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ON SOME TYPES OF PRE-γ-SEPARATION AXIOMS IN FUZZY TOPOLOGICAL SPACES

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Abstract: In topology and its related fields of mathematics, there are many limitations or restrictions on the classes of topological spaces that one desire to regard. Some of these limitations or restrictions are given by the separation axioms. In this paper, we study and analyze the pre-γ-separation axioms in fuzzy topological spaces. Also we introduce notions of pre-γ-homeomorphism and pre*-γ-homeomorphism in fuzzy topological spaces. Further, we prove some fundamental properties and theorems of these separation axioms in fuzzy settings.

Keywords and Phrases: Fuzzy pre- γ - homeomorphism, Fuzzy pre* - γ - homeomorphism, Fuzzy pre*-γ-continuous, Fuzzy pre-γ-separation axioms.

2020 Mathematics Subject Classification: 54A40, 54D10, 54D15.

1. Introduction

The concept of fuzzy sets and their applications were introduced by Zadeh [15] in 1965. After that fuzzy topological space was initiated by Chang [1]. Several authors introduced and studied the concepts of fuzzy separation axioms (e.g. [2], [6], [7], [12], [14]) from different view points. Wadei Faris Al-Omeri [14] introduced mixed b-fuzzy topological Spaces. Separation axioms play crucial role in topology and its related fields of mathematics. Hariwan Z. Ibrahim [5] defined pre-γ-open sets in general topological spaces in 2012. Many concepts in topological space were extended to fuzzy topological space. In this manner, recently Sivashanmugaraja and

Vadivel [11] extended the concept of pre-γ-open sets into fuzzy topological space. By using the notions of pre-γ-Q-neighborhood, fuzzy pre-γ-separation axioms were introduced by Sivashanmugaraja and Vadivel [11] in 2017. In this paper, we redefined the concept of fuzzy pre-γ-separation axioms by using fuzzy pre-γ-open sets. The notion of fuzzy pre-γ-homeomorphism and fuzzy pre*-γ-homeomorphism are introduced. Relationship among pre-γ-separation axioms in fuzzy topological spaces are investigated.

2. Preliminaries

Throughout, by $(X,\ \tau_X)$, $(Y,\ \tau_Y)$ and $(Z,\ \tau_Z)$ (or simply X, Y and Z), we represent fuzzy topological spaces (fts, in short). By $\underline{0}$ and $\underline{1}$, we represent the constant fuzzy sets taking on the values 0 and 1 on X, respectively. Now, we recall some definitions and results used in this paper.

Definition 2.1. *A fuzzy set μ of a fts X is said to be fuzzy pre-γ-open [11] if $\mu \leq \tau_\gamma$-$int(cl(\mu))$. A fuzzy set μ of a fts X is called fuzzy pre-γ-closed [8] if and only if its complement is fuzzy pre-γ-open. The set of all pre-γ-open and pre-γ-closed fuzzy sets are denoted by $FP_\gamma O(X)$ and $FP_\gamma C(X)$ respectively.*

Definition 2.2. [8] *Let μ be a fuzzy set in a fts X. Then the pre-γ-interior of μ is defined as $pint_\gamma(\mu) = \vee\{\lambda : \lambda \leq \mu,\ \lambda \in FP_\gamma O(X)\}$ and the pre-γ-closure of μ is defined as $pcl_\gamma(\mu) = \wedge\{\lambda : \lambda \geq \mu,\ \lambda \in FP_\gamma C(X)\}$.*

Definition 2.3. *A mapping $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is called*

(i) fuzzy continuous [1], if the inverse image of an open fuzzy set in $(Y,\ \tau_Y)$ is an open fuzzy set in $(X,\ \tau_X)$.

(ii) fuzzy pre-γ-continuous [10], if the inverse image of an open fuzzy set in $(Y,\ \tau_Y)$ is a pre-γ-open fuzzy set in $(X,\ \tau_X)$.

(iii) fuzzy pre*-γ-continuous (or fuzzy pre-γ-irresolute) [10], if the inverse image of a pre-γ-open fuzzy set in $(Y,\ \tau_Y)$ is a pre-γ-open fuzzy set in $(X,\ \tau_X)$.

Definition 2.4. [9] *A mapping $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is called*

(i) fuzzy pre-γ-open, if $\theta(\mu)$ is a pre-γ-open fuzzy set in $(Y,\ \tau_Y)$, $\forall$ open fuzzy set μ in $(X,\ \tau_X)$.

(ii) fuzzy pre-γ-closed, if $\theta(\mu)$ is a pre-γ-closed fuzzy set in $(Y,\ \tau_Y)$, $\forall$ closed fuzzy set μ in $(X,\ \tau_X)$.

Definition 2.5. [3] *A fts X is said to be fuzzy T_0 space iff $\forall$ pair of fuzzy singletons $x_\alpha,\ x_\beta$ with different supports, $\exists$ a fuzzy open set μ such that $x_\alpha \leq \mu \leq x^c_\beta$ or*

$x_\beta \leq \mu \leq x_\alpha^c$.

Definition 2.6. [4] *A fuzzy set of a fuzzy topological space X is called fuzzy singleton it takes the value zero for every points x in X except one point. A fuzzy singleton with value 1 is called a crisp fuzzy singleton.*

Proposition 2.1. [8] *If $(X,\ \tau_X)$ is fuzzy door and fuzzy γ-regular space, then every fuzzy pre-γ-open set is a fuzzy open set.*

3. Fuzzy Pre-γ-homeomorphism Mappings

In this section, the notions of pre-γ-homeomorphism and pre*-γ-homeomorphism in fuzzy settings are introduced. Further we discuss the relationships between these homeomorphisms and prove some theorems.

