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COMMON FIXED POINT THEOREMS IN RIGHT COMPLETE DISLOCATED QUASI $\mathcal{G}-$ FUZZY METRIC SPACES

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Abstract: The aim of this paper is to present the ideas of right complete dislocated quasi $\mathcal{G}-$ fuzzy metric spaces and find the common fixed point results for mapping satisfying the $\alpha - \psi$ locally contractive mappings for a couple of such maps in a closed ball in right complete dislocated quasi $\mathcal{G}-$ fuzzy metric spaces. An example is likewise given which outline the predominance of our outcomes.

Keywords and Phrases: Common fixed point, right complete quasi $\mathcal{G}-$ fuzzy metric spaces, $\alpha-\psi$ contractive mappings, $\mathcal{G}-\alpha-$ admissible mapping with respect to η.

2020 Mathematics Subject Classification: 47H10, 54A40.

1. Introduction

In 1965, Zadeh [21] contemplated the idea of a fuzzy set in his original paper. From that point, it was grown widely by numerous scientists, which additionally remember fascinating uses of this hypothesis for various fields. Fuzzy set hypothesis has applications in applied sciences like neural organization hypothesis, steadiness hypothesis, numerical programming, demonstrating hypothesis, designing sciences, clinical sciences (clinical hereditary qualities, sensory system), picture preparing, control hypothesis, and correspondence. In 1975, Kramosil and Michalek [14] presented the idea of fuzzy metric space, which opened a road for additional advancement of examination in such spaces. Further, George and Veeramani [7] modified the idea of fuzzy metric space presented by Kramosil and Michalek [14] and furthermore have prevailing with regards to instigating a Hausdorff geography on a particularly fuzzy metric space which is regularly utilized in ebb and flow research nowadays. Most as of late, Gregori et al. [8] showed a few fascinating instances of fuzzy metrics with regards to the feeling of George and Veeramani [7] and have additionally used such fuzzy metrics to shading picture processing. In 2006, Mustafa and Sims [15] introduced a definition of G-metric space. After that few fixed point results have been demonstrated in G-metric spaces.

The notions of $\alpha - \psi$ contractive mappings and $\alpha-$ admissible mappings in complete metric spaces introduced by Samet et al. [17] in 2012 . In 2013, Salimi et al. [16] modified the notion of $\alpha - \psi$ contractive mappings and improved certain fixed point theorems for such mappings. In 2019, Shoaib et al. [18] introduces $\mathcal{G}-\alpha-$ admissible mappings with respect to η and $\alpha-\psi$ contractive type condition for a pair of such maps.

In this paper, we find the unique common fixed point results for mapping satisfying the $\alpha - \psi$ locally Contractive conditions on a closed ball in an right complete dislocated quasi $\mathcal{G}-$ fuzzy metric space. Our results improve several well known classical results.

2. Preliminaries

Definition 2.1. [20] *A 3-tuple $(\mathcal{X}, \mathcal{G}, *)$ is called a $\mathcal{G}-$ Fuzzy Metric Space if $\mathcal{X}$ is an arbitrary non-empty set, $*$ is a continuous $\mathsf{t}-$ norm and $\mathcal{G}$ is a fuzzy set on $\mathcal{X} \times \mathcal{X} \times \mathcal{X} \times (0, \infty)$ satisfying the following conditions for each $\mathsf{t}, \mathsf{s} > 0$.*

($\mathcal{G}$FMS-1) $\mathcal{G}(\sigma, \sigma, \beta, \mathsf{t}) > 0$ *for all* $\sigma, \beta \in \mathcal{X}$ *with* $\sigma \neq \beta$,

($\mathcal{G}$FMS-2) $\mathcal{G}(\sigma, \sigma, \beta, \mathsf{t}) \geq \mathcal{G}(\sigma, \beta, \gamma, \mathsf{t})$ *for all* $\sigma, \beta, \gamma \in \mathcal{X}$ *with* $\beta \neq \gamma$,

($\mathcal{G}$FMS-3) $\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t}) = 1$ *if and only if* $\sigma = \beta = \gamma$,

($\mathcal{G}$FMS-4) $\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t}) = \mathcal{G}(p\{\sigma, \beta, \gamma\}, \mathsf{t})$, *where p is a permutation function,*

($\mathcal{G}$FMS-5) $\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t} + \mathsf{s}) \geq \mathcal{G}(\sigma, \mathfrak{a}, \mathfrak{a}, \mathsf{t}) * \mathcal{G}(\mathfrak{a}, \beta, \gamma, \mathsf{s})$ *for all* $\sigma, \beta, \gamma, \mathfrak{a} \in \mathcal{X}$,

($\mathcal{G}$FMS-6) $\mathcal{G}(\sigma, \beta, \gamma, \cdot) : (0, \infty) \to [0, 1]$ *is continuous.*

Definition 2.2. [12] *A $\mathcal{G}-$ fuzzy metric space $(\mathcal{X}, \mathcal{G}, *)$ is said to be symmetric if* $\mathcal{G}(\sigma, \sigma, \beta, \mathsf{t}) = \mathcal{G}(\sigma, \beta, \beta, \mathsf{t})$ *for all* $\sigma, \beta \in \mathcal{X}$ *and for each* $\mathsf{t} > 0$.

