3. Fuzzy δ^c-Closure Operator: Some Properties

In this section fuzzy δ^c-closure operator is introduced and studied. It is shown that this operator is coarser than fuzzy δ-closure operator, but these two operators coincide for a fuzzy open set.

Definition 3.1. *A fuzzy point x_t in an fts (X, τ) is called a fuzzy δ^c-cluster point of a fuzzy set A in an fts X if $clUqA$ for every $U \in FRO(X)$ with $x_t qU$.*
The union of all fuzzy δ^c-cluster points of A is called fuzzy δ^c-closure of A, to be denoted by $[A]^c_\delta$. A is called fuzzy δ^c-closed set if $A = [A]^c_\delta$ and the complement of a fuzzy δ^c-closed set in an fts X is called fuzzy δ^c-open set in X.

Note 3.2. *It is clear from definition that for any $A \in I^X$, $[A]_\delta \leq [A]^c_\delta$. But the converse is not necessarily true, follows from the following example.*

Example 3.3. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.5$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$ and the fuzzy point $a_{0.7}$. Here $FRO(X) = \tau$. Now $a_{0.7}qA \in FRO(X)$, but $A \not{q} B \Rightarrow a_{0.7} \notin [B]_\delta$. But as $clA = (1_X \setminus A)qB \Rightarrow a_{0.7} \in [A]^c_\delta$.

The following theorem shows that under which condition $[A]_\delta$ and $[A]^c_\delta$ coincide.

Theorem 3.4. *For a fuzzy open set A in an fts X, $[A]_\delta = [A]^c_\delta$.*
Proof. By Note 3.2, it suffices to show that $[A]^c_\delta \leq [A]_\delta$ for every fuzzy open set A in X. Let $x_t \notin [A]_\delta$. Then there exists $V \in FRO(X)$, $x_t qV, V \not{q} A \Rightarrow V \leq 1_X \setminus A$ where $1_X \setminus A$ is fuzzy closed set in X (as fuzzy regular open set is fuzzy open, A is fuzzy open set and so $1_X \setminus A$ is fuzzy closed). Therefore, $clV \leq cl(1_X \setminus A) = 1_X \setminus A \Rightarrow clV \not{q} A \Rightarrow x_t \notin [A]^c_\delta$. Hence the proof.

The next theorem characterizes fuzzy δ^c-closure operator of a fuzzy set in an fts X.

Theorem 3.5. *For any fuzzy set A in an fts (X, τ),*
$[A]^c_\delta = \bigcap\{[U]^c_\delta : U$ *is fuzzy open set in X with $A \leq U$*$\}$.
Proof. Clearly L.H.S. $\leq$ R.H.S.
If possible, let $x_t \in$ R.H.S, but $x_t \notin$ L.H.S. Then there exists $V \in FRO(X)$ with $x_t qV$ and $clV \not{q} A \Rightarrow A \leq 1_X \setminus clV (\in \tau)$. By hypothesis, $x_t \in [1_X \setminus clV]^c_\delta$. But as $clV \not{q}(1_X \setminus clV)$, $x_t \notin [1_X \setminus clV]^c_\delta$, a contradiction.

Note 3.6. *By Theorem 3.4 and Theorem 3.5, we conclude that $[A]^c_\delta$ is the intersection of fuzzy δ^c-closure of sets in X containing the fuzzy set A in X.*
The next example shows that fuzzy δ^c-closure operator is not an idempotent operator. It is obvious that $[A]^c_\delta \leq [[A]^c_\delta]^c_\delta$ for any $A \in I^X$. But the converse is not true, in general, follows from the next example.

