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SOME DEFINITE INTEGRAL FORMULAE INVOLVING BESSEL FUNCTION, LOG FUNCTION AND HYPERGEOMETRIC FUNCTION

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Abstract: In this paper, we aim to evaluate some definite integrals involving Bessel function and log function in terms of generalized hypergeometric functions.

Keywords and Phrases: Bessel Function, Hypergeometric Function, Pochhammer symbol.

2020 Mathematics Subject Classification: 33B30, 33C10,33C20.

1. Introduction

The following definite integral formulas are recalled (see, e.g., [3, p. 204, Entries 4.7.7-20 and 21]):

$$\int_0^1 x\log x\ J_0^2(ax)dx = -\frac{1}{2}\big[J_0^2(a)+J_1^2(a)-\frac{1}{a}J_0(a)J_1(a)\big]. \tag{1.1}$$

$$\int_0^1 x\log x\ J_1^2(ax)dx = \frac{1}{2a^2}\big[1-(a^2+1)J_0^2(a)+a\ J_0(a)J_1(a)-a^2\ J_1^2(a)\big]. \tag{1.2}$$

Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of Bessel's differential equation that are finite at the origin ($x=0$) for integer or positive α, and diverge as x approaches zero for negative non-integer α (See[11]). It is possible to define the function by its Taylor series expansion around $x=0$.

$$J_\alpha(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\ \Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha} \tag{1.3}$$

where $\Gamma(z)$ is the familiar gamma function and the multi-valued function $(x/2)^{2m+\alpha}$ is assumed to be taken as one of its branches, say, its principal branch ($|arg(x/2)| < \pi$).

The Bessel functions are valid even for complex arguments x, and an important special case is that of a purely imaginary argument (See[11]). In this case, the solutions to the Bessel equation are called the modified Bessel functions (or occasionally the hyperbolic Bessel functions) of the first and second kind. The first kind of modified Bessel function is defined as

$$I_\alpha(x) = \iota^{-\alpha} J_\alpha(\iota x) = \sum_{m=0}^{\infty} \frac{1}{m!\ \Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{1.4}$$

A generalized hypergeometric function ${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(k+b_2)...(k+b_q)(k+1)}\ z. \tag{1.5}$$

Where $k+1$ in the denominator is present for historical reasons of notation [Koepf p. 12 (2.9)], and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{ccc} a_1, a_2, \cdots, a_p & ; & \\ & & z \\ b_1, b_2, \cdots, b_q & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} \tag{1.6}$$

where the parameters $b_1, b_2, \cdots, b_q$ are complex numbers excluding non-positive integers.
Here $(\lambda)_\nu$ denotes the Pochhammer symbol which is defined (for $\lambda, \nu \in \mathbb{C}$) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \\ 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \end{cases}$$

it being understood conventionally that$(0)_0 := 1$ and $\mathbb{C}, \mathbb{N}$ the sets of complex numbers and positive integers, respectively. The ${}_pF_q$ series converges for all finite z if $p \le q$, converges for $|\ z\ | < 1$ if $p = q+1$, diverges for all z , $z \neq 0$ if $p > q+1$ [Luke p.156 (3)].

The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

In mathematics, the falling factorial or Pochhammer symbol (sometimes called the descending factorial,falling sequential product, or lower factorial) is defined as the polynomial [Steffensen p.8]

$$(x)_n = x(x-1)(x-2).......(x-n+1) = \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k) \qquad (1.7)$$