Definition 2.3. [12] *Let $\mathcal{X}$ be a nonempty set and let $\mathcal{G} : \mathcal{X}\times\mathcal{X}\times\mathcal{X}\times[0,\infty) \to [0,1]$ be a function satisfying the following axioms:for all $\sigma, \beta, \gamma, \mathfrak{a} \in \mathcal{X}$*

(i) If $\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t}) = 1$ then $\sigma = \beta = \gamma$,

(ii) $\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t} + \mathsf{s}) \geq \mathcal{G}(\sigma, \mathfrak{a}, \mathfrak{a}, \mathsf{t}) * \mathcal{G}(\mathfrak{a}, \beta, \gamma, \mathsf{s})$.

Then the pair $(\mathcal{X}, \mathcal{G}, *)$ is called the dislocated quasi $\mathcal{G}-$ Fuzzy Metric Space ($D_q\mathcal{G}FMS$).

Definition 2.4. [12] *Let $(\mathcal{X}, \mathcal{G}, *)$ be a $D_q\mathcal{G}FMS$ then*

(i) A sequence $\{\sigma_p\}$ in $\mathcal{G}$ is said to be $\mathcal{G}-$ convergent to σ if for each $\mathsf{t} > 0$, $\mathcal{G}(\sigma, \sigma_p, \sigma_q, \mathsf{t}) \to 1$ as $p, q \to \infty$.

(ii) A sequence $\{\sigma_p\}$ in $\mathcal{G}$ is said to be right $\mathcal{G}-$ Cauchy sequence if for each $\epsilon > 0$ and $\mathsf{t} > 0$, there exists $p_0 \in \mathbb{N}$ with the end goal that $\mathcal{G}(\sigma_p, \sigma_q, \sigma_q, \mathsf{t}) > 1 - \varepsilon$, for each $p, q \geq p_0$.

(iii) A $D_q\mathcal{G}FMS$ is said to be right complete if every right $\mathcal{G}-$ Cauchy sequence is $\mathcal{G}-$ convergent.

Definition 2.5. [12] *Let $(\mathcal{X}, \mathcal{G}, *)$ be a $D_q\mathcal{G}FMS$.*

(i) For $\sigma_0 \in \mathcal{G}$, $\mathsf{r} > 0$, the $\mathcal{G}-$ ball with centre σ_0 and radius r is $B(\sigma_0, r, \mathsf{t}) = \{\beta \in \mathcal{G} : \mathcal{G}(\sigma_0, \beta, \beta, \mathsf{t}) > 1 - \mathsf{r}\}$.

(ii) Also $\overline{B(\sigma_0, r, \mathsf{t})} = \{\beta \in \mathcal{G} : \mathcal{G}(\sigma_0, \beta, \beta, \mathsf{t}) \geq 1 - \mathsf{r}\}$ is closed ball in $\mathcal{G}$.

Lemma 2.6. *Every closed ball in a right complete dislocated quasi $\mathcal{G}-$ fuzzy metric space is right complete.*

Definition 2.7. [18] *Let $\mathcal{P}, \mathcal{Q} : \mathcal{G} \to \mathcal{G}$ and $\alpha, \eta : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to R$ be two functions.*

(i) The pair $(\mathcal{P}, \mathcal{Q})$ is said to be $\mathcal{G} - \alpha-$ admissible with respect to η, if there exists $\sigma, \beta, \gamma \in \mathcal{G}$ such that $\alpha(\sigma, \beta, \gamma) \geq \eta(\sigma, \beta, \gamma)$, then $\alpha(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma) \geq \eta(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma)$ and $\alpha(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma) \geq \eta(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma)$.

(ii) If $\eta(\sigma, \beta, \gamma) = 1$, then $(\mathcal{P}, \mathcal{Q})$ is said to be $\mathcal{G} - \alpha-$ admissible mapping.

(iii) If $\alpha(\sigma, \beta, \gamma) = 1$, then $(\mathcal{P}, \mathcal{Q})$ is said to be $\eta-$ subadmissible mapping.

If $\mathcal{P} = \mathcal{Q}$ then $\mathcal{P}$ is called $\mathcal{G} - \alpha-$admissible mapping with respect to η. If $\mathcal{P} = \mathcal{Q}$ and $\eta(\sigma, \beta, \gamma) = 1$, then $\mathcal{P}$ is called $\mathcal{G} - \alpha-$ admissible mapping.

3. Main Results

Let Ψ denote the class of all functions $\psi : [0, 1] \to [0, 1]$ such that ψ is non-decreasing, continuous and let $\psi(\mathsf{t}) > \mathsf{t}$, for all $\mathsf{t} \in (0, 1)$.

If $\psi(0) = 0$ and $\psi(1) = 1$ additionally hold, then $\psi(\mathsf{t}) \geq \mathsf{t}, \mathsf{t} \in [0, 1]$ for all functions from Ψ.