Definition 3.1. *A one-one and onto mapping $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is called fuzzy pre-γ-homeomorphism, if θ and θ^{-1} are fuzzy pre-γ-continuous.*

Definition 3.2. *A one-one and onto mapping $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is said to be fuzzy pre*-γ-homeomorphism, if θ and θ^{-1} are fuzzy pre*-γ-continuous.*

Theorem 3.1. *Let $(X,\ \tau_X)$ and $(Y,\ \tau_Y)$ are fuzzy door and fuzzy γ-regular spaces. A bijective mapping $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is fuzzy pre*-γ-homeomorphism, if and only if θ is fuzzy pre-γ-homeomorphism.*

Proof. Let θ be a fuzzy pre*-γ-homeomorphism. We prove that θ is pre-γ-continuous. Since θ is fuzzy pre*-γ-homeomorphism, we obtain θ and θ^{-1} are fuzzy pre*-γ-continuous mappings. Let μ be an fuzzy open set in Y. By hypothesis Y is fuzzy door space and fuzzy γ-regular, we obtain μ is fuzzy pre-γ-open set in Y. Since θ is fuzzy pre*-γ-continuous, $\theta^{-1}(\mu)$ is fuzzy pre-γ-open set in X. Therefore θ is pre-γ-continuous. Now we prove θ^{-1} is pre-γ-continuous. Let $\theta^{-1}(\mu)$ is fuzzy pre-γ-open set in X. Since X is fuzzy door space and fuzzy γ-regular, we obtain $\theta^{-1}(\mu)$ is fuzzy open set in X. Since θ^{-1} is fuzzy pre*-γ-continuous, $\theta(\theta^{-1}(\mu)) = \mu$ is fuzzy pre-γ-open set in Y. Thus θ^{-1} is pre-γ-continuous. Hence θ is fuzzy pre-γ-homeomorphism.

Converse is evident.

Remark 3.1. *In fuzzy settings, the concepts of pre-γ-homeomorphism and pre*-γ-homeomorphism are independent as shown in the below example.*

Example 3.1. Let $X = Y = \{a, b\}$ and $\lambda,\ \mu \in I^X$ defined as $\lambda(a) = 1,\ \lambda(b) = 0$; $\mu(a) = 0,\ \mu(b) = 1$. Let $\tau_X = \{\underline{1},\ \underline{0},\ \lambda\}$ and $\tau_Y = \{\underline{1},\ \underline{0},\ \mu\}$. Then $(X,\ \tau_X)$ and $(Y,\ \tau_Y)$ are fts. Define a fuzzy operation $\gamma : \tau_X \to I^X$ by $\gamma(\underline{1}) = \underline{1},\ \gamma(\underline{0}) = \underline{0},\ \gamma(\lambda) = \lambda$ and also define a fuzzy operation $\gamma : \tau_Y \to I^Y$ by $\gamma(\underline{1}) = \underline{1},\ \gamma(\underline{0}) = \underline{0},\ \gamma(\mu) = \mu$. A mapping $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ defined by $\theta(a) = b,\ \theta(b) = a$.

Then θ is fuzzy pre-γ-homeomorphism but not fuzzy pre*-γ-homeomorphism.Since $\theta^{-1} : Y \to X$ is not fuzzy pre*-γ-continuous.

Example 3.2. Let $X = Y = \{a, b\}$ and $\mu_1,\ \mu_2\ \mu_3,\ \mu_4\ \mu_5,\ \mu_6 \in I^X$ defined as $\mu_1(a) = 0.4,\ \mu_1(b) = 0.5;\ \mu_2(a) = 0.1,\ \mu_2(b) = 0.4,\ \mu_3(a) = 0.5,\ \mu_3(b) = 0.6,\ \mu_4(a) = 0.5,\ \mu_4(b) = 0.4,\ \mu_5(a) = 0.4,\ \mu_5(b) = 0.1,\ \mu_6(a) = 0.6,\ \mu_6(b) = 0.5$. Let $\tau_X = \{\underline{1},\ \underline{0},\ \mu_1,\ \mu_2,\ \mu_3\}$ and $\tau_Y = \{\underline{1},\ \underline{0},\ \mu_4,\ \mu_5,\ \mu_6\}$. Then $(X,\ \tau_X)$ and $(Y,\ \tau_Y)$ are fts. Define a fuzzy operation $\gamma : \tau_X \to I^X$ by $\gamma(\underline{1}) = \underline{1},\ \gamma(\underline{0}) = \underline{0},\ \gamma(\mu_1) = cl(\mu_1),\ \gamma(\mu_2) = \mu_2,\ \gamma(\mu_3) = \mu_3$ and also define a fuzzy operation $\gamma : \tau_Y \to I^Y$ by $\gamma(\underline{1}) = \underline{1},\ \gamma(\underline{0}) = \underline{0},\ \gamma(\mu_4) = cl(\mu_4),\ \gamma(\mu_5) = \mu_5,\ \gamma(\mu_6) = \mu_6$. A mapping $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ defined by $\theta(a) = b,\ \theta(b) = a$. Then θ is fuzzy pre*-γ-homeomorphism but not fuzzy pre-γ-homeomorphism. Since θ and θ^{-1} are not fuzzy pre-γ-continuous mappings.

Theorem 3.2. *For a one-one and onto mapping $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$, the below statements are equivalent:*

(i) θ is fuzzy pre-γ-homeomorphism;

(ii) θ is fuzzy pre-γ-continuous and fuzzy pre-γ-open;

(iii) θ is fuzzy pre-γ-continuous and fuzzy pre-γ-closed.