2. Main Formulae of the Integration

$$\int_0^1 x^2 \log x\ J_0^2(ax)dx = \frac{1}{18}\Big[{}_3F_4\Big(\frac{3}{2},\frac{3}{2},\frac{3}{2};1,1,\frac{5}{2},\frac{5}{2};-a^2\Big) - 3\ {}_2F_3\Big(\frac{1}{2},\frac{3}{2};1,1,\frac{5}{2};-a^2\Big)\Big]. \qquad (2.1)$$

$$\int_0^1 x^3 \log x\ J_0^2(ax)dx = -\frac{1}{16}\ {}_3F_4\Big(\frac{1}{2},2,2;1,1,3,3;-a^2\Big). \qquad (2.2)$$

$$\int_0^1 x^4 \log x\ J_0^2(ax)dx = -\frac{1}{25}\ {}_3F_4\Big(\frac{1}{2},\frac{5}{2},\frac{5}{2};1,1,\frac{7}{2},\frac{7}{2};-a^2\Big). \qquad (2.3)$$

$$\int_0^1 x^5 \log x\ J_0^2(ax)dx = -\frac{1}{36}\ {}_3F_4\Big(\frac{1}{2},3,3;1,1,4,4;-a^2\Big). \qquad (2.4)$$

$$\int_0^1 x^6 \log x\ J_0^2(ax)dx = -\frac{1}{49}\ {}_3F_4\Big(\frac{1}{2},\frac{7}{2},\frac{7}{2};1,1,\frac{9}{2},\frac{9}{2};-a^2\Big). \qquad (2.5)$$

$$\int_0^1 x^8 \log x\ J_0^2(ax)dx = -\frac{1}{81}\ {}_3F_4\Big(\frac{1}{2},\frac{9}{2},\frac{9}{2};1,1,\frac{11}{2},\frac{11}{2};-a^2\Big). \qquad (2.6)$$

$$\int_0^1 x^{10} \log x\ J_0^2(ax)dx = -\frac{1}{121}\ {}_3F_4\Big(\frac{1}{2},\frac{11}{2},\frac{11}{2};1,1,\frac{13}{2},\frac{13}{2};-a^2\Big). \qquad (2.7)$$

$\int_0^1 x^2 \log x\ J_1^2(ax)dx$

$$= \frac{1}{200}a^2\Big[3\ {}_3F_4\Big(\frac{5}{2},\frac{5}{2},\frac{5}{2};2,3,\frac{7}{2},\frac{7}{2};-a^2\Big) - 5\ {}_2F_3\Big(\frac{3}{2},\frac{5}{2};2,3,\frac{7}{2};-a^2\Big)\Big]. \qquad (2.8)$$

$$\int_0^1 x^3 \log x\ J_1^2(ax)dx = \frac{1}{8}\Big[{}_1F_2\Big(\frac{1}{2};1,3;-a^2\Big) - {}_2F_3\Big(\frac{1}{2},2;1,3,3;-a^2\Big)\Big]. \qquad (2.9)$$

$$\int_0^1 x^4 \log x \; J_1^2(ax)dx = -\frac{1}{196}a^2 \;{}_3F_4\Big(\frac{3}{2},\frac{7}{2},\frac{7}{2};2,3,\frac{9}{2},\frac{9}{2};-a^2\Big). \tag{2.10}$$

$$\int_0^1 x^6 \log x \; J_1^2(ax)dx = -\frac{1}{324}a^2 \;{}_3F_4\Big(\frac{3}{2},\frac{9}{2},\frac{9}{2};2,3,\frac{11}{2},\frac{11}{2};-a^2\Big). \tag{2.11}$$

$$\int_0^1 x^8 \log x \; J_1^2(ax)dx = -\frac{1}{484}a^2 \;{}_3F_4\Big(\frac{3}{2},\frac{11}{2},\frac{11}{2};2,3,\frac{13}{2},\frac{13}{2};-a^2\Big). \tag{2.12}$$

$$\int_0^1 x^{10} \log x \; J_1^2(ax)dx = -\frac{1}{676}a^2 \;{}_3F_4\Big(\frac{3}{2},\frac{13}{2},\frac{13}{2};2,3,\frac{15}{2},\frac{15}{2};-a^2\Big). \tag{2.13}$$

