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NEW GENERALIZED SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH FRACTIONAL DIFFERENTIAL OPERATOR

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Abstract: Motivated by work of Srivastava-Owa, we define new generalized subclasses of bi-univalent functions defined in the open unit disk which are associated with fractional differential operator. Furthermore, we have obtained estimates of the coefficients $|a_2|$ and $|a_3|$ for the functions belonging to these subclasses.

Keywords and Phrases: Univalent function, Srivastava-Owa fractional operators, bi-univalent function, coefficient estimates.

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1. Introduction

The study of fractional operators (integral and differential) plays a vital and essential role in mathematical analysis. The fractional calculus operators and their various other generalizations have fruitfully been applied in obtaining various things like coefficients estimates, characterization properties and distortion inequalities for various subclasses of analytic and univalent functions. Srivastava and Owa [12] gave definitions for fractional operators (derivative and integral) in the complex Z-plane $\mathbb{C}$ as follows.

Definition 1. [12] *The fractional derivative of order α is defined, for a function f by*

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} d\zeta; 0 \leq \alpha < 1,$$

where the function f is analytic in simply-connected region of the complex Z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2. [12] *The fractional integral of order α is defined, for a function f by*

$$I_z^{\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \alpha > 0,$$

where the function f is analytic in simply-connected region of the complex Z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

From Definitions 1 and 2, we have

$$D_z^{\alpha} z^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \mu > -1; 0 \leq \alpha < 1$$

and

$$I_z^{\alpha} z^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \mu > -1; 0 \leq \alpha < 1.$$

2. Preliminaries

Let $\mathcal{J}$ denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2.1}$$

which are analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let D denote the subclass of $\mathcal{J}$ which consists of functions of the form (2.1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in U.

We know that every function $f \in D$ has an inverse f^{-1}, defined by

$f^{-1}(f(z)) = z, (z \in U)$ and $f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... \tag{2.2}$$

Definition 3. [4] *A function f in D is said to be bi-univalent in U if both f and f^{-1} are univalent in U.*

Let Σ denote the class of bi-univalent functions in U given by(2.1). In [6] Lewin investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$. Subsequently Brannan and Clunie [1] conjectured that $|a_2| \leq \sqrt{2}$. Also Netanyahu [8] showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the Taylor-Maclaurin coefficients $|a_n|$ $(n \geq 3;\ n \in \mathbb{N})$ is still an open problem. Several authors investigated various subclasses of the class Σ and obtained estimates for their coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses (see [2, 3, 5, 7, 9, 10]).

In [11] Srivastava, Mishra and Gochhayat introduced the following two subclasses of the bi-univalent functions class Σ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

Definition 4. [11] *A function f given by (2.1) is said to be in the class $H_{\Sigma}^{(\alpha)}$ $(0 < \alpha \leq 1)$, if the following conditions are satisfied:*

$$f \in \Sigma, \qquad |arg(f'(z))| < \frac{\alpha\pi}{2} \qquad (z \in U)$$

$$and \qquad |arg(g'(w))| < \frac{\alpha\pi}{2} \qquad (w \in U)$$

where the function g is defined by (2.2).

Definition 5. [11] *A function f given by (2.1) is said to be in the class $H_{\Sigma}(\beta)$ $(0 \leq \beta < 1)$, if the following conditions are satisfied:*

$$f \in \Sigma, \qquad \Re\left(f'(z)\right) > \beta \qquad (z \in U)$$

$$and \qquad \Re\left(g'(w)\right) > \beta \qquad (w \in U)$$

where the function g is defined by (2.2).

Motivated by above definitions, we now give two definitions.

Definition 6. *A function f given by (2.1) is said to be in the class $AH_{\Sigma}^{D_z^{\lambda}}(\alpha)$ $(0 < \alpha \leq 1)$, if the following conditions are satisfied:*

$$f \in \Sigma, \qquad \left|arg\left(\frac{\Gamma(2-\lambda)D_z^{\lambda}f(z)}{z^{1-\lambda}}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in U) \tag{2.3}$$

and

$$\left|arg\left(\frac{\Gamma(2-\lambda)D_w^{\lambda}g(w)}{w^{1-\lambda}}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in U) \tag{2.4}$$

where the function g is defined by (2.2).

Definition 7. *A function f given by (2.1) is said to be in the class $RH_{\Sigma}^{D_z^{\lambda}}(\beta)$ $(0 \leq \beta < 1)$, if the following conditions are satisfied:*

$$f \in \Sigma, \quad \Re\left(\frac{\Gamma(2-\lambda)D_z^{\lambda}f(z)}{z^{1-\lambda}}\right) > \beta \qquad (z \in U) \tag{2.5}$$

and

$$\Re\left(\frac{\Gamma(2-\lambda)D_w^{\lambda}g(w)}{w^{1-\lambda}}\right) > \beta \qquad (w \in U) \tag{2.6}$$

where the function g is defined by (2.2).