Theorem 3.1. *Let $(\mathcal{X}, \mathcal{G}, *)$ be a right complete $D_q\mathcal{G}FMS$ and $\mathcal{P}, \mathcal{Q} : \mathcal{G} \to \mathcal{G}$ be two mappings. Suppose there exists two functions $\alpha, \eta : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to R$ such that $(\mathcal{P}, \mathcal{Q})$ is $\mathcal{G} - \alpha-$ admissible mapping with respect to η. For $\mathsf{r} > 0$, $\sigma_0 \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ and $\psi \in \Psi$ assume that,*

$$\begin{aligned}&\sigma, \beta, \gamma \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}, \ \ \alpha(\sigma, \beta, \gamma) \geq \eta(\sigma, \beta, \gamma)\\ &\qquad \Longrightarrow \psi(\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t})) \leq min\big\{\mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t}), \mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t})\big\}, \quad (3.1.1)\end{aligned}$$

and

$$*_{i=0}^{j}\psi^i\big(\mathcal{G}(\sigma_0, \mathcal{P}\sigma_0, \mathcal{P}\sigma_0, \mathsf{t})\big) \geq 1 - \mathsf{r}, \ \ \textit{for all} \ \ j \in \mathbb{N} \cup \{0\}. \quad (3.1.2)$$

Suppose that the following assertions hold:

(i) $\alpha(\sigma_0, \mathcal{P}\sigma_0, \mathcal{P}\sigma_0) \geq \eta(\sigma_0, \mathcal{P}\sigma_0, \mathcal{P}\sigma_0)$,

(ii) for any sequence $\{\sigma_p\}$ in $\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ such that $\alpha(\sigma_p, \sigma_{p+1}, \sigma_{p+1}) \geq \eta(\sigma_p, \sigma_{p+1}, \sigma_{p+1})$, for all $p \in \mathbb{N} \cup \{0\}$ and $\sigma_p \to \upsilon \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ as $p \to \infty$, then $\alpha(\upsilon, \sigma_p, \sigma_p) \geq \eta(\upsilon, \sigma_p, \sigma_p)$, for all $p \in \mathbb{N} \cup \{0\}$.

Then there exist a point $\sigma^\star$ in $\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ such that $\sigma^\star = \mathcal{P}\sigma^\star = \mathcal{Q}\sigma^\star$.

Proof. Let $\sigma_1 \in \mathcal{G}$ be such that $\sigma_1 = \mathcal{P}\sigma_0$ and $\sigma_2 = \mathcal{Q}\sigma_1$. Continuing this process, we construct a sequence σ_p of points in $\mathcal{G}$ such that $\sigma_{2i+1} = \mathcal{P}\sigma_{2i}$ and $\sigma_{2i+2} = \mathcal{Q}\sigma_{2i+1}$ where $i = 0, 1, 2, \cdots$.

By assumption $\alpha(\sigma_0, \sigma_1, \sigma_1) \geq \eta(\sigma_0, \sigma_1, \sigma_1)$ and $(\mathcal{P}, \mathcal{Q})$ is $\mathcal{G}-\alpha-$admissible with respect to η, we have $\alpha(\mathcal{P}\sigma_0, \mathcal{Q}\sigma_1, \mathcal{Q}\sigma_1) \geq \eta(\mathcal{P}\sigma_0, \mathcal{Q}\sigma_1, \mathcal{Q}\sigma_1)$ from which we deduce that $\alpha(\sigma_1, \sigma_2, \sigma_2) \geq \eta(\sigma_1, \sigma_2, \sigma_2)$ which also implies that $\alpha(\mathcal{Q}\sigma_1, \mathcal{P}\sigma_2, \mathcal{P}\sigma_2) \geq \eta(\mathcal{Q}\sigma_1, \mathcal{P}\sigma_2, \mathcal{P}\sigma_2)$.

Continuing in this way we obtain $\alpha(\sigma_p, \sigma_{p+1}, \sigma_{p+1}) \geq \eta(\sigma_p, \sigma_{p+1}, \sigma_{p+1})$, for all $p \in \mathbb{N} \cup \{0\}$.

First we show that $\sigma_p \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$, for all $p \in \mathbb{N}$. Using inequality (3.1.2) we have, $*_{i=0}^{p}\psi^i\big(\mathcal{G}(\sigma_0, \mathcal{P}\sigma_0, \mathcal{P}\sigma_0, \mathsf{t})\big) \geq 1 - \mathsf{r}$, for all $p \in \mathbb{N} \cup \{0\}$. It follows that,

$\sigma_1 \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$. Let $\sigma_2, \sigma_3, \cdots, \sigma_j \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ for some $j \in \mathbb{N}$. If $j = 2i+1$, where $i = 0, 1, \cdots, \frac{j-1}{2}$, so using inequality (3.1.1) we obtain,

$$\begin{aligned}
\mathcal{G}(\sigma_{2i+1}, \sigma_{2i+2}, \sigma_{2i+2}, \mathsf{t}) &= \mathcal{G}(\mathcal{P}\sigma_{2i}, \mathcal{Q}\sigma_{2i+1}, \mathcal{Q}\sigma_{2i+1}, \mathsf{t}),\\
&\geq \psi\big(\mathcal{G}(\sigma_{2i}, \sigma_{2i+1}, \sigma_{2i+1}, \mathsf{t})\big),\\
&\geq \psi^2\big(\mathcal{G}(\sigma_{2i-1}, \sigma_{2i}, \sigma_{2i}, \mathsf{t})\big),\\
&\vdots\\
&\geq \psi^{2i+1}\big(\mathcal{G}(\sigma_0, \sigma_1, \sigma_1, \mathsf{t})\big).
\end{aligned}$$