$\int_0^1 x^2 \log x \; J_2^2(ax)dx$

$$= \frac{1}{6272}a^4\Big[5 \;{}_3F_4\Big(\frac{7}{2},\frac{7}{2},\frac{7}{2};3,5,\frac{9}{2},\frac{9}{2};-a^2\Big) - 7 \;{}_2F_3\Big(\frac{5}{2},\frac{7}{2};3,5,\frac{9}{2};-a^2\Big)\Big]. \tag{2.14}$$

$$\int_0^1 x^4 \log x \; J_2^2(ax)dx = -\frac{1}{5184}a^4 \;{}_3F_4\Big(\frac{5}{2},\frac{9}{2},\frac{9}{2};3,5,\frac{11}{2},\frac{11}{2};-a^2\Big). \tag{2.15}$$

$$\int_0^1 x^6 \log x \; J_2^2(ax)dx = -\frac{1}{7744}a^4 \;{}_3F_4\Big(\frac{5}{2},\frac{11}{2},\frac{11}{2};3,5,\frac{13}{2},\frac{13}{2};-a^2\Big). \tag{2.16}$$

$$\int_0^1 x^8 \log x \; J_2^2(ax)dx = -\frac{1}{10816}a^4 \;{}_3F_4\Big(\frac{5}{2},\frac{13}{2},\frac{13}{2};3,5,\frac{15}{2},\frac{15}{2};-a^2\Big). \tag{2.17}$$

$$\int_0^1 x^{10} \log x \; J_2^2(ax)dx = -\frac{1}{14400}a^4 \;{}_3F_4\Big(\frac{5}{2},\frac{15}{2},\frac{15}{2};3,5,\frac{17}{2},\frac{17}{2};-a^2\Big). \tag{2.18}$$

$\int_0^1 x^2 \log x \; J_3^2(ax)dx$

$$= \frac{1}{373248}a^6\Big[7 \;{}_3F_4\Big(\frac{9}{2},\frac{9}{2},\frac{9}{2};4,7,\frac{11}{2},\frac{11}{2};-a^2\Big) - 9 \;{}_2F_3\Big(\frac{7}{2},\frac{9}{2};4,7,\frac{11}{2};-a^2\Big)\Big]. \tag{2.19}$$

$$\int_0^1 x^4 \log x \; J_3^2(ax)dx = -\frac{1}{278784}a^6 \;{}_3F_4\Big(\frac{7}{2},\frac{11}{2},\frac{11}{2};4,7,\frac{13}{2},\frac{13}{2};-a^2\Big). \tag{2.20}$$

$$\int_0^1 x^6 \log x \; J_3^2(ax)dx = -\frac{1}{389376}a^6 \;{}_3F_4\Big(\frac{7}{2},\frac{13}{2},\frac{13}{2};4,7,\frac{15}{2},\frac{15}{2};-a^2\Big). \tag{2.21}$$

$$\int_0^1 x^8 \log x \; J_3^2(ax)dx = -\frac{1}{518400}a^6 \;{}_3F_4\Big(\frac{7}{2},\frac{15}{2},\frac{15}{2};4,7,\frac{17}{2},\frac{17}{2};-a^2\Big). \tag{2.22}$$

$$\int_0^1 x^{10} \log x \; J_3^2(ax)dx = -\frac{1}{665856}a^6 \;{}_3F_4\Big(\frac{7}{2},\frac{17}{2},\frac{17}{2};4,7,\frac{19}{2},\frac{19}{2};-a^2\Big). \tag{2.23}$$

$$\int_0^1 x^2 \log x\ J_4^2(ax)dx$$
$$= \frac{1}{35684352}a^8\Big[9\ {}_3F_4\Big(\frac{11}{2},\frac{11}{2},\frac{11}{2};5,9,\frac{13}{2},\frac{13}{2};-a^2\Big)-11\ {}_2F_3\Big(\frac{9}{2},\frac{11}{2};5,9,\frac{13}{2};-a^2\Big)\Big]. \tag{2.24}$$

$$\int_0^1 x^4 \log x\ J_4^2(ax)dx = -\frac{1}{24920064}a^8\ {}_3F_4\Big(\frac{9}{2},\frac{13}{2},\frac{13}{2};5,9,\frac{15}{2},\frac{15}{2};-a^2\Big). \tag{2.25}$$

