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ON CAUCHY'S BOUND FOR ZEROS OF TRANSCENDENTAL ENTIRE FUNCTIONS

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Abstract: The prime concern of this paper is to derive bound for the moduli of the zeros of a transcendental entire function. A few examples are given here to validate the results obtained.

Keywords and Phrases: Transcendental entire function, Cauchy's bound, zero.

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1. Introduction, Definitions and Notations

Fundamental theorem of algebra only gives information about the number of zeros of a polynomial but not location of the zeros. All zeros of a quadratic polynomial can be derived algebraically for all possible values of its coefficients. But, difficulty arises when degree of polynomial increases. So, it is desirable to know a region where the zeros of a polynomial lie.

Problem of finding a region containing all the zeros of a polynomial has a rich old history {cf.[7]}. In 1829, Cauchy {cf.[7]} develop the following classical result:

Theorem A. *Let $P(z) = \sum_{j=0}^{n} a_j z^n$ be a polynomial of degree n, then all the zeros of $P(z)$ lie in $|z| \leq r$ where r is the unique positive root of the equation*

$$|a_n|z^n - (|a_{n-1}|z^{n-1} + ... + |a_0|) = 0.$$

So many improvements and generalizations of Theorem A for polynomials exist in the literature {[2], [3] & [4]}. We think that there is no such results for transcendental entire functions. The main aim of this paper is to develop some Cauchy's type bounds for the moduli of zeros of transcendental entire functions. We do not explain the standard theories, notations and definitions of entire functions as those are available in [6] & [9].

Some well known definitions are given below.

Definition 1. [6] *The order ρ of an entire function $f(z)$ is defined as*

$$\rho = \inf\{k > 0 : M_f(r) < e^{r^k}, r > r_0\}$$

where $M(r, f) := M_f(r) = \max_{|z|=r} |f(z)|$.

Definition 1 can be alternatively stated as:

Definition 2. [6] *The order ρ of an entire function $f(z)$ is defined as*

$$\rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r}$$

where $\exp^{[k]} x = \log^{[-k]} x = \exp(\exp^{[k-1]} x) = \log(\log^{[-k-1]} x)$ *for* $k = \pm 1, \pm 2, \pm 3, ...$ *and* $\exp^{[0]} x = \log^{[0]} x = x$.

For $\rho = \infty$, Satto [8] defined the concept of 'index' of an entire function as follows:

Definition 3. [8] *An entire function $f(z)$ is said to be of index q if*

$$\rho(q) = \limsup_{r\to\infty} \frac{\log^{[q]} M(r, f)}{\log r}$$

with $\rho(q - 1) = \infty$ and $\rho(q) < \infty$.

$\rho(q)$ is called the rate of growth of $f(z)$ of index q.

Juneja, Kapoor and Bajpai [5] introduced (p, q)th order of a non constant entire function as:

$$\rho(p, q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r}$$

where p & q are integers with $p \geq q \geq 0$.

Clearly, $\rho(q) = \rho(q, 1)$.

2. Lemmas

Here, we state some lemmas which are essential for the presentation of the work.

Lemma 2.1. [1] *If $g(z)$ is analytic in $|z| \leq r$ and $|g(z)| < K$ for $|z| = r$, then*

$$|g^{(n)}(0)| \leq \frac{K.n!}{r^n}\ ,\ n = 0, 1, 2, \ldots\ .$$

Lemma 2.1 is known as Cauchy's inequality.

Lemma 2.2. [1] *If $f_1(z)$ and $f_1(z)$ are analytic within and on a simple closed curve Γ with $|f_2(z)| < |f_1(z)|$ on Γ, then $f_1(z) + f_2(z)$ and $f_1(z)$ have the same number of zeros inside Γ.*

Lemma 2.2 is called Rouche's theorem.