Example 3.7. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.5$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = B(b) = 0.4$ and the fuzzy points $a_{0.5}$ and $a_{0.7}$. We claim that $a_{0.7} \in [a_{0.5}]^c_\delta$, $a_{0.5} \in [B]^c_\delta$, but $a_{0.7} \notin [B]^c_\delta$. Now $a_{0.7}qA \in FRO(X)$ and $a_{0.5}q1_X \in FRO(X)$ only and so $a_{0.5} \in [B]^c_\delta$. Here $FRO(X) = \tau$. Now $a_{0.7}qA$, but $clA = (1_X \setminus A) \not{q} B \Rightarrow a_{0.7} \notin [B]^c_\delta$. Therefore, $a_{0.7} \in [a_{0.5}]^c_\delta \subseteq [[B]^c_\delta]^c_\delta$, but $a_{0.7} \notin [B]^c_\delta \Rightarrow [[B]^c_\delta]^c_\delta \not\subseteq [B]^c_\delta$.

Theorem 3.8. *In an fts (X, τ), the following statements are true :*
(a) 0_X and 1_X are fuzzy δ^c-closed sets in X,
(b) for any two fuzzy sets $A, B \in X$, $A \leq B \Rightarrow [A]^c_\delta \leq [B]^c_\delta$,
(c) for any two $A, B \in I^X$, $[A \bigcup B]^c_\delta = [A]^c_\delta \bigcup [B]^c_\delta$,
(d) for any two $A, B \in I^X$, $[A \bigcap B]^c_\delta \leq [A]^c_\delta \bigcap [B]^c_\delta$, the equality does not hold, in general, follows from the next example,
(e) union of any two fuzzy δ^c-closed sets in X is also so,
(f) intersection of any two fuzzy δ^c-closed sets in X is also so.
Proof. (a) and (b) are obvious.
(c) By (b), we can write, $[A]^c_\delta \bigcup [B]^c_\delta \leq [A \bigcup B]^c_\delta$.
To prove the converse, let $x_t \in [A \bigcup B]^c_\delta$. Then for any $U \in FRO(X)$ with $x_t qU$, $clUq(A \bigcup B)$. Then there exists $y \in X$ such that $(clU)(y) + max\{A(y), B(y)\} > 1 \Rightarrow$ either $(clU)(y) + A(y) > 1$ or $(clU)(y) + B(y) > 1 \Rightarrow$ either $clUqA$ or $clUqB \Rightarrow$ either $x_t \in [A]^c_\delta$ or $x_t \in [B]^c_\delta \Rightarrow x_t \in [A]^c_\delta \bigcup [B]^c_\delta$.
(d) Follows from (b).
(e) Follows from (c).
(f) From (d), we have $[A \bigcap B]^c_\delta \leq [A]^c_\delta \bigcap [B]^c_\delta$ for any two fuzzy sets $A, B \in X$.
Conversely, let A, B be two fuzzy δ^c-closed sets in X. Then $[A]^c_\delta = A, [B]^c_\delta = B$. Now $[A]^c_\delta \bigcap [B]^c_\delta = A \bigcap B \leq [A \bigcap B]^c_\delta \Rightarrow [A]^c_\delta \bigcap [B]^c_\delta \leq [A \bigcap B]^c_\delta$.

Example 3.9. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.5$. Then (X, τ) is an fts. Consider two fuzzy sets C and D defined by $C(a) = 0.5, C(b) = 0.1, D(a) = 0.1, D(b) = 0.6$ and the fuzzy point $a_{0.61}$. Now $FRO(X) = \tau$. Here $a_{0.61}qA \in FRO(X)$, $clA = (1_X \setminus A)$ and so $clAqC, clAqD \Rightarrow a_{0.61} \in [C]^c_\delta$ and $a_{0.61} \in [D]^c_\delta$, i.e., $a_{0.61} \in [C]^c_\delta \bigcap [D]^c_\delta$. Let $E = C \bigcap D$. Then $E(a) = E(b) = 0.1$. Now $clA \not{q} E \Rightarrow a_{0.61} \notin [E]^c_\delta = [C \bigcap D]^c_\delta$.