$$\int_0^1 x^6 \log x\ J_4^2(ax)dx = -\frac{1}{33177600}a^8\ {}_3F_4\Big(\frac{9}{2},\frac{15}{2},\frac{15}{2};5,9,\frac{17}{2},\frac{17}{2};-a^2\Big). \tag{2.26}$$

$$\int_0^1 x^8 \log x\ J_4^2(ax)dx = -\frac{1}{42614784}a^8\ {}_3F_4\Big(\frac{9}{2},\frac{17}{2},\frac{17}{2};5,9,\frac{19}{2},\frac{19}{2};-a^2\Big). \tag{2.27}$$

$$\int_0^1 x^{10} \log x\ J_4^2(ax)dx = -\frac{1}{53231616}a^8\ {}_3F_4\Big(\frac{9}{2},\frac{19}{2},\frac{19}{2};5,9,\frac{21}{2},\frac{21}{2};-a^2\Big). \tag{2.28}$$

$$\int_0^1 x^2 \log x\ J_5^2(ax)dx = \frac{1}{4984012800}a^{10}\Big[11\ {}_3F_4\Big(\frac{13}{2},\frac{13}{2},\frac{13}{2};6,11,\frac{15}{2},\frac{15}{2};-a^2\Big)-$$
$$-13\ {}_2F_3\Big(\frac{11}{2},\frac{13}{2};6,11,\frac{15}{2};-a^2\Big)\Big]. \tag{2.29}$$

$$\int_0^1 x^4 \log x\ J_5^2(ax)dx = -\frac{1}{3317760000}a^{10}\ {}_3F_4\Big(\frac{11}{2},\frac{15}{2},\frac{15}{2};6,11,\frac{17}{2},\frac{17}{2};-a^2\Big). \tag{2.30}$$

$$\int_0^1 x^6 \log x\ J_5^2(ax)dx = -\frac{1}{4261478400}a^{10}\ {}_3F_4\Big(\frac{11}{2},\frac{17}{2},\frac{17}{2};6,11,\frac{19}{2},\frac{19}{2};-a^2\Big). \tag{2.31}$$

$$\int_0^1 x^8 \log x\ J_5^2(ax)dx = -\frac{1}{5323161600}a^{10}\ {}_3F_4\Big(\frac{11}{2},\frac{19}{2},\frac{19}{2};6,11,\frac{21}{2},\frac{21}{2};-a^2\Big). \tag{2.32}$$

$$\int_0^1 x^{10} \log x\ J_5^2(ax)dx = -\frac{1}{6502809600}a^{10}\ {}_3F_4\Big(\frac{11}{2},\frac{21}{2},\frac{21}{2};6,11,\frac{23}{2},\frac{23}{2};-a^2\Big). \tag{2.33}$$

$$\int_0^1 x^2 \log x\ J_6^2(ax)dx = \frac{1}{955514880000}a^{12}$$
$$\times\Big[11\ {}_3F_4\Big(\frac{15}{2},\frac{15}{2},\frac{15}{2};7,13,\frac{17}{2},\frac{17}{2};-a^2\Big)-15\ {}_2F_3\Big(\frac{15}{2},\frac{17}{2};7,13,\frac{19}{2};-a^2\Big)\Big]. \tag{2.34}$$

$$\int_0^1 x^4 \log x\ J_6^2(ax)dx = -\frac{1}{613652889600}a^{12}\ {}_3F_4\Big(\frac{13}{2},\frac{17}{2},\frac{17}{2};7,13,\frac{19}{2},\frac{19}{2};-a^2\Big). \tag{2.35}$$

$$\int_0^1 x^6 \log x \ J_6^2(ax)dx = -\frac{1}{766535270400} a^{12} \ {}_3F_4\Big(\frac{13}{2}, \frac{19}{2}, \frac{19}{2}; 7, 13, \frac{21}{2}, \frac{21}{2}; -a^2\Big). \tag{2.36}$$