Proof. (i)$\Rightarrow$ (ii) Assume that θ be fuzzy pre-γ-homeomorphism. Therefore by definition of fuzzy pre-γ-homeomorphism, θ and θ^{-1} are fuzzy pre-γ-continuous. To prove that θ is fuzzy pre-γ-open. Let μ be a fuzzy open set in X. Since θ^{-1} is fuzzy pre-γ-continuous, we obtain $(\theta^{-1})^{-1}(\mu) = \theta(\mu)$ is fuzzy pre-γ-open set in Y. Thus θ is fuzzy pre-γ-open mapping.

(ii)$\Rightarrow$ (i) Assume that θ be fuzzy pre-γ-continuous and fuzzy pre-γ-open. To prove that $\theta^{-1} : (Y,\ \tau_Y) \to (X,\ \tau_X)$ is fuzzy pre-γ-continuous. Let μ be a fuzzy open set in X. Since θ is fuzzy pre-γ-open, we obtain $\theta(\mu)$ is a fuzzy pre-γ-open set in Y. But $\theta(\mu) = (\theta^{-1})^{-1}(\mu)$. Therefore $(\theta^{-1})^{-1}(\mu)$ is a fuzzy pre-γ-open set in Y. Thus θ^{-1} is fuzzy pre-γ-continuous. Hence θ is fuzzy pre-γ-homeomorphism.

(ii)$\Rightarrow$(iii) Assume that θ be fuzzy pre-γ-continuous and fuzzy pre-γ-open. To prove that $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is fuzzy pre-γ-closed. Let η be a fuzzy closed set in X. So η^c is a fuzzy open set in X. Since θ is fuzzy pre-γ-open mapping, we obtain $\theta(\eta^c)$ is fuzzy pre-γ-open set in Y. But $\theta(\eta^c) = (\theta(\eta))^c$. Therefore $(\theta(\eta))^c$ is fuzzy pre-γ-open set in Y. This implies $\theta(\eta)$ is fuzzy pre-γ-closed set in Y. Thus θ is fuzzy pre-γ-closed mapping.

Theorem 3.3. *For a one-one and onto mapping $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$, the below statements are equivalent:*

(i) θ is fuzzy pre*-γ-homeomorphism;

(ii) θ is fuzzy pre*-γ-continuous and fuzzy pre*-γ-open;

(iii) θ is fuzzy pre*-γ-continuous and fuzzy pre*-γ-closed.

Proof. Evident.

Remark 3.2. *The composition of two fuzzy pre-γ-homeomorphism mappings need not be fuzzy pre-γ-homeomorphism as shown in the below example.*

Example 3.3. Let $X = Y = Z = \{a, b\}$ and $\mu_1,\ \mu_2,\ \mu_3 \in I^X$ which are defined as $\mu_1 = \underline{0.4},\ \mu_2 = \underline{0.7},\ \mu_3 = \underline{0.6}$. Let $\tau_X = \{\underline{1},\ \underline{0},\ \mu_1\}$, $\tau_Y = \{\underline{1},\ \underline{0},\ \mu_2\}$, and $\tau_Z = \{\underline{1},\ \underline{0},\ \mu_3\}$. Then $(X,\ \tau_X)$, $(Y,\ \tau_Y)$ and $(Z,\ \tau_Z)$ are fts. Define an fuzzy operation γ on $\tau_X,\ \tau_Y$ and τ_Z by $\gamma(\lambda) = \lambda$, for every open fuzzy sets of X, Y and Z. Let $\theta_1 : (X,\ \tau_X) \to (Y,\ \tau_Y)$ and $\theta_2 : (Y,\ \tau_Y) \to (Z,\ \tau_Z)$ be the identity mappings. Clearly, θ_1 and θ_2 are fuzzy pre-γ-homeomorphism mappings but $(\theta_2 \circ \theta_1)$ is not fuzzy pre-γ-homeomorphism mapping. Since, μ_3 is an open fuzzy set of $(Z,\ \tau_Z)$, but $(\theta_2 \circ \theta_1)(\mu_3) \notin FP_\gamma O(X)$.

Theorem 3.4. *If $\theta_1 : (X,\ \tau_X) \to (Y,\ \tau_Y)$ and $\theta_2 : (Y,\ \tau_Y) \to (Z,\ \tau_Z)$ be two fuzzy pre*-γ-homeomorphism mappings, then the composite mapping $(\theta_2 \circ \theta_1)$ is fuzzy pre*-γ-homeomorphism.*

Proof. First we prove $(\theta_2 \circ \theta_1)$ is fuzzy pre*-γ-continuous. Let λ be a fuzzy pre-γ-open set in Z. By hypothesis $\theta_2 : (Y,\ \tau_Y) \to (Z,\ \tau_Z)$ is fuzzy pre*-γ-homeomorphism, θ_2 is fuzzy pre*-γ-continuous. Therefore $\theta_2^{-1}(\lambda)$ is fuzzy pre-γ-open set in Y. Also since $\theta_1 : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is fuzzy pre*-γ-homeomorphism, θ_1 is fuzzy pre*-γ-continuous. Therefore $(\theta_1^{-1}\theta_2^{-1}(\lambda))$ is fuzzy pre-γ-open set in X. But $\theta_1^{-1}(\theta_2^{-1}(\lambda)) = (\theta_2 \circ \theta_1)^{-1}(\lambda)$. Therefore $(\theta_2 \circ \theta_1)^{-1}(\lambda)$ is fuzzy pre-γ-open set in X. Hence $(\theta_2 \circ \theta_1)$ is fuzzy pre*-γ-continuous.