Note 3.10. *Infact, intersection of any collection of fuzzy δ^c-closed sets in an fts (X, τ) is fuzzy δ^c-closed set in X. So we can conclude that fuzzy δ^c-open sets in an fts (X, τ) form a fuzzy topology τ_{δ^c} (say) which is coarser than fuzzy topology τ of (X, τ).*

Result 3.11. *We conclude that $x_t \in [y_{t'}]^c_\delta$ does not imply $y_{t'} \in [x_t]^c_\delta$ where $x_t, y_{t'}$*

$(0 < t, t' \leq 1)$ *are fuzzy points in* X *as shown from the following example.*

Example 3.12. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.2, B(a) = 0.7, B(b) = 0.2$. Then (X, τ) is an fts. Consider two fuzzy points $a_{0.4}$ and $b_{0.9}$. We claim that $a_{0.4} \in [b_{0.9}]^c_\delta$, but $b_{0.9} \notin [a_{0.4}]^c_\delta$. Here $FRO(X) = \{0_X, 1_X, A\}$. Now $1_X \in FRO(X)$ only such that $a_{0.4} q 1_X$ and so $a_{0.4} \in [b_{0.9}]^c_\delta$. But $b_{0.9} q A \in FRO(X)$, $clA = (1_X \setminus A) \not{q} a_{0.4} \Rightarrow b_{0.9} \notin [a_{0.4}]^c_\delta$.

4. Mutual Relationship and Fuzzy Almost Regular Space

In this section we first recall several types of fuzzy closure-like operators from [2, 3, 4, 6, 7, 8, 9, 11] and then establish the mutual relationship between these closure operators with fuzzy δ^c-closure operator. Next we recall fuzzy almost regular space in which fuzzy δ^c-closure operator will be an idempotent operator.

Definition 4.1. *A fuzzy point* x_t *in an fts* (X, τ) *is called fuzzy* s^**-cluster point* [3] *(resp., fuzzy* p^**-cluster point* [4], *fuzzy* β^**-cluster point* [2], *fuzzy* γ^**-cluster point* [6]*) of a fuzzy set* A *in* X *if for every* $U \in FSO(X)$ *(resp.,* $U \in FPO(X)$, $U \in F\beta O(X)$, $U \in F\gamma O(X)$*) with* $x_t q U$, $sclU q A$ *(resp.,* $pclU q A$, $\beta clU q A$, $\gamma clU q A$*).*
The union of all fuzzy s^**-cluster (resp., fuzzy* p^**-cluster, fuzzy* β^**-cluster, fuzzy* γ^**-cluster) points of a fuzzy set* A *is called fuzzy* s^**-closure* [4] *(resp., fuzzy* p^**-closure* [4], *fuzzy* β^**-closure* [2], *fuzzy* γ^**-closure* [6]*) of* A*, denoted by* $[A]_s$ *(resp.,*$[A]_p$, $[A]_\beta$, $[A]_\gamma$*).*

Definition 4.2. *A fuzzy point* x_t *in an fts* (X, τ) *is called a fuzzy* s^c*-cluster point* [7] *(resp., fuzzy* β^c*-cluster point* [8]*) of a fuzzy set* A *in* X *if* $clU q A$ *(resp.,* $clU q A$*) for every fuzzy semiopen (resp., fuzzy* β*-open) set* U *in* X *with* $x_t q U$.
The union of all fuzzy s^c*-cluster (resp., fuzzy* β^c*-cluster) points of a fuzzy set* A *in an fts* X *is called fuzzy* s^c*-closure* [7] *(resp., fuzzy* β^c*-closure* [8]*) of* A*, denoted by* $[A]^c_s$ *(resp.,* $[A]^c_\beta$*).*

Note 4.3. *It is clear from discussion that for any fuzzy set* A *in an fts* (X, τ), $clA, [A]_s, [A]_\beta, [A]_p, [A]_\gamma, [A]^c_s, [A]^c_\beta \subseteq [A]^c_\delta$. *But the reverse implications are not true, in general, follow from the following examples.*