$$\int_0^1 x^8 \log x \ J_6^2(ax)dx = -\frac{1}{936404582400} a^{12} \ {}_3F_4\Big(\frac{13}{2}, \frac{21}{2}, \frac{21}{2}; 7, 13, \frac{23}{2}, \frac{23}{2}; -a^2\Big). \tag{2.37}$$

$$\int_0^1 x^{10} \log x \ J_6^2(ax)dx = -\frac{1}{1123260825600} a^{12} \ {}_3F_4\Big(\frac{13}{2}, \frac{23}{2}, \frac{23}{2}; 7, 13, \frac{25}{2}, \frac{25}{2}; -a^2\Big). \tag{2.38}$$

$$\int_0^1 x^2 \log x \ J_7^2(ax)dx = \frac{1}{240551932723200} a^{14}$$

$$\times \Big[15 \ {}_3F_4\Big(\frac{17}{2}, \frac{17}{2}, \frac{17}{2}; 8, 15, \frac{19}{2}, \frac{19}{2}; -a^2\Big) - 17 \ {}_2F_3\Big(\frac{15}{2}, \frac{17}{2}; 8, 15, \frac{19}{2}; -a^2\Big)\Big]. \tag{2.39}$$

$$\int_0^1 x^4 \log x \ J_7^2(ax)dx = -\frac{1}{150240912998400} a^{14} \ {}_3F_4\Big(\frac{15}{2}, \frac{19}{2}, \frac{19}{2}; 8, 15, \frac{21}{2}, \frac{21}{2}; -a^2\Big). \tag{2.40}$$

$$\int_0^1 x^6 \log x \ J_7^2(ax)dx = -\frac{1}{183535298150400} a^{14} \ {}_3F_4\Big(\frac{15}{2}, \frac{21}{2}, \frac{21}{2}; 8, 15, \frac{23}{2}, \frac{23}{2}; -a^2\Big). \tag{2.41}$$

$$\int_0^1 x^8 \log x \ J_7^2(ax)dx = -\frac{1}{220159121817600} a^{14} \ {}_3F_4\Big(\frac{15}{2}, \frac{23}{2}, \frac{23}{2}; 8, 15, \frac{25}{2}, \frac{25}{2}; -a^2\Big). \tag{2.42}$$

$$\int_0^1 x^{10} \log x \ J_7^2(ax)dx = -\frac{1}{260112384000000} a^{14}$$

$$\times \ {}_3F_4\Big(\frac{15}{2}, \frac{25}{2}, \frac{25}{2}; 8, 15, \frac{27}{2}, \frac{27}{2}; -a^2\Big). \tag{2.43}$$

$$\int_0^1 x^2 \log x \ J_8^2(ax)dx = \frac{1}{76923347455180800} a^{16}$$

$$\times \Big[17 \ {}_3F_4\Big(\frac{19}{2}, \frac{19}{2}, \frac{19}{2}; 9, 17, \frac{21}{2}, \frac{21}{2}; -a^2\Big) - 19 \ {}_2F_3\Big(\frac{17}{2}, \frac{19}{2}; 9, 17, \frac{21}{2}; -a^2\Big)\Big]. \tag{2.44}$$

$$\int_0^1 x^4 \log x \ J_8^2(ax)dx = -\frac{1}{46985036326502400} a^{16}$$

$$\times \ {}_3F_4\Big(\frac{17}{2}, \frac{21}{2}, \frac{21}{2}; 9, 17, \frac{23}{2}, \frac{23}{2}; -a^2\Big). \tag{2.45}$$

$$\int_0^1 x^6 \log x \ J_8^2(ax)dx = -\frac{1}{56360735185305600} a^{16}$$

$$\times\ {}_3F_4\Big(\frac{17}{2},\frac{23}{2},\frac{23}{2};9,17,\frac{25}{2},\frac{25}{2};-a^2\Big). \tag{2.46}$$

$$\int_0^1 x^8\log x\ J_8^2(ax)dx=-\frac{1}{665887703040000000}a^{16}$$
$$\times\ {}_3F_4\Big(\frac{17}{2},\frac{25}{2},\frac{25}{2};9,17,\frac{27}{2},\frac{27}{2};-a^2\Big). \tag{2.47}$$