Thus we have ,

$$\mathcal{G}(\sigma_{2i+1}, \sigma_{2i+2}, \sigma_{2i+2}, \mathsf{t}) \geq \psi^{2i+1}\big(\mathcal{G}(\sigma_0, \sigma_1, \sigma_1, \mathsf{t})\big). \tag{3.1.3}$$

If $j = 2i+2$, then as $\sigma_1, \sigma_2, \cdots, \sigma_j \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ where $i = 0, 1, \cdots, \frac{j-2}{2}$. We obtain,

$$\mathcal{G}(\sigma_{2i+2}, \sigma_{2i+3}, \sigma_{2i+3}, \mathsf{t}) \geq \psi^{2i+2}\big(\mathcal{G}(\sigma_0, \sigma_1, \sigma_1, \mathsf{t})\big). \tag{3.1.4}$$

Thus from inequality (3.1.3) and (3.1.4), we have

$$\mathcal{G}(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \mathsf{t}) \geq \psi^j\big(\mathcal{G}(\sigma_0, \sigma_1, \sigma_1, \mathsf{t})\big). \tag{3.1.5}$$

Now,

$$\begin{aligned}
\mathcal{G}(\sigma_0, \sigma_{j+1}, \sigma_{j+1}, \mathsf{t}) &\geq \mathcal{G}\Big(\sigma_0, \sigma_1, \sigma_1, \frac{\mathsf{t}}{j+1}\Big) * \mathcal{G}\Big(\sigma_1, \sigma_2, \sigma_2, \frac{\mathsf{t}}{j+1}\Big) * \cdots *\\
&\qquad \mathcal{G}\Big(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \frac{\mathsf{t}}{j+1}\Big),\\
&\geq \psi^0\Big(\mathcal{G}\Big(\sigma_0, \sigma_1, \sigma_1, \frac{\mathsf{t}}{j+1}\Big)\Big) * \psi^1\Big(\mathcal{G}\Big(\sigma_0, \sigma_1, \sigma_1, \frac{\mathsf{t}}{j+1}\Big)\Big) * \cdots *\\
&\qquad \psi^j\Big(\mathcal{G}\Big(\sigma_0, \sigma_1, \sigma_1, \frac{\mathsf{t}}{j+1}\Big)\Big),\\
&\geq *_{i=0}^{j}\psi^i\Big(\mathcal{G}\Big(\sigma_0, \sigma_1, \sigma_1, \frac{\mathsf{t}}{j+1}\Big)\Big)
\end{aligned}$$

$$\mathcal{G}(\sigma_0, \sigma_{j+1}, \sigma_{j+1}, \mathsf{t}), \geq 1 - \mathsf{r}.$$

Thus $\sigma_{j+1} \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$. Hence $\sigma_p \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ for all $p \in \mathbb{N}$. Now inequality (3.1.5) can be written as

$$\mathcal{G}(\sigma_p, \sigma_{p+1}, \sigma_{p+1}, \mathsf{t}) \geq \psi^p\big(\mathcal{G}(\sigma_0, \sigma_1, \sigma_1, \mathsf{t})\big), \quad \text{for all } p \in \mathbb{N}. \tag{3.1.6}$$

Fix $\varepsilon > 0$ and let $p(\varepsilon) \in \mathbb{N}$ such that $*_{p \geq p(\varepsilon)} \psi^p\big(\mathcal{G}(\sigma_0, \sigma_1, \sigma_1, \mathsf{t})\big) > 1 - \varepsilon$. Let $p, q \in \mathbb{N}$ with $q > p > p(\varepsilon)$ we obtain,

$$\begin{aligned}
\mathcal{G}(\sigma_p, \sigma_q, \sigma_q, \mathsf{t}) &\geq \mathcal{G}\Big(\sigma_p, \sigma_{p+1}, \sigma_{p+1}, \frac{\mathsf{t}}{q}\Big) * \mathcal{G}\Big(\sigma_{p+1}, \sigma_{p+2}, \sigma_{p+2}, \frac{\mathsf{t}}{q}\Big) * \cdots * \\
&\qquad \mathcal{G}\Big(\sigma_{q-1}, \sigma_q, \sigma_q, \frac{\mathsf{t}}{q}\Big), \\
&\geq \psi^p\left(\mathcal{G}\Big(\sigma_0, \sigma_1, \sigma_1, \frac{\mathsf{t}}{q}\Big)\right) * \psi^{p+1}\left(\mathcal{G}\Big(\sigma_0, \sigma_1, \sigma_1, \frac{\mathsf{t}}{q}\Big)\right) * \cdots * \\
&\qquad \psi^{q-1}\left(\mathcal{G}\Big(\sigma_0, \sigma_1, \sigma_1, \frac{\mathsf{t}}{q}\Big)\right), \\
&\geq *_{p \geq p(\varepsilon)}\ \psi^p\big(\mathcal{G}(\sigma_0, \sigma_1, \sigma_1, \mathsf{t})\big), \\
\mathcal{G}(\sigma_p, \sigma_q, \sigma_q, \mathsf{t}) &> 1 - \varepsilon.
\end{aligned}$$