Remark 1. *For $\lambda = 1$, the classes $AH_{\Sigma}^{D_z^{\lambda}}(\alpha)$ and $RH_{\Sigma}^{D_z^{\lambda}}(\beta)$ reduces to the classes $H_{\Sigma}^{(\alpha)}, H_{\Sigma}(\beta)$ respectively which was introduced by Srivastava et al.* [11].

3. Coefficient bounds for the function class $AH_{\Sigma}^{D_z^{\lambda}}(\alpha)$

We begin with the following useful lemma.

Lemma 1. [10] *Let $h \in P$, the family of all functions h analytic in U for which $\Re\{h(z)\} > 0$ and have the form $h(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$ for $z \in U$. Then $|p_n| \leq 2$ for each $n \in \mathbb{N}$.*

We first state and prove the following result.

Theorem 1. *Let f be in the function class $AH_{\Sigma}^{D_z^{\lambda}}(\alpha)$, then*

$$|a_2| \leq 2\alpha(2-\lambda)\sqrt{\frac{3-\lambda}{12\alpha(2-\lambda) - 4(\alpha-1)(3-\lambda)}} \tag{3.7}$$

and

$$|a_3| \leq \frac{\alpha(2-\lambda)(3-\lambda)}{3} + (2-\lambda)^2\alpha^2. \tag{3.8}$$

Proof. We can write the argument in (2.3) and (2.4) as

$$\left(\frac{\Gamma(2-\lambda)D_z^{\lambda}f(z)}{z^{1-\lambda}}\right) = [p(z)]^{\alpha} \tag{3.9}$$

and

$$\left(\frac{\Gamma(2-\lambda)D_w^{\lambda}g(w)}{w^{1-\lambda}}\right) = [q(w)]^{\alpha} \tag{3.10}$$

respectively, where p, $q \in P$ have the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \tag{3.11}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots \ . \tag{3.12}$$

Equating the coefficients in (3.9) and (3.10), we get

$$\frac{2a_2}{(2-\lambda)} = \alpha p_1, \tag{3.13}$$

$$\frac{6}{(3-\lambda)(2-\lambda)} a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \tag{3.14}$$

$$\frac{-2a_2}{(2-\lambda)} = \alpha q_1, \tag{3.15}$$

$$\frac{6}{(3-\lambda)(2-\lambda)} (2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2. \tag{3.16}$$

From (3.13) and (3.15), we get

$$p_1 = -q_1 \tag{3.17}$$

and

$$\frac{8a_2^2}{(2-\lambda)^2} = \alpha^2 (p_1^2 + q_1^2)\,. \tag{3.18}$$

Adding (3.14) and (3.16), and using (3.18), we obtain

$$a_2^2 = \frac{\alpha^2 (3-\lambda)(2-\lambda)^2 (p_2 + q_2)}{12\alpha(2-\lambda) - 4(\alpha-1)(3-\lambda)}. \tag{3.19}$$

Using Lemma 1, we get

$$|a_2| \leq 2\alpha(2-\lambda)\sqrt{\frac{3-\lambda}{12\alpha(2-\lambda) - 4(\alpha-1)(3-\lambda)}}. \tag{3.20}$$

Next in order to find the bound on $|a_3|$, by subtracting (3.16) from (3.14) and then using (3.17), we obtain

$$a_3 - a_2^2 = \frac{\alpha(p_2 - q_2)(2-\lambda)(3-\lambda)}{12}. \tag{3.21}$$

By (3.16), we get

$$a_3 = \frac{\alpha(p_2 - q_2)(2-\lambda)(3-\lambda)}{12} + \frac{\alpha^2 (2-\lambda)^2 (p_1^2 + q_1^2)}{8}. \tag{3.22}$$

Using Lemma 1, we get

$$|a_3| \leq \frac{\alpha(2-\lambda)(3-\lambda)}{3} + (2-\lambda)^2\alpha^2. \tag{3.23}$$

This completes the proof of the theorem.
Putting λ =1 in Theorem 1, we have

Corollary 1. *Let f be in the function class $AH_{\Sigma}^{D_z^1}(\alpha)$, then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+2}}$$

and

$$|a_3| \leq \frac{\alpha(3\alpha+2)}{3}.$$

4. Coefficient bounds for the function class $RH_{\Sigma}^{D_z^{\lambda}}(\beta)$

We now state and prove the following result.

