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NEW TRIPLED FIXED POINT THEOREMS IN CONE METRIC SPACE

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Abstract: We prove some tripled fixed point theorems on Cone Metric Space. Our results generalise the fixed point results due to Erdal Karapinar [E. Karapinar, Couple Fixed Point on Cone Metric Space, GU. J. Sci. 24(1): 51-58(2011)].

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1. Introduction

Bhaskar and Lakshmikanthan [3] proved the existence of a new fixed point theorem for a mixed monotone mapping in a metric space with the help of partial order, using a weak contractivity type of assumption. They named this new type of fixed point as coupled fixed point. Since then this new concept is extended and used in various directions. This concept is extended to tripled fixed point by Berinde and Borcut [2]. They obtained the existence and uniqueness theorems for contractive mappings in partially ordered complete metric spaces. This concept is extended to quadrupled fixed point by Karapinar [9].

The concept of cone metric space introduced by Huang and Zang [7] in 2007 as generalizations of metric space. They generalized metric space by replacing the set of real numbers with an ordering Banach space. Thus, cone naturally induces a partial order in Banach spaces. In recent years many authors established various triple fixed point theorems in cone metric space ([1], [15], [6]) and references there in. More results on coupled ,tripled and quadrupled results can be refereed to ([14], [17], [4], [5], [13], [16], [10], [11], [12]).

Throughout the manuscript, we assume that $X \neq \phi$ and $X^3 = X \times X \times X$. Let E be a real Banach Space. A subset P of E is called a cone if

1. P is closed, non-empty and $P \neq 0$
2. $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$
3. $P \cap (-P) = 0$.

Given a cone $P \subset E$ we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to denote that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in int.P$(interior of P).

There are two kinds of cone. They are normal cone and non-normal cones. A cone $P \subset E$ is normal if there is a number $K > 0$ such that for all $x, y \in P$, $0 \leq x \leq y \Rightarrow \| x \| \leq K \| y \|$. In other words if $x_n \leq y_n \leq z_n$ and $lim_{n\to\infty} x_n = lim_{n\to\infty} z_n = x$ imply $lim_{n\to\infty} y_n = x$. Also, a cone $P \subset E$ is regular if every increasing sequence which is bounded above is convergent.

In this manuscript, we prove triple fixed point theorems on cone metric space without using normality or regularity of cone.

2. Preliminaries

Definition 2.1. [7] *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies the following conditions:*

1. *$0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$.*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*
3. *$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2. [7] *Let (X, d) be a cone metric space(CMS), $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then,*

1. *$\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x) << c$ for all $n \geq N$. It is denoted by $lim_{n\to\infty} x_n = x$ or $x_n \to x$.*

2. *$\{x_n\}_{n\geq 1}$ is a Cauchy Sequence whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x_m) << c$ for all $n, m \geq N$*

3. *(X, d) is a complete cone metric space if every Cauchy sequence is convergent.*

Definition 2.3. *Let (X, d) be a cone metric space(CMS), P be a normal cone with normal constant K and $\{x_n\}$ be a sequence in X. Then, the sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ or $\| d(x_n, x) \| \to 0$ as $n \to \infty$.*

Definition 2.4. [8] *Let (X, d) be a complete cone metric space and Let $F : X \times X \times X \to X$. The mapping F is said to has the mixed monotone property if for any $x, y, z \in X$ such that*

$$\begin{aligned} x_1, x_2 \in X, x_1 \leq x_2 &\Rightarrow F(x_1, y, z) \leq F(x_2, y, z) \\ y_1, y_2 \in X, y_1 \leq y_2 &\Rightarrow F(x, y_1, z) \geq F(x, y_2, z) \\ z_1, z_2 \in X, z_1 \leq z_2 &\Rightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{aligned}$$

Definition 2.5. [8] *Let $F : X^3 \to X$. An element (x,y,z) is called a tripled fixed point of F if $F(x, y, z) = x, F(y, x, y) = y$ and $F(z, y, x) = z$.*

Definition 2.6. [8] *Let (X, d) be a complete cone metric space and Let $F : X^3 \to X$. The mapping F is said to have the mixed monotone property if $F(x, y, z)$ is monotone non-decreasing in x, z and is monotone non-increasing in y, that is, for any $x, y, z \in X$*

$$\begin{aligned} x_1, x_2 \in X, x_1 \leq x_2 &\Rightarrow F(x_1, y, z) \leq F(x_2, y, z) \\ y_1, y_2 \in X, y_1 \leq y_2 &\Rightarrow F(x, y_1, z) \geq F(x, y_2, z) \\ z_1, z_2 \in X, z_1 \leq z_2 &\Rightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{aligned}$$