Thus $\{\sigma_p\}$ is a Cauchy sequence in $\big(\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}, \mathcal{G}\big)$. Using Lemma (2.6) there exist $\sigma^\star \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ such that $\sigma_p \to \sigma^\star$. Also

$$\lim_{p \to \infty} \mathcal{G}(\sigma^\star, \sigma_p, \sigma_p, \mathsf{t}) = 1. \tag{3.1.7}$$

On the other hand, from (ii) we have

$$\alpha(\sigma^\star, \sigma_p, \sigma_p) \geq \eta(\sigma^\star, \sigma_p, \sigma_p) \ \text{ for all } \ p \in \mathbb{N} \cup \{0\}. \tag{3.1.8}$$

Now using inequalities (3.1.1) and (3.1.8) we get

$$\mathcal{G}(\mathcal{P}\sigma^\star, \sigma_{2i+2}, \sigma_{2i+2}, \mathsf{t}) \geq \psi\Big(\mathcal{G}(\sigma^\star, \sigma_{2i+1}, \sigma_{2i+1}, \mathsf{t})\Big) > \mathcal{G}(\sigma^\star, \sigma_{2i+1}, \sigma_{2i+1}, \mathsf{t}).$$

Letting $i \to \infty$ and using (3.1.7), we obtain $\mathcal{G}(\mathcal{P}\sigma^\star, \sigma^\star, \sigma^\star, \mathsf{t}) > 1$. That is $\mathcal{P}\sigma^\star = \sigma^\star$. Similarly by using

$$\mathcal{G}(\mathcal{Q}\sigma^\star, \sigma_{2i+1}, \sigma_{2i+1}, \mathsf{t}) \geq \psi\Big(\mathcal{G}(\sigma^\star, \sigma_{2i}, \sigma_{2i}, \mathsf{t})\Big) > \mathcal{G}(\sigma^\star, \sigma_{2i}, \sigma_{2i}, \mathsf{t}).$$

As $i \to \infty$ we get $\mathcal{G}(\mathcal{Q}\sigma^\star, \sigma^\star, \sigma^\star, \mathsf{t}) > 1$. That is $\mathcal{Q}\sigma^\star = \sigma^\star$. Hence $\mathcal{P}$ and $\mathcal{Q}$ have a common fixed point in $\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$.

Corollary 3.2. *Let $(\mathcal{X}, \mathcal{G}, *)$ be a right complete $D_q\mathcal{G}FMS$ and $\mathcal{P}, \mathcal{Q} : \mathcal{G} \to \mathcal{G}$ be* *two mappings. Suppose there exist a function* $\alpha : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to R$ *such that* $(\mathcal{P}, \mathcal{Q})$ *is* $\mathcal{G} - \alpha -$ *admissible mapping. For* $\mathsf{r} > 0,\ \sigma_0 \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ *and* $\psi \in \Psi$ *assume that,*

$$\sigma, \beta, \gamma \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})},\ \ \alpha(\sigma, \beta, \gamma) \geq 1$$
$$\Longrightarrow \psi(\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t})) \leq min\big\{\mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t}), \mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t})\big\},$$

and

$$*_{i=0}^{j} \psi^i\big(\mathcal{G}(\sigma_0, \mathcal{P}\sigma_0, \mathcal{P}\sigma_0, \mathsf{t})\big) \geq 1 - \mathsf{r},\ \ \textit{for all}\ \ \ j \in \mathbb{N} \cup \{0\}.$$

Suppose that the following assertions hold:

(i) $\alpha(\sigma_0, \mathcal{P}\sigma_0, \mathcal{P}\sigma_0) \geq 1$,

(ii) for any sequence $\{\sigma_p\}$ *in* $\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ *such that* $\alpha(\sigma_p, \sigma_{p+1}, \sigma_{p+1}) \geq 1$, *for all* $p \in \mathbb{N} \cup \{0\}$ *and* $\sigma_p \to \upsilon \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ *as* $p \to \infty$, *then* $\alpha(\upsilon, \sigma_p, \sigma_p) \geq 1$, *for all* $p \in \mathbb{N} \cup \{0\}$.

Then there exist a point $\sigma^\star$ *in* $\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ *such that* $\sigma^\star = \mathcal{P}\sigma^\star = \mathcal{Q}\sigma^\star$.

Corollary 3.3. *Let* $(\mathcal{X}, \mathcal{G}, *)$ *be a right complete* $D_q\mathcal{G}FMS$ *and* $\mathcal{P}, \mathcal{Q} : \mathcal{G} \to \mathcal{G}$ *be two mappings. Suppose there exists a function* $\eta : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to R$ *such that* $(\mathcal{P}, \mathcal{Q})$ *is* $\eta-$ *subadmissible mapping. For* $\mathsf{r} > 0,\ \sigma_0 \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ *and* $\psi \in \Psi$ *assume that,*

$$\sigma, \beta, \gamma \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})},\ \ 1 \geq \eta(\sigma, \beta, \gamma)$$
$$\Longrightarrow \psi(\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t})) \leq min\big\{\mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t}), \mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t})\big\},$$

and

$$*_{i=0}^{j} \psi^i\big(\mathcal{G}(\sigma_0, \mathcal{P}\sigma_0, \mathcal{P}\sigma_0, \mathsf{t})\big) \geq 1 - \mathsf{r},\ \ \textit{for all}\ \ \ j \in \mathbb{N} \cup \{0\}.$$

Suppose that the following assertions hold:

(i) $1 \geq \eta(\sigma_0, \mathcal{P}\sigma_0, \mathcal{P}\sigma_0)$,

(ii) for any sequence $\{\sigma_p\}$ *in* $\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ *such that* $1 \geq \eta(\sigma_p, \sigma_{p+1}, \sigma_{p+1})$, *for all* $p \in \mathbb{N} \cup \{0\}$ *and* $\sigma_p \to \upsilon \in \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ *as* $p \to \infty$, *then* $1 \geq \eta(\upsilon, \sigma_p, \sigma_p)$, *for all* $p \in \mathbb{N} \cup \{0\}$.