Lemma 2.3. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a transcendental entire function of finite order ρ. Then for a positive integer $N_\rho > \rho$,*

$$|a_n| < (e^{\frac{1}{N_\rho}})^n \text{ for } n \geq N_\rho.$$

Proof. For all sufficiently large values of r, it follows from Definition 1 that

$$M(r, f) < e^{r^{N_\rho}}.$$

Hence for all sufficiently large values of r, we get by Lemma 2.1 that

$$|a_n| < \frac{e^{r^{N_\rho}}}{r^n}\ ,\ n = 0, 1, 2, \ldots\ .$$

Let $h(r) = \frac{e^{r^{N_\rho}}}{r^n}$. Then, $h'(r) = \frac{e^{r^{N_\rho}}}{r^{n+1}}(N_\rho r^{N_\rho} - n)$.
Now, $h'(r) = 0$ if and only if $(N_\rho r^{N_\rho} - n) = 0$.
Let r_1 be the unique positive root of this equation. Then, we have $r_1 \geq 1$ if $n \geq N_\rho$.
Clearly, $h(r)$ is minimum at $r = r_1$ and $\min\limits_{r>0} h(r) = \frac{e^{r_1^{N_\rho}}}{r_1^n} \leq (e^{\frac{1}{N_\rho}})^n$.
Hence,

$$|a_n| < (e^{\frac{1}{N_\rho}})^n \text{ for } n \geq N_\rho.$$

This completes the proof of the lemma.

Lemma 2.4. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a transcendental entire function of the rate of growth $\rho(q)$ of index $q(\geq 3)$. Then for a positive integer $N_{\rho(q)} > \rho(q)$,*

$$|a_n| < (e^{\frac{1}{N_{\rho(q)}}})^n \textit{ for } n \geq N_{\rho(q)} E_{[q-2]}(1)$$

where $E_{[m]}(x) = \prod_{i=0}^{m} \exp^{[i]} x$.

Proof. From Definition 3, we obtain for all sufficiently large values of r that

$$M(r, f) < \exp^{[q-1]} r^{N_{\rho(q)}}.$$

Therefore in view of Lemma 2.1, it follows for all sufficiently large values of r that

$$|a_n| < \frac{\exp^{[q-1]} r^{N_{\rho(q)}}}{r^n} \ , \ n = 0, 1, 2, ... \ .$$

Let $h(r) = \frac{\exp^{[q-1]} r^{N_{\rho(q)}}}{r^n}$.

Then,

$$\begin{aligned} h'(r) &= \frac{\exp^{[q-1]} r^{N_{\rho(q)}}}{r^{n+1}} \left\{ N_{\rho(q)} r^{N_{\rho(q)}} (\exp^{[q-2]} r^{N_{\rho(q)}}).(\exp^{[q-3]} r^{N_{\rho(q)}})...(\exp r) - n \right\} \\ &= \frac{\exp^{[q-1]} r^{N_{\rho(q)}}}{r^{n+1}} \left\{ N_{\rho(q)} E_{[q-2]}(r^{N_{\rho(q)}}) - n \right\}. \end{aligned}$$

Now, $h'(r) = 0$ gives

$$N_{\rho(q)} E_{[q-2]}(r^{N_{\rho(q)}}) - n = 0.$$

Let t_1 be the unique positive root of this equation. If $n \geq N_{\rho(q)} E_{[q-2]}(1)$, we get that $t_1 \geq 1$.

Hence,

$$\exp^{[q-2]} t_1^{N_{\rho(q)}} = \frac{n}{N_{\rho(q)} E_{[q-3]}(t_1^{N_{\rho(q)}})} \leq \frac{n}{N_{\rho(q)}}.$$

Clearly, $h(r)$ attains its minimum value at $r = t_1$.

Now,

$$\begin{aligned} \min_{r>0} h(r) &= \frac{\exp^{[q-1]} t_1^{N_{\rho(q)}}}{t_1^n} \\ &= \frac{\exp(\exp^{[q-2]} t_1^{N_{\rho(q)}})}{t_1^n} \\ &\leq (e^{\frac{1}{N_{\rho(q)}}})^n. \end{aligned}$$

Hence,

$$|a_n| < (e^{\frac{1}{N_{\rho(q)}}})^n \text{ for } n \geq N_{\rho(q)} E_{[q-2]}(1).$$

Thus the lemma is proved.

3. Theorems

The main results of this paper are being presented here.