Next we prove $(\theta_2 \circ \theta_1)^{-1}$ is fuzzy pre*-γ-continuous. Let λ be a fuzzy pre-γ-open set in X. Since $\theta_1 : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is fuzzy pre*-γ-homeomorphism, θ_1^{-1} is fuzzy pre*-γ-continuous. Therefore $(\theta_1^{-1})^{-1}(\lambda) = \theta_1(\lambda)$ is fuzzy pre-γ-open set in Y. Also since $\theta_2 : (Y,\ \tau_Y) \to (Z,\ \tau_Z)$ is fuzzy pre*-γ-homeomorphism, θ_2^{-1} is fuzzy pre*-γ-continuous. Therefore $(\theta_2^{-1})^{-1}(\theta_1(\lambda))$ is fuzzy pre-γ-open set in Z. But $(\theta_2^{-1})^{-1}(\theta_1(\lambda)) = \theta_2(\theta_1(\lambda)) = (\theta_2 \circ \theta_1)(\lambda)$. Thus $(\theta_2 \circ \theta_1)(\lambda)$ is fuzzy pre-γ-open set in Z. Thus $(\theta_2 \circ \theta_1)^{-1}$ is fuzzy pre*-γ-continuous. Hence $(\theta_2 \circ \theta_1)$ is fuzzy pre*-γ-homeomorphism.

4. Fuzzy Pre-γ-Separation Axioms

Definition 4.1. *A fts $(X,\ \tau_X)$ is called fuzzy pre-γ-T_0 iff $\forall$ pair of fuzzy singletons*

x_α and x_β with different supports in X, $\exists$ a fuzzy pre-γ-open set λ such that either $x_\alpha \leq \lambda \leq x_\beta^c$ or $x_\beta \leq \lambda \leq x_\alpha^c$.

Example 4.1. Let $X = \{a, b\}$ and $\mu_1, \mu_2,\ \mu_3 \in I^X$ which are defined as $\mu_1(a) = 0,\ \mu_1(b) = 1;\ \mu_2(a) = 0.4,\ \mu_2(b) = 0;\ \mu_3(a) = 0.4,\ \mu_3(b) = 1;$ Let $\tau_X = \{\underline{1},\ \underline{0},\ \mu_1,\ \mu_2,\ \mu_3\}$. Then $(X,\ \tau_X)$ is a fts. Define $\gamma : \tau_X \to I^X$ by $\gamma(\underline{1}) = \underline{1},\ \gamma(\underline{0}) = \underline{0},\ \gamma(\mu_1) = cl(\mu_1),\ \gamma(\mu_2) = \mu_2,\ \gamma(\mu_3) = \mu_3$. Then the fuzzy topological space $(X,\ \tau_X)$ is fuzzy pre-γ-T_0.

Theorem 4.1. *A fuzzy topological space $(X,\ \tau_X)$ is fuzzy pre-γ-T_0 iff fuzzy pre-γ-closure of any two crisp fuzzy singletons with different supports are distinct.*
Proof. Let X be fuzzy pre-γ-T_0 and $x_\alpha,\ x_\beta$ be two crisp fuzzy singletons with two different supports y_α and y_β respectively. By hypothesis X is fuzzy pre-γ-T_0, $\exists$ a pre-γ-open fuzzy set λ such that $x_\alpha \leq \lambda \leq x_\beta^c$. Therefore $x_\beta \leq \lambda^c$. But $x_\beta \leq pcl_\gamma(x_\beta) \leq \lambda^c$, where λ^c is pre-γ-closed fuzzy set. Now $x_\beta \leq \lambda^c$, this implies $x_\alpha \leq pcl_\gamma(x_\beta)$. But $x_\alpha \leq pcl_\gamma(x_\alpha)$. Thus $pcl_\gamma(x_\alpha) \neq pcl_\gamma(x_\beta)$.

Conversely, consider $x_\alpha\ x_\beta$ are any two fuzzy singletons with different supports $y_\alpha\ y_\beta$ respectively. Let $z_\alpha,\ z_\beta$ are two crisp fuzzy singletons such that $z_\alpha(y_\alpha) = 1$ and $z_\beta(y_\beta) = 1$. Since $z_\alpha \leq pcl_\gamma(z_\alpha)$, we obtain $(pcl_\gamma(z_\alpha))^c \leq z_\alpha^c \leq x_\alpha^c$. Since each crisp fuzzy singleton is pre-γ-closed fuzzy set, $(pcl_\gamma(z_\alpha))^c$ is pre-γ-open fuzzy set and $x_\beta \leq (pcl_\gamma(z_\alpha))^c \leq x_\alpha^c$. Thus X is fuzzy pre-γ-T_0.

Definition 4.2. *A fts $(X,\ \tau_X)$ is called fuzzy pre-γ-T_1 if $\forall$ pair of fuzzy singletons $x_\alpha,\ x_\beta$ with different supports $y_\alpha,\ y_\beta$, $\exists$ fuzzy pre-γ-open sets $\lambda,\ \mu$ such that $x_\alpha \leq \lambda \leq x_\beta^c$ and $x_\beta \leq \mu \leq x_\alpha^c$.*

Theorem 4.2. *A fts X is fuzzy pre-γ-T_1 if and only if every crisp fuzzy singleton is pre-γ-closed fuzzy set.*
Proof. Let X be fuzzy pre-γ-T_1 and x_α be crisp fuzzy singleton with support y_α. For any fuzzy singleton x_β with support $y(\neq y_\alpha)$ $\exists$ fuzzy pre-γ-open sets λ and μ such that $x_\alpha \leq \lambda \leq x_\beta^c$ and $x_\beta \leq \mu \leq x_\alpha^c$. Therefore $x_\alpha^c = \bigvee_{x_\beta \leq x_\alpha^c} x_\beta = 0$. Hence x_α^c is pre-γ-open fuzzy set. Thus x_α is a pre-γ-closed fuzzy set.