Theorem 2. *Let f be in the function class $RH_{\Sigma}^{D_z^{\lambda}}(\beta)$, then*

$$|a_2| \leq \sqrt{\frac{(1-\beta)(2-\lambda)(3-\lambda)}{3}} \tag{4.24}$$

and

$$|a_3| \leq \frac{(2-\lambda)(1-\beta)}{3}\left((3-\lambda)+3(2-\lambda)(1-\beta)\right). \tag{4.25}$$

Proof. First of all the argument inequalities in (2.5) and (2.6) can be written in their equivalent forms as,

$$\left(\frac{\Gamma(2-\lambda)D_z^{\lambda}f(z)}{z^{1-\lambda}}\right) = \beta + (1-\beta)p(z) \tag{4.26}$$

and

$$\left(\frac{\Gamma(2-\lambda)D_w^{\lambda}g(w)}{w^{1-\lambda}}\right) = \beta + (1-\beta)q(z) \tag{4.27}$$

respectively, where $p(z), q(z)$ are given by (3.11) and (3.12). Now equating the coefficients in (4.26) and (4.27), we get

$$\frac{2a_2}{(2-\lambda)} = (1-\beta)p_1, \tag{4.28}$$

$$\frac{6a_3}{(3-\lambda)(2-\lambda)} = (1-\beta)p_2, \tag{4.29}$$

$$\frac{-2a_2}{(2-\lambda)} = (1-\beta)q_1 \tag{4.30}$$

and

$$\frac{6(2a_2^2-a_3)}{(3-\lambda)(2-\lambda)} = (1-\beta)q_2. \tag{4.31}$$

Using (4.28) and (4.30), we obtain

$$p_1 = -q_1 \tag{4.32}$$

and

$$(1-\beta)^2(p_1^2+q_1^2) = \frac{8a_2^2}{(2-\lambda)^2}. \tag{4.33}$$

By (4.29) and (4.31), we get

$$\frac{12a_2^2}{(3-\lambda)(2-\lambda)} = (1-\beta)(p_2+q_2) \tag{4.34}$$

or equivalently,

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)(3-\lambda)(2-\lambda)}{12}. \tag{4.35}$$

Applying Lemma 1, we get

$$|a_2| \leq \sqrt{\frac{(1-\beta)(2-\lambda)(3-\lambda)}{3}} \tag{4.36}$$

which is the bound on $|a_2|$ as given in (4.24). Next in order to find the bound on $|a_3|$, subtracting (4.31) from (4.29), we get

$$\frac{12}{(3-\lambda)(2-\lambda)}(a_3-a_2^2) = (1-\beta)(p_2-q_2) \tag{4.37}$$

or equivalently,

$$a_3 = a_2^2 + \frac{(3-\lambda)(2-\lambda)}{12}(1-\beta)(p_2-q_2). \tag{4.38}$$

Using (4.33), we obtain

$$a_3 = \frac{(1-\beta)^2(2-\lambda)^2(p_1^2+q_1^2)}{8} + \frac{(3-\lambda)(2-\lambda)}{12}(1-\beta)(p_2-q_2). \tag{4.39}$$

Using Lemma 1, we get

$$|a_3| \leq \frac{(2-\lambda)(1-\beta)}{3}\left[(3-\lambda)+3(2-\lambda)(1-\beta)\right]. \tag{4.40}$$

This completes the proof of the theorem.

Putting λ =1 in Theorem 2, we have

Corollary 2. *Let f be in the function class $RH_{\Sigma}^{D_z^1}(\beta)$, then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}}$$

and

$$|a_3| \leq \frac{(1-\beta)(5-\beta)}{3}.$$

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References

[1] Brannan D. A., Clunie J., Aspects of Contemporary Complex Analysis, Academic Press, New York, London, (1980).

[2] Brannan D. A. and Taha T. S., On some classes of bi-univalent functions, Studia Universitatis Babes-Bolyai Mathematica, 31, no. 2 (1986), 70-77.

[3] Bulut S., Coefficient estimates for a class of analytic and bi-univalent function, Novi. Sad. J. Math., 43, no. 2 (2013), 59-65.

[4] Duren P. L., Univalent Functions, Grundlehren der Mathematischen Wissenschaften, New York, (1983).

[5] Girgaonkar V. B., Joshi S. B., Yadav P. P., Certain special subclasses of analytic Function associated with bi-univalent Functions, Palestine J. Math, 6, no. 1 (2017), 1–7.

[6] Lewin M., On a coefficient problem for bi-univalent function, Proc. Amer. Math. Soc., 18 (1967), 63-68.

[7] Naik U. H. and Patil A. B., On initial coefficient inequalities for certain new subclasses of bi-univalent functions, J. Egypt. Math. Soc., 25 (2017), 291-293.

[8] Netanyahu E., The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech. Anal., 32 (1969), 100-112.

[9] Pommerenke Ch., Univalent Functions, Vandenhoeck and Rupercht, Göttingen, (1975).

[10] Srivastava H. M., Bansal D., Coefficient estimates for a subclass of analytic and bi-univalent functions, J. Egypt. Math. Soc., 23 (2014), 1-4.

[11] Srivastava H. M., Mishra A. K., Gochhayat P., Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23 (2010), 1188-1192.

[12] Srivastava H. M., Owa S., Univalent Functions, Fractional Calculus and their Applications, Halsted Press, John Wiley and Sons, New York Chichester, Brisban and Toronto, (1989).