$$\int_0^1 x^{10}\log x\ J_8^2(ax)dx=-\frac{1}{776691416825856000}a^{16}$$
$$\times\ {}_3F_4\Big(\frac{17}{2},\frac{27}{2},\frac{27}{2};9,17,\frac{29}{2},\frac{29}{2};-a^2\Big). \tag{2.48}$$

$$\int_0^1 x^2\log x\ J_9^2(ax)dx=\frac{1}{30446303539573555200}a^{18}$$
$$\times\Big[19\ {}_3F_4\Big(\frac{21}{2},\frac{21}{2},\frac{21}{2};10,19,\frac{23}{2},\frac{23}{2};-a^2\Big)-21\ {}_2F_3\Big(\frac{19}{2},\frac{21}{2};10,19,\frac{23}{2};-a^2\Big)\Big]. \tag{2.49}$$

$$\int_0^1 x^4\log x\ J_9^2(ax)dx=-\frac{1}{1826087820003901440 0}a^{18}$$
$$\times\ {}_3F_4\Big(\frac{19}{2},\frac{23}{2},\frac{23}{2};10,19,\frac{25}{2},\frac{25}{2};-a^2\Big). \tag{2.50}$$

$$\int_0^1 x^6\log x\ J_9^2(ax)dx=-\frac{1}{21574761578496000000}a^{18}$$
$$\times\ {}_3F_4\Big(\frac{19}{2},\frac{25}{2},\frac{25}{2};10,19,\frac{27}{2},\frac{27}{2};-a^2\Big). \tag{2.51}$$

$$\int_0^1 x^8\log x\ J_9^2(ax)dx=-\frac{1}{25164801905157734400}a^{18}$$
$$\times\ {}_3F_4\Big(\frac{19}{2},\frac{27}{2},\frac{27}{2};10,19,\frac{29}{2},\frac{29}{2};-a^2\Big). \tag{2.52}$$

$$\int_0^1 x^{10}\log x\ J_9^2(ax)dx=-\frac{1}{29030999180242176000}a^{18}$$
$$\times\ {}_3F_4\Big(\frac{19}{2},\frac{29}{2},\frac{29}{2};10,19,\frac{31}{2},\frac{31}{2};-a^2\Big). \tag{2.53}$$

$$\int_0^1 x^2\log x\ J_{10}^2(ax)dx=\frac{1}{146087025600312115200000}a^{20}$$

$$\times\Big[21\ {}_3F_4\Big(\frac{23}{2},\frac{23}{2},\frac{23}{2};11,21,\frac{25}{2},\frac{25}{2};-a^2\Big)-23\ {}_2F_3\Big(\frac{21}{2},\frac{23}{2};11,21,\frac{25}{2};-a^2\Big)\Big]. \tag{2.54}$$

$$\int_0^1 x^4\log x\ J_{10}^2(ax)dx=-\frac{1}{862990463139840000000}a^{20} \times {}_3F_4\Big(\frac{21}{2},\frac{25}{2},\frac{25}{2};11,21,\frac{27}{2},\frac{27}{2};-a^2\Big). \tag{2.55}$$

$$\int_0^1 x^6\log x\ J_{10}^2(ax)dx=-\frac{1}{1006592076206309376000}a^{20} \times {}_3F_4\Big(\frac{21}{2},\frac{27}{2},\frac{27}{2};11,21,\frac{29}{2},\frac{29}{2};-a^2\Big). \tag{2.56}$$

$$\int_0^1 x^8\log x\ J_{10}^2(ax)dx=-\frac{1}{1161239967200968704000}a^{20} \times {}_3F_4\Big(\frac{21}{2},\frac{29}{2},\frac{29}{2};11,21,\frac{31}{2},\frac{31}{2};-a^2\Big). \tag{2.57}$$

$$\int_0^1 x^{10}\log x\ J_{10}^2(ax)dx=-\frac{1}{1326934136123817984000}a^{20} \times {}_3F_4\Big(\frac{21}{2},\frac{31}{2},\frac{31}{2};11,21,\frac{33}{2},\frac{33}{2};-a^2\Big). \tag{2.58}$$