Conversely, let x_α and x_β are any pair of fuzzy singletons with different supports $y_\alpha,\ y_\beta$ respectively. Let $z_\alpha,\ z_\beta$ be two crisp fuzzy singletons with different supports $y_\alpha,\ y_\beta$ such that $z_\alpha(y_\alpha) = 1$ and $z_\beta(y_\beta) = 1$. Since each crisp fuzzy singleton is fuzzy pre-γ-closed set in X, z_α^c and z_β^c are fuzzy pre-γ-open sets such that $x_\alpha \leq z_\alpha^c \leq x_\beta^c$ and $x_\beta \leq z_\beta^c \leq x_\alpha^c$. Thus X is fuzzy pre-γ-T_1.

Remark 4.1. *Every fuzzy pre-γ-T_1 space is fuzzy pre-γ-T_0.*
Proof. Evident.

The converse of the remark 4.1, may not be true as shown in the below example.

Example 4.2. Let $X = \{a, b\}$ and $\lambda,\ \mu,\ \eta \in I^X$ defined as $\lambda(a) = 0.3,\ \lambda(b) = 0$; $\mu(a) = 0,\ \mu(b) = 1$; $\eta(a) = 0.3,\ \eta(b) = 1$; Let $\tau_X = \{\underline{1},\ \underline{0},\ \lambda,\ \mu,\ \eta\}$. Then the space $(X,\ \tau_X)$ is a fts. Define $\gamma : \tau_X \to I^X$ by $\gamma(\underline{1}) = \underline{1},\ \gamma(\underline{0}) = \underline{0},\ \gamma(\lambda) = \lambda,\ \gamma(\mu) = cl(\mu),\ \gamma(\eta) = \eta$. Then $(X,\ \tau_X)$ is fuzzy pre-γ-T_0 but not fuzzy pre-γ-T_1.

Definition 4.3. *A fts $(X,\ \tau_X)$ is called fuzzy pre-γ-strong T_1 iff $\forall$ fuzzy singleton is a pre-γ-closed fuzzy set.*

Remark 4.2. *Every fuzzy pre-γ-strong T_1 space is fuzzy pre-γ-T_1.*
Proof. Evident.

The converse of the remark 4.2, may not be true as shown in the below examples.

Example 4.3. Let $X = \{a, b\}$ and $\lambda,\ \mu,\ \eta,\ \nu \in I^X$ defined as $\lambda(a) = 0.4,\ \lambda(b) = 0$; $\mu(a) = 0.2,\ \mu(b) = 0.8$; $\eta(a) = 0.2,\ \eta(b) = 0$; $\nu(a) = 0.4,\ \nu(b) = 0.8$; Let $\tau_X = \{\underline{1},\ \underline{0},\ \lambda,\ \mu,\ \eta,\ \nu\}$. Then $(X,\ \tau_X)$ is a fts. Define $\gamma : \tau_X \to I^X$ by $\gamma(\underline{1}) = \underline{1},\ \gamma(\underline{0}) = \underline{0},\ \gamma(\lambda) = \lambda,\ \gamma(\mu) = \mu,\ \gamma(\eta) = cl(\eta),\ \gamma(\nu) = \nu$. Then the space $(X,\ \tau_X)$ is fuzzy pre-γ-T_1 but not fuzzy pre-γ-strong T_1. Since fuzzy singletons λ and η are not pre-γ-closed fuzzy sets.

Definition 4.4. *A fts $(X,\ \tau_X)$ is called fuzzy pre-γ-Hausdorff or fuzzy pre-γ-T_2 iff $\forall$ pair of fuzzy singletons $x_\alpha,\ x_\beta$ with different supports, $\exists$ fuzzy pre-γ-open sets $\lambda,\ \mu$ such that $x_\alpha \le \lambda \le x_\beta^c$, $x_\beta \le \mu \le x_\alpha^c$ and $\lambda \le \mu^c$.*

Theorem 4.3. *A fts $(X,\ \tau_X)$ is fuzzy pre-γ-T_2 iff $\forall$ pair of fuzzy singletons, with different supports, $\exists$ a pre-γ-open fuzzy set λ such that $x_\alpha \le \lambda \le pcl_\gamma(\lambda) \le x_\beta^c$.*
Proof. Let X be fuzzy pre-γ-T_2 and let $x_\alpha,\ x_\beta$ be two fuzzy singletons with two different supports $y_\alpha,\ y_\beta$ respectively. Let $\lambda,\ \mu$ be fuzzy pre-γ-open sets. Then $x_\alpha \le \lambda \le x_\beta^c$, $x_\beta \le \mu \le x_\alpha^c$ and $\lambda \le \mu^c$. By definition 2.2, $pcl_\gamma(\lambda) = \wedge\{\mu^c : \lambda \le \mu^c,\ \mu^c \in FP_\gamma C(X)\}$. Also $\lambda \le pcl_\gamma(\lambda)$. Hence $x_\alpha \le \lambda \le pcl_\gamma(\lambda) \le \mu^c \le x_\beta^c$, that implies $x_\alpha \le \lambda \le pcl_\gamma(\lambda) \le x_\beta^c$.