Example 4.4. $x_t \in [A]^c_\delta$, but $x_t \notin [A]_\beta, [A]_s, [A]_\gamma, [A]^c_s, [A]^c_\beta, clA$ for any $A \in I^X$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.5$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = B(b) = 0.4$ and the fuzzy point $a_{0.5}$. Here $FRO(X) = \tau$. Now $1_X \in \tau$ only such that $a_{0.5} q 1_X$ and so $a_{0.5} \in [B]^c_\delta$. Let us consider the fuzzy set U defined by $U(a) = 0.51, U(b) = 0.5$. Then $U \in FSO(X)$ (also $U \in FSC(X)$) and $U \in F\beta O(X)$ (also $U \in F\beta C(X)$) and $U \in F\gamma O(X)$ (also $U \in F\gamma C(X)$). Then $sclU = \beta clU = \gamma clU = U \not{q} B \Rightarrow a_{0.5} \notin [B]_s$, $a_{0.5} \notin [B]_\beta$, $a_{0.5} \notin [B]_\gamma$. Also $clU = (1_X \setminus A) \not{q} B \Rightarrow a_{0.5} \notin [B]^c_s, [B]^c_\beta$.

Next consider the fuzzy point $b_{0.6}$ and the fuzzy set C defined by $C(a) = C(b) = 0.5$. Now $b_{0.6}qA \in FRO(X)$ and $clA = (1_X \setminus A)qC \Rightarrow b_{0.6} \in [C]^c_\delta$. But $clC = (1_X \setminus A) \not\ni_{0.6}$.

Example 4.5. $x_t \in [A]^c_\delta$, but $x_t \notin [A]_p$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = A(b) = 0.4, B(a) = 0.6, B(b) = 0.5$. Then (X, τ) is an fts. Here $FRO(X) = \tau$. Now $FPO(X) = \{0_X, 1_X, U, V\}$ where $U \leq A, 0.4 < V(a) \leq 0.6, V(b) \leq 0.5$. Consider the fuzzy set C defined by $C(a) = C(b) = 0.45$ and the fuzzy point $a_{0.51}$. $a_{0.51}qB \in FRO(X)$ and $clB = (1_X \setminus A)qC \Rightarrow a_{0.51} \in [C]^c_\delta$. Now $a_{0.51}qD \in FPO(X)$ where $D(a) = D(b) = 0.5$. Since $D \in FPC(X)$ also, $pclD = D \not{q}C \Rightarrow a_{0.51} \notin [C]_p$.

Let us now recall the following separation axiom from [14] for ready references.

Definition 4.6. [14] *An fts (X, τ) is called fuzzy almost regular space if for each fuzzy point x_t and each fuzzy set $U \in FRO(X)$ with x_tqU, there exists $V \in FRO(X)$ such that $x_tqV \leq clV \leq U$.*

Theorem 4.7. *For an fts (X, τ), the following statements are equivalent :*
(a) X is fuzzy almost regular space,
(b) for any $A \in I^X$, $[A]_\delta = [A]^c_\delta$,
(c) for each fuzzy point x_t and each $U \in FRC(X)$ with $x_t \notin U$, there exists $V \in FRO(X)$ such that $x_t \notin clV$ and $U \leq V$,
(d) for each fuzzy point x_t and each $U \in FRC(X)$ with $x_t \notin U$, there exist $V, W \in FRO(X)$ such that x_tqV, $U \leq W$ and $V \not{q}W$,
(e) for any $A \in I^X$ and any $U \in FRC(X)$ with $A \not\leq U$, there exist $V, W \in FRO(X)$ such that $AqV, U \leq W$ and $V \not{q}W$,
(f) for any $A \in I^X$ and any $U \in FRO(X)$ with AqU, there exists $V \in FRO(X)$ such that $AqV \leq clV \leq U$.