$$\int_0^1 x^2\log x\ J_{11}^2(ax)dx=\frac{1}{835374768319365120000000}a^{22}$$
$$\times\Big[23\ {}_3F_4\Big(\frac{25}{2},\frac{25}{2},\frac{25}{2};12,23,\frac{27}{2},\frac{27}{2};-a^2\Big)-25\ {}_2F_3\Big(\frac{23}{2},\frac{25}{2};12,23,\frac{27}{2};-a^2\Big)\Big]. \tag{2.59}$$

$$\int_0^1 x^4\log x\ J_{11}^2(ax)dx=-\frac{1}{487190564883853737984000}a^{22} \times {}_3F_4\Big(\frac{23}{2},\frac{27}{2},\frac{27}{2};12,23,\frac{29}{2},\frac{29}{2};-a^2\Big). \tag{2.60}$$

$$\int_0^1 x^6\log x\ J_{11}^2(ax)dx=-\frac{1}{562040144125268852736000}a^{22} \times {}_3F_4\Big(\frac{23}{2},\frac{29}{2},\frac{29}{2};12,23,\frac{31}{2},\frac{31}{2};-a^2\Big). \tag{2.61}$$

$$\int_0^1 x^8\log x\ J_{11}^2(ax)dx=-\frac{1}{642236121883927904256000}a^{22}$$

$$\times\ {}_3F_4\Big(\frac{23}{2},\frac{31}{2},\frac{31}{2};12,23,\frac{33}{2},\frac{33}{2};-a^2\Big). \tag{2.62}$$

$$\int_0^1 x^{10}\log x\ J_{11}^2(ax)dx=-\frac{1}{7277784981598308925440000}a^{22}$$
$$\times\ {}_3F_4\Big(\frac{23}{2},\frac{33}{2},\frac{33}{2};12,23,\frac{35}{2},\frac{35}{2};-a^2\Big). \tag{2.63}$$

$$\int_0^1 x^{2}\log x\ J_{12}^2(ax)dx=\frac{1}{5612435307461995061575680000}a^{24}$$
$$\times\Big[25\ {}_3F_4\Big(\frac{27}{2},\frac{27}{2},\frac{27}{2};13,25,\frac{29}{2},\frac{29}{2};-a^2\Big)-27\ {}_2F_3\Big(\frac{25}{2},\frac{27}{2};13,25,\frac{29}{2};-a^2\Big)\Big]. \tag{2.64}$$

$$\int_0^1 x^{4}\log x\ J_{12}^2(ax)dx=-\frac{1}{3237351230161548591759360000}a^{24}$$
$$\times\ {}_3F_4\Big(\frac{25}{2},\frac{29}{2},\frac{29}{2};13,25,\frac{31}{2},\frac{31}{2};-a^2\Big). \tag{2.65}$$

$$\int_0^1 x^{6}\log x\ J_{12}^2(ax)dx=-\frac{1}{3699280062051424728514560000}a^{24}$$
$$\times\ {}_3F_4\Big(\frac{25}{2},\frac{31}{2},\frac{31}{2};13,25,\frac{33}{2},\frac{33}{2};-a^2\Big). \tag{2.66}$$

$$\int_0^1 x^{8}\log x\ J_{12}^2(ax)dx=-\frac{1}{4192004149400625941053440000}a^{24}$$
$$\times\ {}_3F_4\Big(\frac{25}{2},\frac{33}{2},\frac{33}{2};13,25,\frac{35}{2},\frac{35}{2};-a^2\Big). \tag{2.67}$$

$$\int_0^1 x^{10}\log x\ J_{12}^2(ax)dx=-\frac{1}{47155234922091522293760000000}a^{24}$$
$$\times\ {}_3F_4\Big(\frac{25}{2},\frac{35}{2},\frac{35}{2};13,25,\frac{37}{2},\frac{37}{2};-a^2\Big). \tag{2.68}$$

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