Conversely, for a pair of fuzzy singletons $x_\alpha,\ x_\beta$ with different supports and for any pre-γ-open fuzzy set λ, let $x_\alpha \le \lambda \le pcl_\gamma(\lambda) \le x_\beta^c$. That implies $x_\alpha \le \lambda \le x_\beta^c$. Also since $x_\alpha \le pcl_\gamma(\lambda) \le x_\beta^c$, we obtain $x_\beta \le pcl_\gamma(\lambda) \le x_\alpha$. Thus $(pcl_\gamma(\lambda))^c$ is fuzzy pre-γ-open set. Also $pcl_\gamma(\lambda) \le (pcl_\gamma(\mu))^c$. Hence X is fuzzy pre-γ-T_2.

Definition 4.5. *A fts $(X,\ \tau_X)$ is called fuzzy pre-γ-regular if and only if for a fuzzy singleton x_α and a fuzzy closed set η, $\exists$ two fuzzy pre-γ-open sets $\lambda,\ \mu$ such that $\eta \le \lambda$ and $x_\alpha \le \mu$ and $\lambda \le \mu^c$.*

Theorem 4.4. *A fts $(X,\ \tau_X)$ is fuzzy pre-γ-regular if and only if for a fuzzy singleton x_α and a fuzzy open set λ such that $x_\alpha \le \lambda$, $\exists$ a fuzzy pre-γ-open set μ*

such that $x_\alpha \leq \mu \leq pcl_\gamma(\mu) \leq \lambda$.

Theorem 4.5. *For every closed fuzzy set* η *in a fuzzy pre-*γ*-regular spaces and a fuzzy singleton* x_α *such that* $x_\alpha \leq \eta^c$, $\exists$ *fuzzy pre-*γ*-open sets* $\lambda,\ \mu$ *such that* $x_\alpha \leq \lambda,\ \eta \leq \mu$ *and* $pcl_\gamma(\lambda) \leq (pcl_\gamma(\mu))^c$.

Proof. Let x_α be a fuzzy singleton and η be a fuzzy closed set such that $x_\alpha \leq \eta^c$. Since the fts X is fuzzy pre-γ-regular, $\exists$ fuzzy pre-γ-open set λ such that $x_\alpha \leq \lambda \leq pcl_\gamma(\lambda) \leq \eta^c$. Let $\mu = (pcl_\gamma(\lambda))^c$ be a fuzzy pre-γ-open set such that $\eta \leq (pcl_\gamma(\lambda))^c$. Now $\mu \leq pcl_\gamma(\mu)$. Hence $pcl_\gamma(\lambda) \leq (pcl_\gamma(\mu))^c$.

Definition 4.6. *A fts* $(X,\ \tau_X)$ *is called fuzzy pre-*γ_1*-normal iff* $\forall$ *pair of fuzzy closed sets* $\mu_1,\ \mu_2$, *such that* $\mu_1\ \overline{q}\ \mu_2$, $\exists$ *two fuzzy pre-*γ*-open sets* η_1 *and* η_2 *such that* $\mu_1 \leq \eta_1,\ \mu_2 \leq \eta_2$ *and* $\eta_1 \wedge \eta_2 = 0$.

Definition 4.7. *A fts* $(X,\ \tau_X)$ *is called fuzzy pre-*γ_2*-normal iff* $\forall$ *pair of fuzzy closed sets* $\mu_1,\ \mu_2$, *such that* $\mu_1 \wedge \mu_2 = 0$, $\exists$ *two fuzzy pre-*γ*-open sets* $\eta_1,\ \eta_2$ *such that* $\mu_1 \leq \eta_1,\ \mu_2 \leq \eta_2$ *and* $\eta_1 \wedge \eta_2 = 0$.

Definition 4.8. *A fts* $(X,\ \tau_X)$ *is called fuzzy pre-*γ_3*-normal iff* $\forall$ *pair of fuzzy closed sets* μ_1 *and* μ_2, *such that* $\mu_1\ \overline{q}\ \mu_2$, $\exists$ *two fuzzy pre-*γ*-open sets* η_1 *and* η_2 *such that* $\mu_1 \leq \eta_1,\ \mu_2 \leq \eta_2$ *and* $\eta_1\ \overline{q}\ \eta_2$.

Definition 4.9. *A fts* $(X,\ \tau_X)$ *is called fuzzy pre-*γ_4*-normal iff* $\forall$ *pair of fuzzy closed sets* $\mu_1,\ \mu_2$, *such that* $\mu_1 \wedge \mu_2 = 0$, $\exists$ *two fuzzy pre-*γ*-open sets* η_1 *and* η_2 *such that* $\mu_1 \leq \eta_1,\ \mu_2 \leq \eta_2$ *and* $\eta_1\ \overline{q}\ \eta_2$.

Remark 4.3.

(i) pre-γ_1*-normal* $\Rightarrow$ *pre-*γ_2*-normal* $\Rightarrow$ *pre-*γ_4*-normal.*

(ii) pre-γ_1*-normal* $\Rightarrow$ *pre-*γ_3*-normal* $\Rightarrow$ *pre-*γ_4*-normal.*

The converse of the remark 4.3 may not be true as shown in the below examples.