Proof. (a) $\Rightarrow$ (b) By Note 3.2, it suffices to show that $[A]^c_\delta \subseteq [A]_\delta$, for any $A \in I^X$. Let $x_t \in [A]^c_\delta$ be arbitrary and $V \in FRO(X)$ with x_tqV. By (a), there exists $U \in FRO(X)$ such that $x_tqU \leq clU \leq V$. By hypothesis, $clUqA \Rightarrow VqA \Rightarrow x_t \in [A]_\delta$. Since x_t is taken arbitrarily, $[A]^c_\delta \subseteq [A]_\delta$.
(b) $\Rightarrow$ (a) Let x_t be a fuzzy point in X and $U \in FRO(X)$ with x_tqU. Then $U(x) + t > 1 \Rightarrow x_t \notin 1_X \setminus U(\in FRC(X))$. By (b), $[1_X \setminus U]_\delta = [1_X \setminus U]^c_\delta$. As $U \not{q}(1_X \setminus U)$, $x_t \notin [1_X \setminus U]_\delta \Rightarrow x_t \notin [1_X \setminus U]^c_\delta$. Then there exists $V \in FRO(X)$ with x_tqV, $clV \not{q}(1_X \setminus U) \Rightarrow clV \leq U$. Therefore, $x_tqV \leq clV \leq U \Rightarrow X$ is fuzzy almost regular space.
(a) $\Rightarrow$ (c) Let x_t be a fuzzy point in X and $U \in FRC(X)$ with $x_t \notin U$. Then $x_tq(1_X \setminus U) \in FRO(X)$. By (a), there exists $V \in FRO(X)$ such that $x_tqV \leq clV \leq 1_X \setminus U$. Therefore, $U \leq 1_X \setminus clV$ $(= W$, say). Then $W \in FRO(X)$. Now

$x_tqV = intV \Rightarrow x_tqintV \leq V \leq intclV \Rightarrow x_tq(intclV) \Rightarrow (intclV)(x) + t > 1 \Rightarrow 1 - (intclV)(x) < t \Rightarrow x_t \notin 1_X \setminus intclV = cl(1_X \setminus clV) = clW$.
(c) $\Rightarrow$ (d) Let x_t be a fuzzy point in X and $U \in FRC(X)$ with $x_t \notin U$. By (c), there exists $V \in FRO(X)$ such that $U \leq V$ and $x_t \notin clV$. Now $x_tq(1_X \setminus clV)$ ($= W$, say). Then $W \in FRO(X)$. Also $V \not{q} W$.
(d) $\Rightarrow$ (e) Let $A \in I^X$ and $U \in FRC(X)$ with $A \not\leq U$. Then there exists $x \in X$ such that $A(x) > U(x)$. Let $A(x) = t$. Then $x_t \notin U$. By (d), there exist $V, W \in FRO(X)$ such that $x_tqV, U \leq W$ and $V \not{q} W$. Again $V(x) + t > 1 \Rightarrow V(x) + A(x) > 1 - t + t = 1 \Rightarrow AqV$.
(e) $\Rightarrow$ (f) Let $A \in I^X$ and $U \in FRO(X)$ with AqU. Then $A \not\leq 1_X \setminus U \in FRC(X)$. By (e), there exist $V, W \in FRO(X)$ such that $AqV, 1_X \setminus U \leq W$ and $V \not{q} W \Rightarrow V \leq 1_X \setminus W \in FRC(X) \Rightarrow clV \leq cl(1_X \setminus W) = 1_X \setminus W \leq U$. Therefore, $AqV \leq clV \leq U$.
(f) $\Rightarrow$ (a) Obvious.

Corollary 4.8. *An fts (X, τ) is fuzzy almost regular if and only if every fuzzy δ-closed set in X is fuzzy δ^c-closed set in X.*

Corollary 4.9. *In a fuzzy almost regular space, $[A]^c_\delta = [[A]^c_\delta]^c_\delta$, for any $A \in I^X$.*
Proof. From [14], we have $[[A]_\delta]_\delta = [A]_\delta$ in a fuzzy almost regular space. So by Theorem 4.7 (a)$\Leftrightarrow$ (b), $[A]^c_\delta = [[A]^c_\delta]^c_\delta$.