Then there exist a point $\sigma^\star$ *in* $\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$ *such that* $\sigma^\star = \mathcal{P}\sigma^\star = \mathcal{Q}\sigma^\star$.

Theorem 3.4. *Let* $(\mathcal{X}, \mathcal{G}, *)$ *be a right complete* $D_q\mathcal{G}FMS$ *and* $\mathcal{P}, \mathcal{Q} : \mathcal{G} \to \mathcal{G}$ *be two mappings. Suppose there exists two functions* $\alpha, \eta : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to R$ *such that*

$(\mathcal{P},\mathcal{Q})$ is $\mathcal{G}-\alpha-$ admissible mapping with respect to η. For $\mathsf{r}>0$, $\sigma_0\in\overline{B(\sigma_0,\mathsf{r},\mathsf{t})}$ and $\psi\in\Psi$ assume that,

$$\sigma,\beta,\gamma\in\overline{B(\sigma_0,\mathsf{r},\mathsf{t})},\quad \alpha(\sigma,\beta,\gamma)\geq\eta(\sigma,\beta,\gamma)$$
$$\implies \psi(\mathcal{G}(\sigma,\beta,\gamma,\mathsf{t}))\leq min\big\{\mathcal{G}(\mathcal{P}\sigma,\mathcal{Q}\beta,\mathcal{Q}\gamma,\mathsf{t}),\mathcal{G}(\mathcal{Q}\sigma,\mathcal{P}\beta,\mathcal{P}\gamma,\mathsf{t})\big\},$$

and

$$*_{i=0}^{j}\psi^i\big(\mathcal{G}(\sigma_0,\mathcal{P}\sigma_0,\mathcal{P}\sigma_0,\mathsf{t})\big)\geq 1-\mathsf{r},\quad for\ all\quad j\in\mathbb{N}\cup\{0\}.$$

Suppose that the following assertions hold:

(i) $\alpha(\sigma_0,\mathcal{P}\sigma_0,\mathcal{P}\sigma_0)\geq\eta(\sigma_0,\mathcal{P}\sigma_0,\mathcal{P}\sigma_0)$,

(ii) for any sequence $\{\sigma_p\}$ in $\overline{B(\sigma_0,\mathsf{r},\mathsf{t})}$ such that $\alpha(\sigma_p,\sigma_{p+1},\sigma_{p+1})\geq\eta(\sigma_p,\sigma_{p+1},\sigma_{p+1})$, for all $p\in\mathbb{N}\cup\{0\}$ and $\sigma_p\to\upsilon\in\overline{B(\sigma_0,\mathsf{r},\mathsf{t})}$ as $p\to\infty$, then $\alpha(\upsilon,\sigma_p,\sigma_p)\geq\eta(\upsilon,\sigma_p,\sigma_p)$, for all $p\in\mathbb{N}\cup\{0\}$,

(iii) if $\sigma^\star$ is any common fixed point in $\overline{B(\sigma_0,\mathsf{r},\mathsf{t})}$ of $\mathcal{P}$ and $\mathcal{Q}$, σ be any fixed point of $\mathcal{P}$ or $\mathcal{Q}$ in $\overline{B(\sigma_0,\mathsf{r},\mathsf{t})}$, then $\alpha(\sigma^\star,\sigma,\sigma)\geq\eta(\sigma^\star,\sigma,\sigma)$.

Then $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point $\sigma^\star$.

Proof. By Theorem (3.1) we get $\mathcal{P}$ and $\mathcal{Q}$ have a common fixed point $\sigma^\star$. Assume that $\beta^\star$ be another fixed point of $\mathcal{P}$ in $\overline{B(\sigma_0,\mathsf{r},\mathsf{t})}$, then by assumption $\alpha(\sigma^\star,\beta^\star,\beta^\star)\geq\eta(\sigma^\star,\beta^\star,\beta^\star)$. Also,

$$\begin{aligned}\mathcal{G}(\sigma^\star,\beta^\star,\beta^\star,\mathsf{t})&=\mathcal{G}(\mathcal{Q}\sigma^\star,\mathcal{P}\beta^\star,\mathcal{P}\beta^\star,\mathsf{t}),\\&=min\{\mathcal{G}(\mathcal{Q}\sigma^\star,\mathcal{P}\beta^\star,\mathcal{P}\beta^\star,\mathsf{t}),\mathcal{G}(\mathcal{P}\sigma^\star,\mathcal{Q}\beta^\star,\mathcal{Q}\beta^\star,\mathsf{t})\},\\&\geq\psi\big(\mathcal{G}(\sigma^\star,\beta^\star,\beta^\star,\mathsf{t})\big).\end{aligned}$$

A contraction to the fact that for each $\mathsf{t}>0$, $\psi(\mathsf{t})>\mathsf{t}$. So $\sigma^\star=\beta^\star$. Hence $\mathcal{P}$ has no fixed point other that $\sigma^\star$. Similarly, $\mathcal{Q}$ has no fixed point other that $\sigma^\star$.