Theorem 3.1. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a transcendental entire function of finite order ρ with $f(0) = a_0 \neq 0$. Also, let $N_\rho > \rho$ be the least positive integer such that $a_{N_\rho - 1} \neq 0$ and $|a_{N_\rho-1}| < e^{\frac{N_\rho-1}{N_\rho}}$. Then no zeros of $f(z)$ lie in*

$$|z| < \min\left\{t_0, \frac{1}{e^{\frac{1}{N_\rho}}}\right\}$$

where t_0 is the least positive root of the equation $|a_0| - (|a_1| + e^{\frac{1}{N_\rho}}|a_0|)t - (|a_2| - e^{\frac{1}{N_\rho}}|a_1|)t^2 - ... - (|a_{N_\rho-1}| - e^{\frac{1}{N_\rho}}|a_{N_\rho-2}|)t^{N_\rho-1} - (e - e^{\frac{1}{N_\rho}}|a_{N_\rho-1}|)t^{N_\rho} = 0$.

Proof. For the least positive integer $N_\rho > \rho$, it follows by Lemma 2.3 that

$$\begin{aligned}
|f(z)| =&|a_0 + a_1 z + ... + a_{N_\rho-1} z^{N_\rho-1} + \sum_{j=N_\rho}^{\infty} a_j z^j| \\
\geq&|a_0| - (|a_1 z + ... + a_{N_\rho-1} z^{N_\rho-1} + \sum_{j=N_\rho}^{\infty} a_j z^j|) \\
\geq&|a_0| - |a_1||z| - ... - |a_{N_\rho-1}||z|^{N_\rho-1} - \sum_{j=N_\rho}^{\infty} |a_j||z|^j \\
\geq&|a_0| - |a_1||z| - ... - |a_{N_\rho-1}||z|^{N_\rho-1} - \sum_{j=N_\rho}^{\infty} (e^{\frac{1}{N_\rho}})^j |z|^j.
\end{aligned}$$

Now,

$$\sum_{j=N_\rho}^{\infty} (e^{\frac{1}{N_\rho}})^j |z|^j = \sum_{j=N_\rho}^{\infty} (e^{\frac{1}{N_\rho}}|z|)^j = \frac{e|z|^{N_\rho}}{1 - e^{\frac{1}{N_\rho}}|z|} \text{ if } |z| < \frac{1}{e^{\frac{1}{N_\rho}}}.$$

Hence for $N_\rho > \rho$ and $|z| < \frac{1}{e^{\frac{1}{N_\rho}}}$, we get that

$$|f(z)| \geq |a_0| - |a_1||z| - ... - |a_{N_\rho-1}||z|^{N_\rho-1} - \frac{e|z|^{N_\rho}}{1 - e^{\frac{1}{N_\rho}}|z|}$$

$$= \frac{\left[\begin{array}{r} |a_0| - (|a_1| + e^{\frac{1}{N_\rho}}|a_0|)|z| - (|a_2| - e^{\frac{1}{N_\rho}}|a_1|)|z|^2 - ... - (|a_{N_\rho-1}| - e^{\frac{1}{N_\rho}} \\ |a_{N_\rho-2}|)|z|^{N_\rho-1} - (e - e^{\frac{1}{N_\rho}}|a_{N_\rho-1}|)|z|^{N_\rho} \end{array}\right]}{1 - e^{\frac{1}{N_\rho}}|z|}.$$

Let t_0 be the least positive root of the equation
$g(t) = |a_0| - (|a_1| + e^{\frac{1}{N_\rho}}|a_0|)t - (|a_2| - e^{\frac{1}{N_\rho}}|a_1|)t^2 - ... - (|a_{N_\rho-1}| - e^{\frac{1}{N_\rho}}|a_{N_\rho-2}|)t^{N_\rho-1} - (e - e^{\frac{1}{N_\rho}}|a_{N_\rho-1}|)t^{N_\rho} = 0.$
Then $g(t) > 0$ if $t < t_0$, otherwise there will be another positive root.
Consequently, $|f(z)| > 0$ if $|z| < \min\left\{t_0, \frac{1}{e^{\frac{1}{N_\rho}}}\right\}$.
This proves the theorem.