5. Fuzzy δ^c-Closure Operator : More characterizations Via Fuzzy Net

In this section we first introduce fuzzy δ^c-cluster point and fuzzy δ^c-convergence of a fuzzy net and then fuzzy δ^c-closure operator of fuzzy sets is characterized in terms of these concepts.

Definition 5.1. *A fuzzy point x_t in an fts (X, τ) is called a fuzzy δ^c-cluster point of a fuzzy net $\{S_n : n \in (D, \geq)\}$ if for every $U \in FRO(X)$ with x_tqU and for any $n \in D$, there exists $m \in D$ with $m \geq n$ such that S_mqclU.*

Definition 5.2. *A fuzzy net $\{S_n : n \in (D, \geq)\}$ in an fts (X, τ) is said to δ^c-converge to a fuzzy point x_t if for every $U \in FRO(X)$ with x_tqU, there exists $m \in D$ such that S_nqclU, for all $n \geq m$ $(n \in D)$. This is denoted by $S_n \overrightarrow{\delta^c} x_t$.*

Theorem 5.3. *A fuzzy point x_t is a fuzzy δ^c-cluster point of a fuzzy net $\{S_n : n \in (D, \geq)\}$ in an fts (X, τ) iff there exists a fuzzy subnet of $\{S_n : n \in (D, \geq)\}$ which δ^c-converges to x_t.*
Proof. Let x_t be a fuzzy δ^c-cluster point of the fuzzy net $\{S_n : n \in (D, \geq)\}$. Let $C(Q_{x_t})$ denote the set of fuzzy closures of all fuzzy regular open sets of X q-coincident with x_t. Then for any $A \in C(Q_{x_t})$ and any $m \in D$, there exists $n \in D$ such that $n \geq m$ and S_nqA. Let E denote the set of all ordered pairs

(n, A) such that $n \in D$, $A \in C(Q_{x_t})$ and $S_n q A$. Then $(E, \gg)$ is a directed set, where $(m, A) \gg (n, B)$ $((m, A), (n, B) \in E)$ iff $m \geq n$ in D and $A \leq B$. Then $T : (E, \gg) \to (X, \tau)$ given by $T(m, A) = S_m$ is clearly a fuzzy subnet of $\{S_n : n \in (D, \geq)\}$. We claim that $T \overrightarrow{\delta^c} x_t$. Let $V \in FRO(X)$ with $x_t q V$. Then there exists $n \in D$ such that $(n, clV) \in E$ and so $S_n q clV$. Now for any $(m, A) \gg (n, clV)$, $T(m, A) = S_m q A \leq clV \Rightarrow T(m, A) q clV$. Consequently, $T \overrightarrow{\delta^c} x_t$.

Conversely, let x_t be not a fuzzy δ^c-cluster point of the fuzzy net $\{S_n : n \in (D, \geq)\}$. Then there exists $U \in FRO(X)$ with $x_t q U$ and an $n \in D$ such that $S_m \not{q} clU$, for all $m \geq n$. Then clearly no fuzzy subnet of the net $\{S_n : n \in (D, \geq)\}$ can δ^c-converge to x_t.