3. Main Results

Theorem 3.1. *Let (X, d) be a complete cone metric space. Let $F : X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y, z), F(u, v, w)) \leq \frac{k}{3}[d(x, u) + d(y, v) + d(z, w)]$$

for all $u \leq x, y \leq v, z \leq w$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0), F(y_0, x_0, y_0) \leq y_0$ and $z_0 \leq F(z_0, y_0, x_0)$, there exist $x, y, z \in X$ such that $x = F(x, y, z), y = F(y, x, y)$ and $z = F(z, y, x)$.

Proof. Set $x_1 = F(x_0, y_0, z_0)$, $y_1 = F(y_0, x_0, y_0)$ and $z_1 = F(z_0, y_0, x_0)$, $x_0 \leq x_1, y_1 \leq y_0$ and $z_0 \leq z_1$ and

$$\begin{aligned}
x_2 &= F(x_1, y_1, z_1) \\
&= F(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)) \\
&= F^2(x_0, y_0, z_0) \\
y_2 &= F(x_1, y_1, z_1) = F^2(y_0, x_0, y_0) \\
z_2 &= F(z_1, y_1, x_1) = F^2(z_0, y_0, x_0).
\end{aligned}$$

For $n = 1, 2, \ldots$ the general form of the sequence are as follows:

$$\begin{aligned}
x_{n+1} &= F^{n+1}(x_n, y_n, z_n) = F^{n+1}(x_0, y_0, z_0) \\
y_{n+1} &= F^{n+1}(y_0, x_0, y_0) \\
z_{n+1} &= F^{n+1}(z_0, y_0, x_0).
\end{aligned}$$

Now,

$$\begin{aligned}
x_0 \leq x_1 \leq F(x_1, y_1, z_1) = F^2(x_0, y_0, z_0) \leq x_2 \leq \cdots \leq F^{n+1}(x_0, y_0, z_0) \\
F^{n+1}(y_0, x_0, y_0) \leq \cdots \leq F^2(y_0, x_0, y_0) = F(y_1, x_1, y_1) = y_2 \leq y_1 = F(y_0, x_0, y_0) \leq y_0 \\
z_0 \leq F(z_0, y_0, x_0) = z_1 \leq F(z_1, y_1, x_1) \\
= F^2(z_0, y_0, x_0) \leq z_2 \leq \cdots \leq F^{n+1}(z_0, y_0, x_0).
\end{aligned}$$

$\therefore x_0 \leq F(x_0, y_0, z_0), F(y_0, x_0, y_0) \leq y_0$ and $z_0 \leq F(z_0, y_0, x_0)$.
Then

$$\begin{aligned}
& d(F^2(x_0, y_0, z_0), F(x_0, y_0, z_0)) \\
& = d(F(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)), F(x_0, y_0, z_0)) \\
& \leq \frac{k}{3}[d(F(x_0, y_0, z_0), x_0) + d(F(y_0, x_0, y_0), y_0) + d(F(z_0, y_0, x_0)), z_0)].
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
& d(F^2(y_0, x_0, y_0), F(y_0, x_0, y_0)) \\
& \leq \frac{k}{3}[d(F(y_0, x_0, y_0), y_0) + d(F(z_0, y_0, x_0), z_0) + d(F(x_0, y_0, z_0), x_0)].
\end{aligned}$$

and

$$\begin{aligned}
& d(F^2(z_0, y_0, x_0), F(z_0, y_0, x_0)) \\
& \leq \frac{k}{3}[d(F(z_0, y_0, x_0), z_0) + d(F(y_0, x_0, y_0), y_0) + d(F(x_0, y_0, z_0), z_0))].
\end{aligned}$$

Hence by induction, we can prove the following:

$$\begin{aligned} & d(F^{n+1}(x_0,y_0,z_0),F^n(x_0,y_0,z_0)) \\ & \leq \frac{k^n}{3}[d(F(x_0,y_0,z_0),x_0)+d(F(y_0,x_0,y_0),y_0)+d(F(z_0,y_0,x_0)),z_0)]. \end{aligned}$$

$$\begin{aligned} & d(F^{n+1}(y_0,x_0,y_0),F^n(y_0,x_0,y_0)) \\ & \leq \frac{k^n}{3}[d(F(y_0,x_0,y_0),x_0)+d(F(z_0,y_0,x_0),y_0)+d(F(x_0,y_0,z_0)),x_0)]. \end{aligned}$$

$$\begin{aligned} & d(F^{n+1}(z_0,y_0,x_0),F^n(z_0,y_0,x_0)) \\ & \leq \frac{k^n}{3}[d(F(z_0,y_0,x_0),z_0)+d(F(y_0,x_0,y_0),y_0)+d(F(x_0,y_0,x_0)),x_0)]. \end{aligned}$$