Example 3.5. Let $\mathcal{X}=\mathbb{R}^+\cup\{0\}$ and $\mathcal{G}(\sigma,\beta,\gamma,\mathsf{t})=\dfrac{\mathsf{t}}{\mathsf{t}+\mathcal{G}(\sigma,\beta,\gamma)}$ where $\mathcal{G}(\sigma,\beta,\gamma)=\sigma+\beta+\gamma$. Clearly $(\mathcal{X},\mathcal{G},*)$ be a right complete $D_q\mathcal{G}FMS$. Let $\mathcal{P},\mathcal{Q}:\mathcal{X}\to\mathcal{X}$ be defined by,

$$\mathcal{P}(\sigma)=\begin{cases}\dfrac{\sigma}{8}, & \sigma\in[0,1],\\[2ex] \dfrac{3\sigma}{2}, & \sigma\in(1,\infty)\end{cases}\qquad\text{and}\qquad \mathcal{Q}(\sigma)=\begin{cases}\dfrac{5\sigma}{7}, & \sigma\in[0,1],\\[2ex] 3\sigma, & \sigma\in(1,\infty).\end{cases}$$

Take $\sigma_0 = 1$, $\mathsf{r} = 4$, then $\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})} = [0, 1]$.
Define $\alpha(\sigma, \beta, \gamma) = 2\sigma - \beta + \gamma$, $\eta(\sigma, \beta, \gamma) = \sigma - 2\beta + \gamma$. Clearly $(\mathcal{P}, \mathcal{Q})$ is $\mathcal{G} - \alpha-$ admissible mapping with respect to η for all $\sigma, \beta, \gamma \in \mathcal{X}$. Let $\psi(\mathsf{t}) = \frac{t}{5}$. Now

$$\mathcal{G}(\sigma_0, \mathcal{P}\sigma_0, \mathcal{P}\sigma_0, \mathsf{t}) = \mathcal{G}(1, \frac{1}{8}, \frac{1}{8}, \mathsf{t}) = \frac{\mathsf{t}}{\mathsf{t} + 1 + \frac{1}{8} + \frac{1}{8}} = \frac{\mathsf{t}}{\mathsf{t} + \frac{10}{8}}$$

Also,

$$*_{i=0}^{p} \psi^i\big(\mathcal{G}(\sigma_0, \mathcal{P}\sigma_0, \mathcal{P}\sigma_0, \mathsf{t})\big) > -3.$$

If $\sigma, \beta, \gamma \in (1, \infty)$, then
Case 1. If $\min\big\{\mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t}), \mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t})\big\} = \mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t})$ then

$$\begin{aligned}\mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t}) = \mathcal{G}\Big(\frac{3\sigma}{2}, 3\beta, 3\gamma, \mathsf{t}\Big) &= \frac{\mathsf{t}}{\mathsf{t} + \frac{3\sigma}{2} + 3\beta + 3\gamma}, \\ &< \frac{\mathsf{t}}{\mathsf{t} + \sigma + \beta + \gamma} < \psi\big(\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t})\big).\end{aligned}$$

Case 2. If $\min\big\{\mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t}), \mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t})\big\} = \mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t})$ then

$$\begin{aligned}\mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t}) = \mathcal{G}\Big(3\sigma, \frac{3\beta}{2}, \frac{3\gamma}{2}, \mathsf{t}\Big) &= \frac{\mathsf{t}}{\mathsf{t} + 3\sigma + \frac{3\beta}{2} + \frac{3\gamma}{2}}, \\ &< \frac{\mathsf{t}}{\mathsf{t} + \sigma + \beta + \gamma} < \psi\big(\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t})\big).\end{aligned}$$

So the contractive conditions does not hold on $\mathcal{G}$.
If $\sigma, \beta, \gamma \in [0, 1] = \overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$, then
Case 3. If $\min\big\{\mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t}), \mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t})\big\} = \mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t})$ then

$$\begin{aligned}\mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t}) = \mathcal{G}\Big(\frac{\sigma}{8}, \frac{5\beta}{7}, \frac{5\gamma}{7}, \mathsf{t}\Big) &= \frac{\mathsf{t}}{\mathsf{t} + \frac{\sigma}{8} + \frac{5\beta}{7} + \frac{5\gamma}{7}}, \\ &\geq \frac{\mathsf{t}}{\mathsf{t} + \sigma + \beta + \gamma} \geq \psi\big(\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t})\big).\end{aligned}$$