Remark 3.1. *The following example justifies the validity of Theorem 3.1.*

Example 3.1. Let $f(z) = e^{z^2} + \frac{1}{2}\sin z + 1$.
Then, the Taylor's series expansion of $f(z)$ is

$$f(z) = 2 + \frac{z}{2} + z^2 - \frac{z^3}{12} + ...$$

Here, $\rho = 2,\ a_0 = 2,\ a_1 = \frac{1}{2}$ & $a_2 = 1$.
Now, taking $N_\rho = 3$, the equation of Theorem 3.1 reduces to

$$2 - \left(\frac{1}{2} + 2e^{\frac{1}{3}}\right)t - \left(1 - \frac{1}{2}e^{\frac{1}{3}}\right)t^2 - \left(e - e^{\frac{1}{3}}\right)t^3 = 0$$

and the least positive root $t_0 \approx 0.52$.
Hence by Theorem 3.1, no zeros of $f(z)$ lie in

$$|z| < 0.52 .$$

Remark 3.2. *Taking $f_1(z) = e^{z^2} + 1,\ f_2(z) = \frac{1}{2}\sin z$ and using Lemma 2.2, it can be easily verified that $e^{z^2} + \frac{1}{2}\sin z + 1$ does not vanishes in $|z| < 0.52$ as $f_1(z) = 0$ for $z^2 = (2n+1)\pi i,\ n = 0,\ \pm 1,\ \pm 2...$.*

Theorem 3.2. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a transcendental entire function of the rate of growth $\rho(q)$ of index $q(\geq 3)$ with $a_0 \neq 0$. Also, let $N_q < N_{\rho(q)}E_{[q-2]}(1)$ be the greatest positive integer for the positive integer $N_{\rho(q)} > \rho(q)$ such that $a_{N_q} \neq 0$ with $|a_{N_q}| < e^{E_{[q-2]}(1)}$. Then zeros of $f(z)$ do not lie in*

$$|z| < \min\left\{t_0', \frac{1}{e^{\frac{1}{N_q}}}\right\}$$

where t_0' is the least positive root of the equation.
$|a_0| - (|a_1| + e^{\frac{1}{N_{\rho(q)}}}|a_0|)t - (|a_2| - e^{\frac{1}{N_{\rho(q)}}}|a_1|)t^2 - ... - (|a_{N_q}| - e^{\frac{1}{N_{\rho(q)}}}|a_{N_q-1}|)t^{N_q} -$

$e^{\frac{1}{N_{\rho(q)}}}(e^{E_{[q-2]}(1)}-|a_{N_q}|)t^{N_q+1}=0.$

Proof. For the greatest positive integer $N_q < N_{\rho(q)}E_{[q-2]}(1)$ with $N_{\rho(q)} > \rho(q)$, we get by Lemma 2.4 that

$$\begin{aligned}
|f(z)| =&|a_0 + a_1 z + ... + a_{N_q} z^{N_q} + \sum_{j=N_q+1}^{\infty} a_j z^j| \\
\geq&|a_0| - (|a_1 z + ... + a_{N_q} z^{N_q} + \sum_{j=N_q+1}^{\infty} a_j z^j|) \\
\geq&|a_0| - |a_1||z| - ... - |a_{N_q}||z|^{N_q} - \sum_{j=N_q+1}^{\infty} |a_j||z|^j \\
\geq&|a_0| - |a_1||z| - ... - |a_{N_q}||z|^{N_q} - \sum_{j=N_q+1}^{\infty} (e^{\frac{1}{N_{\rho(q)}}})^j |z|^j.
\end{aligned}$$

Now, we obtain for $|z| < \frac{1}{e^{\frac{1}{N_{\rho(q)}}}}$ that

$$\begin{aligned}
\sum_{j=N_q+1}^{\infty} (e^{\frac{1}{N_{\rho(q)}}})^j |z|^j =& \frac{e^{\frac{N_q+1}{N_{\rho(q)}}} |z|^{N_q+1}}{1 - e^{\frac{1}{N_{\rho(q)}}} |z|} \\
<& \frac{e^{E_{[q-2]}(1)+\frac{1}{N_{\rho(q)}}} |z|^{N_q+1}}{1 - e^{\frac{1}{N_{\rho(q)}}} |z|}.
\end{aligned}$$