Example 4.4. Let $X = \{a, b\}$ and $\mu_1,\ \mu_2,\ \mu_3\ \mu_4, \mu_5,\ \mu_6,\ \mu_7\ \mu_8 \in I^X$ defined as $\mu_1(a) = 0.7,\ \mu_1(b) = 0.4$; $\mu_2(a) = 0.4,\ \mu_2(b) = 0.7$; $\mu_3(a) = 0.4,\ \mu_3(b) = 0$; $\mu_4(a) = 0,\ \mu_4(b) = 0.4$; $\mu_5(a) = 0.7,\ \mu_5(b) = 0$; $\mu_6(a) = 0,\ \mu_6(b) = 0.7$; $\mu_7(a) = 0.7,\ \mu_7(b) = 1$; $\mu_8(a) = 1,\ \mu_8(b) = 0.7$; Let $\tau_X = \{\underline{1},\ \underline{0},\ \mu_1,\ \mu_2,\ \mu_1 \vee \mu_2,\ \mu_3,\ \mu_4,\ \mu_3 \vee \mu_4,\ \mu_5,\ \mu_6,\ \mu_7,\ \mu_8\}$. Then $(X,\ \tau_X)$ is a fts. Define a fuzzy operation $\gamma : \tau_X \to I^X$ on τ_X by $\gamma(\underline{1}) = \underline{1},\ \gamma(\underline{0}) = \underline{0},\ \gamma(\lambda_i) = \lambda_i$, for $i = 1, 2, 3, 4, 5, 6$ and $\gamma(\lambda_i) = cl(\lambda_i)$, for $i = 7, 8$. Then $(X,\ \tau_X)$ is pre-γ_2-normal, but not pre-γ_3-normal. For two closed fuzzy sets $\mu_1 = \underline{0.6}$ and $\mu_2 = \underline{0.3}$, there are no pre-γ-open fuzzy sets $\eta_1,\ \eta_2$ containing $\mu_1,\ \mu_2$ such that $\eta_1\ \overline{q}\ \eta_2$.

Example 4.5. Let $X = \{a, b\}$ and $\mu_1,\ \mu_2,\ \mu_3\ \mu_4 \in I^X$ defined as $\mu_1(a) = 0.6,\ \mu_1(b) = 1;\ \mu_2(a) = 1,\ \mu_2(b) = 0.6;\ \mu_3(a) = 0.6,\ \mu_3(b) = 0.4;\ \mu_4(a) = 0.4,\ \mu_4(b) = 0.6$. Let $\tau_X = \{\underline{1},\ \underline{0},\ \mu_1,\ \mu_2,\ \mu_1 \wedge \mu_2,\ \mu_3,\ \mu_4,\ \mu_3 \wedge \mu_4\}$. Then $(X,\ \tau_X)$ is a fts. Define a fuzzy operation $\gamma : \tau_X \to I^X$ on τ_X by $\gamma(\underline{1}) = \underline{1},\ \gamma(\underline{0}) = \underline{0},\ \gamma(\lambda) = \lambda,\ \forall\ \lambda \in \tau_X$. Then $(X,\ \tau_X)$ is pre-γ_3-normal, but not pre-γ_2-normal. For two closed fuzzy sets $\mu_1 = \{0,\ 0.4\}$ and $\mu_2 = \{0.4,\ 0\}$, there are no pre-γ-open fuzzy sets $\eta_1,\ \eta_2$ containing $\mu_1,\ \mu_2$ such that $\eta_1 \wedge \eta_2 = 0$.

Theorem 4.6. *A fuzzy topological space $(X,\ \tau_X)$ is a fuzzy pre-γ_2-normal if for a fuzzy closed set μ and a fuzzy open set λ such that $\eta \leq \lambda$, $\exists$ a fuzzy pre-γ-open set η such that $\mu \leq \eta \leq pcl_\gamma(\eta) \leq \lambda$.*

Theorem 4.7. *A fts $(X,\ \tau_X)$ is a fuzzy pre-γ_3-normal iff for a closed fuzzy set μ and an open fuzzy set λ such that $\mu \leq \lambda$, $\exists$ a pre-γ-open fuzzy set η such that $\mu \leq \eta \leq pcl_\gamma(\eta) \leq \lambda$.*

Proof. Suppose that X is fuzzy pre-γ_3-normal. Let μ be a fuzzy closed set and λ be a fuzzy open set in X such that $\mu \leq \lambda$. Therefore λ^c is a fuzzy closed set and $\mu\ \overline{q}\ \lambda^c$. Let $\eta_1,\ \eta_2$ be fuzzy pre-γ-open sets. Since X is fuzzy pre-γ_3-normal, $\mu \leq \eta_1$. But $\eta_1 \leq pcl_\gamma(\eta_1) = \bigwedge\{\eta_2^c : \eta_2^c$ are fuzzy pre-γ-closed sets and $\eta_1 \leq \eta_2^c\}$ and $\eta_2^c \leq \lambda$. Hence $pcl_\gamma(\eta_1) \leq \lambda$. Thus $\mu \leq \eta_1 \leq pcl_\gamma(\eta_1) \leq \lambda$.

Converse is evident.

Theorem 4.8. *Let the mapping $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is an injective fuzzy pre-γ-continuous. Then the below statements are hold:*

(i) If Y is fuzzy T_0, then X is fuzzy pre-γ-T_0;

(ii) If Y is fuzzy T_1, then X is fuzzy pre-γ-T_1;

(iii) If Y is fuzzy Hausdorff, then X is fuzzy pre-γ-Hausdorff.