Case 4. If $\min\big\{\mathcal{G}(\mathcal{P}\sigma, \mathcal{Q}\beta, \mathcal{Q}\gamma, \mathsf{t}), \mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t})\big\} = \mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t})$ then

$$\begin{aligned}\mathcal{G}(\mathcal{Q}\sigma, \mathcal{P}\beta, \mathcal{P}\gamma, \mathsf{t}) = \mathcal{G}\Big(\frac{5\sigma}{7}, \frac{\beta}{8}, \frac{\gamma}{8}, \mathsf{t}\Big) &= \frac{\mathsf{t}}{\mathsf{t} + \frac{5\sigma}{7} + \frac{\beta}{8} + \frac{\gamma}{8}}, \\ &\geq \frac{\mathsf{t}}{\mathsf{t} + \sigma + \beta + \gamma} \geq \psi\big(\mathcal{G}(\sigma, \beta, \gamma, \mathsf{t})\big).\end{aligned}$$

Then the contractive condition holds on $\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$. Hence all the conditions of Theorem 3.1 are satisfied and $\mathcal{P}$ and $\mathcal{Q}$ have a common fixed point in $\overline{B(\sigma_0, \mathsf{r}, \mathsf{t})}$.

4. Conclusion

In this paper, we using this turn of events and compression condition to show the sequence in a closed ball in right complete $D_q\mathcal{G}FMS$ is a Cauchy sequences. In the process we have generalized a few notable, later and traditional outcomes from the writing.

References

[1] Afshari H., Solution of fractional differential equations in quasi-b-metric and b-metric-like spaces, Adv Differ Equ, 285 (2019). https://doi.org/10.1186/s13662-019-2227-9.

[2] Afshari H., Hojjat, Kalantari S., and Hassen A., Fixed point results for generalized $\alpha - \Psi-$ Suzuki-contractions in quasi-b-metric-like spaces, Asian-European journal of mathematics, 11.01 (2018): 1850012. https://doi.org/10.1142/S1793557118500122.

[3] Afshari H., Fahd Jarad, and Abdeljawad T., On a new fixed point theorem with an application on a coupled system of fractional differential equations, Advances in Difference Equations, 2020.1 (2020): 1-13. https://doi.org/10.1186/s13662-020-02926-0.

[4] Agarwal R. P., Karapinar E., Remarks on some coupled fixed point theorems in G-metric spaces, Fixed Point Theory Appl., 2 (2013).

[5] Amiri P., Afshari H., Common fixed point results for multi-valued mappings in complex-valued double controlled metric spaces and their applications to the existence of solution of fractional integral inclusion systems, Chaos, Solitons & Fractals, 154 (2022): 111622. https://doi.org/10.1016/j.chaos.2021.111622.

[6] Arshad M., Shoaib A., Beg I., Fixed point of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space, Fixed Point Theory Appl., 115 (2013).

[7] George A., Veeramani P., On some result in fuzzy metric spaces, Fuzzy Sets and Systems, 64 (1994), 395-399.

[8] Gregori V., Morillas S., and Sapena A., Examples of fuzzy metrics and applications, Fuzzy Sets and Systems, vol. 170 (2011), 95–111.

[9] Hussain N., Karapinar E., Salimi P., Akbar F., $\alpha-$ Admissible mappings and related fixed point theorems, J. Inequal. Appl., 114, (2013).

[10] Jeyaraman M., Muthuraj R., Sornavalli M. and Jeyabharathi M., Common fixed point Theorems in G- Fuzzy Metric Spaces, Journal of New Theory, 10 (2016), 12-18.

[11] Jeyaraman M. and Barveenbanu M., Fixed point Results on G - Fuzzy Metric Spaces, Roots International Journal of Multidisciplinary Researches, 3 (1) (2016), 72-76.

[12] Jeyaraman M., Poovaragavan D., Sowndrarajan S., and Manro S., Fixed Point Theorems for Dislocated Quasi G- Fuzzy Metric Spaces, Commun. Nonlinear Anal., 1 (2019), 23-31.

[13] Jleli M., Samet B., Remarks on G-metric spaces and fixed point theorems, Fixed Point Theory Appl., 210, (2012).

[14] Kramosil I., Michalek J., Fuzzy metric and statistical metric spaces, Kybernetica, 11 (1975), 326-334.

[15] Mustafa Z. and Sims B., A new approch to generalized metric spaces, J. Nonlinear Convex Anal., 7 (2006), 289-297.

[16] Salimi P., Latif A., Hussain N., Modified $\alpha - \psi$ contractive mappings with applications, Fixed Point Theory Appl., 151, (2013).

[17] Samet B., Vetro C., Vetro P., Fixed point theorems for $\alpha - \psi$ contractive type mappings, Nonlinear Anal., 75 (2012), 2154-2165.

[18] Shoaib A., Uddin F., Arshad M. and Usman Ali M., Common Fixed Point Results for $\alpha - \psi$ Locally Contractive Type Mappings in Right Complete Dislocated Quasi G-Metric Spaces, Thai Journal of Mathematics, Vol. 17, No. 3 (2019), 627-638.

[19] Shoaib A., Arshad M., Rasham T., Abbas M., Unique fixed point results on closed ball for dislocated quasi G-metric spaces, Transactions of A. Razmadze Mathematical Institute, 2017.

[20] Sun G. and Yang K., Generalized fuzzy metric spaces with properties, Research journal of Applied Sciences, Engineering and Technology, vol. 2, no. 7 (2010), 673–678.

[21] Zadeh L. A., Fuzzy Sets, Inform and Control, 8 (1965), 338-358.