Therefore, it follows for $|z| < \frac{1}{e^{\frac{1}{N_{\rho(q)}}}}$ that

$$\begin{aligned}
|f(z)| >&|a_0| - |a_1||z| - ... - |a_{N_q}||z|^{N_q} - \frac{e^{E_{[q-2]}(1)+\frac{1}{N_{\rho(q)}}} |z|^{N_q+1}}{1 - e^{\frac{1}{N_{\rho(q)}}} |z|} \\
=& \frac{\left[\begin{array}{r} |a_0| - (|a_1| + e^{\frac{1}{N_{\rho(q)}}}|a_0|)|z| - (|a_2| - e^{\frac{1}{N_{\rho(q)}}}|a_1|)|z|^2 - ... - (|a_{N_q}| - e^{\frac{1}{N_{\rho(q)}}} \\ |a_{N_q-1}|)|z|^{N_q} - e^{\frac{1}{N_{\rho(q)}}}(e^{E_{[q-2]}(1)} - |a_{N_q}|)|z|^{N_q+1} \end{array}\right]}{1 - e^{\frac{1}{N_{\rho(q)}}} |z|}.
\end{aligned}$$

Let

$h(t) = |a_0| - (|a_1| + e^{\frac{1}{N_{\rho(q)}}}|a_0|)t - (|a_2| - e^{\frac{1}{N_{\rho(q)}}}|a_1|)t^2 - ... - (|a_{N_q}| - e^{\frac{1}{N_{\rho(q)}}}|a_{N_q-1}|)t^{N_q} -$

$e^{\frac{1}{N_{\rho(q)}}}(e^{E_{[q-2]}(1)} - |a_{N_q}|)t^{N_q+1}$.
Clearly, if t'_0 is the least positive root of $h(t) = 0$, then $h(t) > 0$ for $t < t'_0$.
Hence $|f(z)| > 0$ if $|z| < \min\left\{t'_0, \frac{1}{e^{\frac{1}{N_q}}}\right\}$
Thus the theorem is established.

Remark 3.3. *The following example ensures the validity of Theorem 3.2.*

Example 3.2. Taking $f(z) = e^{\cos z} + \sin z - 1$, we have the Taylor's series expansion of $f(z)$ as

$$f(z) = e - 1 + z - \frac{e}{2}z^2 - \frac{1}{6}z^3 + \frac{e}{6}z^4 + \frac{1}{120}z^5 - \dots .$$

Here, $\rho(q) = 1$ for $q = 3$ and $N_q = 5$ for $N_{\rho(q)} = 2$.
Now, the equation of Theorem 3.2 becomes
$e - 1 - \left\{1 + (e-1)e^{\frac{1}{2}}\right\}t - \left\{\frac{1}{2}e - e^{\frac{1}{2}}\right\}t^2 - \left\{\frac{1}{6} - \frac{1}{2}e^{\frac{3}{2}}\right\}t^3 - \frac{1}{6}\left\{e - e^{\frac{1}{2}}\right\}t^4$
$- \left\{\frac{1}{120} - \frac{1}{6}e^{\frac{1}{2}}\right\}t^5 - e^{\frac{1}{2}}\left\{e^e - \frac{1}{120}\right\}t^6 = 0$
and the least positive root $t'_0 \approx 0.46$.
Hence by Theorem 3.2, $f(z)$ does not vanish in

$$|z| < 0.46\ .$$

Remark 3.4. *If $f_1(z) = e^{\cos z}$, $f_2(z) = \sin z + 1$, then it is easily verifiable by Lemma 2.2 that $e^{\cos z} + \sin z - 1$ does not vanish in $|z| < 0.46$.*

4. Conclusion and Future Prospect

A zero free region about the origin in $\mathbb{C}$ has been established in the paper for a transcendental entire function. To get a region about an arbitrary point having no zeros for a transcendental entire function, one can use the theorems of this paper considering Taylor's series expansion about that point. In the line of the works as carried out in the paper it may be thought of proving the results in case of entire functions of several complex variables.

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