Proof. (i) Consider x_α and x_β are any two fuzzy singletons in X with different supports.By hypothesis θ is a one-one, $\theta(x_\alpha)$ and $\theta(x_\beta)$ are in Y and $\theta(x_\alpha) \neq \theta(x_\beta)$. Also by hypothesis Y is fuzzy T_0 space, $\exists$ a fuzzy open set λ in Y such that $\theta(x_\alpha) \leq \lambda \leq (\theta(x_\beta))^c$ or $\theta(x_\beta) \leq \lambda \leq (\theta(x_\alpha))^c$. So $x_\alpha \leq \theta^{-1}(\lambda) \leq x_\beta^c$ or $x_\beta \leq \theta^{-1}(\lambda) \leq x_\alpha^c$. Since θ is fuzzy pre-γ-continuous, $\theta^{-1}(x_\beta)$ is fuzzy pre-γ-open set in X. Thus, X is fuzzy pre-γ-T_0.

(ii) Consider x_α and x_β are any two fuzzy singletons in X with different supports. By hypothesis θ is a one-one, we obtain $\theta(x_\alpha)$ and $\theta(x_\beta)$ are in Y and $\theta(x_\alpha) \neq \theta(x_\beta)$. Also since Y is fuzzy T_1 space, $\exists$ fuzzy open sets λ_1 and λ_2 in Y such that $\theta(x_\alpha) \leq \lambda_1 \leq (\theta(x_\beta))^c$ and $\theta(x_\beta) \leq \lambda_2 \leq (\theta(x_\alpha))^c$ which implies that

$x_\alpha \leq \theta^{-1}(\lambda_1) \leq x_\beta^c$ and $x_\beta \leq \theta^{-1}(\lambda_2) \leq x_\alpha^c$. Since θ is fuzzy pre-γ-continuous, $\theta^{-1}(\lambda_1)$ and $\theta^{-1}(\lambda_2)$ are pre-γ-open fuzzy sets in X. Thus X is fuzzy pre-γ-T_1.

(iii) Let x_α and x_β be any two fuzzy singletons in X with different supports. By hypothesis θ is a one-one, we obtain $\theta(x_\alpha)$ and $\theta(x_\beta)$ are in Y and $\theta(x_\alpha) \neq \theta(x_\beta)$. Also since Y is fuzzy Hausdorff, $\exists$ fuzzy open set λ in Y such that $\theta(x_\alpha) \leq \lambda \leq cl(\lambda) \leq (\theta(x_\beta))^c$. Thus $x_\alpha \leq \theta^{-1}(\lambda) \leq \theta^{-1}(cl(\lambda)) \leq x_\beta^c$. By hypothesis θ is fuzzy pre-γ-continuous, $\theta^{-1}(\lambda)$ is fuzzy pre-γ-open set in X. Thus $\forall$ pair of fuzzy singletons x_α and x_β with different supports, $\exists$ fuzzy pre-γ-open set λ such that $x_\alpha \leq \theta^{-1}(\lambda) \leq \theta^{-1}(cl(\lambda)) \leq x_\beta^c$. Thus X is a Hausdorff.

Theorem 4.9. *If $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is fuzzy closed one-one pre-γ-continuous and Y is fuzzy regular, then X is fuzzy pre-γ-regular.*

Proof. Evident.

Theorem 4.10. *If $\theta : (X,\ \tau_X) \to (Y,\ \tau_Y)$ is a one-one, closed fuzzy pre-γ-continuous and Y is normal, then X is fuzzy pre-γ_4-normal.*

Proof. Assume that μ_1 and μ_2 be closed fuzzy sets of X such that $\mu_1 \wedge \mu_2 = 0$. Since θ is a one-one fuzzy closed, $\theta(\mu_1)$ and $\theta(\mu_2)$ are closed fuzzy sets in Y such that $\theta(\mu_1) \leq \theta(\mu_2) = 0$. Since Y is fuzzy normal, $\exists$ fuzzy open sets λ_1 and λ_2 are in Y such that $\theta(\mu_1) \leq \lambda_1$ and $\theta(\mu_2) \leq \lambda_2$ and $\lambda_1 \wedge \lambda_2 = 0$. Thus we get, $\mu_1 \leq \theta^{-1}(\mu_2)$ and $\mu_2 \leq \theta^{-1}(\lambda_2)$ and $\theta^{-1}(\lambda_1 \wedge \lambda_2) = 0$. By hypothesis θ is fuzzy pre-γ-continuous, $\theta^{-1}(\lambda_1)$ and $\theta^{-1}(\lambda_2)$ are fuzzy pre-γ-open set in X. Therefore $\forall$ pair of fuzzy closed sets $\mu_1,\ \mu_2$ and pre-γ-open fuzzy sets $\theta^{-1}(\lambda_1),\ \theta^{-1}(\lambda_2)$ and $\mu_1 \wedge \mu_2 = 0$ such that $\mu_1 \leq \theta^{-1}(\mu_2)$, $\mu_2 \leq \theta^{-1}(\lambda_2)$ and $\theta^{-1}(\lambda_1) \leq (\theta^{-1}(\lambda_2))^c$. Therefore $\theta^{-1}(\lambda_1)\ \overline{q}\ \theta^{-1}(\lambda_2) = 0$. Hence X is fuzzy pre-γ_4-normal.

5. Conclusion

In this paper, we introduced the concept of pre-γ-homeomorphism, pre*-γ-homeomorphism and pre-γ-separation axioms in fuzzy topological spaces. We stated that the concept of fuzzy pre-γ-homeomorphism and fuzzy pre * - γ - homeomorphism are independent. The relationships among pre-γ-T_0, pre-γ-T_1, pre-γ-strong T_1 and pre-γ-T_2 are investigated. We have shown heriditariness of pre-γ-separation in fuzzy topological spaces. There is a scope to extend these separation axioms.

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