Theorem 5.4. *Let A be a fuzzy set in an fts (X, τ). A fuzzy point $x_t \in [A]^c_\delta$ iff there is a fuzzy net $\{S_n : n \in (D, \geq)\}$ in A, which δ^c-converges to x_t.*

Proof. Let $x_t \in [A]^c_\delta$. Then for any $U \in FRO(X)$ with $x_t q U$, $clU q A$, i.e., there exists $y^U \in suppA$ and a real number s_U with $0 < s_U \leq A(y^U)$ such that the fuzzy point $y^U_{s_U}$ with support y^U and the value s_U belongs to A and $y^U_{s_U} q clU$. We choose and fix one such $y^U_{s_U}$ for each U. Let $\mathcal{D}$ denote the set of all fuzzy regular open set in X q-coincident with x_t. Then $(\mathcal{D}, \succeq)$ is a directed set under inclusion relation, i.e., $B, C \in \mathcal{D}$, $B \succeq C$ iff $B \leq C$. Then $\{y^U_{s_U} \in A : y^U_{s_U} q clU, U \in \mathcal{D}\}$ is a fuzzy net in A such that it δ^c-converges to x_t. Indeed, for any fuzzy regular open set U in X with $x_t q U$, if $V \in \mathcal{D}$ and $V \succeq U$ (i.e., $V \leq U$) then $y^V_{s_V} q clV \leq clU \Rightarrow y^V_{s_V} q clU$.

Conversely, let $\{S_n : n \in (D, \geq)\}$ be a fuzzy net in A such that $S_n \overrightarrow{\delta^c} x_t$. Then for any $U \in FRO(X)$ with $x_t q U$, there exists $m \in D$ such that $n \geq m \Rightarrow S_n q clU \Rightarrow A q clU$ (since $S_n \in A$). Hence $x_t \in [A]^c_\delta$.

Remark 5.5. *It is clear that an improved version of the converse of the last theorem can be written as "$x_t \in [A]^c_\delta$ if there exists a fuzzy net in A with x_t as a fuzzy δ^c-cluster point".*

References

[1] Azad, K. K., On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl., 82 (1981), 14-32.

[2] Bhattacharyya, Anjana, β^*-closure operator in fuzzy setting, Jour. Fuzzy Math., 22, No. 1 (2014), 203-212.

[3] Bhattacharyya, Anjana, s^*-closure operator and s^*-regularity in fuzzy setting, International Journal of Pure and Applied Mathematics, 96, No. 2 (2014), 279-288.

[4] Bhattacharyya, Anjana, p^*-closure operator and p^*-regularity in fuzzy setting, Mathematica Moravica, 19, No. 1 (2015), 131-139.

[5] Bhattacharyya, Anjana, Fuzzy γ-continuous Multifunction, International Journal of Advanced Research in Science and Engineering, 4, No. 2 (2015), 195-209.

[6] Bhattacharyya, Anjana, γ^*-regular space, Jour. Fuzzy Math., 28 No. 3 (2020), 709-719.

[7] Bhattacharyya, Anjana, A new type of closure-like operator via fuzzy semiopen sets, "Vasile Alecsandri" University of Bacău, Faculty of Sciences, Scientific Studies and Research, Series Mathematics and Informatics, 30, No. 1 (2020), 45-56.

[8] Bhattacharyya, Anjana, β^c-closure operator in fuzzy setting, J. of Ramanujan Society of Mathematics and Mathematical Sciences, 9, No. 1 (2021), 115-124.

[9] Chang, C. L., Fuzzy topological spaces, J. Math. Anal. Appl., 24 (1968), 182-190.

[10] Fath Alla, M. A., On fuzzy topological spaces, Ph. D. Thesis, Assiut Univ., Sohag, Egypt, 1984.

[11] Ganguly, S. and Saha, S., A note on δ-continuity and δ-connected sets in fuzzy set theory, Simon Stevin, 62 (1988), 127-141.

[12] Nanda, S., Strongly compact fuzzy topological spaces, Fuzzy Sets and Systems, 42 (1991), 259-262.

[13] Pu, Pao Ming and Liu, Ying Ming, Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moore-Smith convergence, Jour. Math. Anal. Appl., 76 (1980), 571-599.

[14] Sinha, S. P., Separation axioms in fuzzy topological spaces, Ph. D. Thesis, University of Calcutta, 1991.

[15] Zadeh, L. A., Fuzzy Sets, Inform. Control, 8 (1965), 338-353.