Therefore,

$$\begin{aligned} F^n(x_0,y_0,z_0) &\leq F^{n+1}(x_0,y_0,z_0) \\ F^{n+1}(y_0,y_0,y_0) &\leq F^n(y_0,y_0,y_0) \\ F^n(z_0,y_0,x_0) &\leq F^{n+1}(z_0,y_0,x_0). \end{aligned}$$

Again,

$$\begin{aligned} & d(F^{n+2}(x_0,y_0,z_0),F^{n+1}(x_0,y_0,z_0)) \\ &\quad = d(F(F^{n+1}(x_0,y_0,z_0),F^{n+1}(y_0,x_0,y_0) \\ &\qquad F^{n+1}(z_0,y_0,x_0)),F(F^n(x_0,y_0,z_0) \\ &\qquad F^n(y_0,x_0,y_0),F^n(x_0,y_0,z_0))) \\ &\quad \leq \frac{k}{3}[d(F^{n+1}(x_0,y_0,z_0),F^n(x_0,y_0,z_0)) \\ &\quad + d(F^{n+1}(y_0,x_0,y_0),F^n(y_0,x_0,y_0)) \\ &\quad + d(F^{n+1}(z_0,y_0,x_0)),F^{n+1}(z_0,y_0,x_0)))] \\ &\quad \leq \frac{k^{n+1}}{3}[d(F(x_0,y_0,z_0),x_0))+d(F^{n+1}(y_0,x_0,y_0),y_0)) \\ &\quad + d(F^{n+1}(z_0,y_0,x_0)),z_0))] \end{aligned}$$

Analogously, we get

$$\begin{aligned} & d(F^{n+2}(y_0,x_0,y_0),F^{n+1}(y_0,x_0,y_0)) \\ & \leq \frac{k^{n+1}}{3}[d(F(y_0,x_0,y_0),y_0)+d(F(z_0,y_0,x_0),z_0)+d(F(x_0,y_0,z_0)),x_0))] \end{aligned}$$

$$d(F^{n+2}(z_0,y_0,x_0),F^{n+1}(z_0,y_0,x_0))$$
$$\leq \frac{k^{n+1}}{3}[d(F(z_0,y_0,x_0),z_0)+d(F(y_0,x_0,y_0),y_0)+d(F(x_0,y_0,z_0)),x_0))]$$

We shall prove that $\{F^n(x_0,y_0,z_0)\},\{F^n(y_0,x_0,y_0)\}$ and $\{F^n(z_0,y_0,x_0)\}$ are Cauchy sequences in X. Suppose that $m>n$. Let $0<<c$ be given. Choose $\delta>0$ such that $c+B_\delta(0)\subset P$ where $B_\delta(0)=y\in E:\parallel y\parallel<\delta$. Now, choose a natural no. such that

$$\frac{k^n}{2(1-k)}[d(x_0,y_0,z_0),x_0)+d(y_0,x_0,y_0),y_0+d(z_0,y_0,x_0),z_0]\in B_\delta(0).$$

for all $n\geq N_0$. Then

$$\frac{k^n}{2(1-k)}[d(F(x_0,y_0,z_0),x_0))+d(F(y_0,x_0,y_0),y_0))+d(F(z_0,y_0,x_0),z_0))]<<c.$$

Thus

$$\begin{aligned}
&d(F^m(x_0,y_0,z_0),F^n(x_0,y_0,z_0))\\
&\quad\leq\ d(F^m(x_0,y_0,z_0),F^{m-1}(x_0,y_0,z_0))\\
&\quad+\ \dots d(F^{n+1}(x_0,y_0,z_0),F^n(x_0,y_0,z_0))].\\
&\quad\leq\ \frac{k^{m-1}+\cdots+k^n}{3}[d(F(x_0,y_0,z_0)),x_0))+d(F(y_0,x_0,y_0),y_0)\\
&\quad+\ d(F(z_0,y_0,x_0),z_0)].\\
&\quad\leq\ \frac{k^n(1+k+\cdots+k^{m-n-1}}{3}[d(F(x_0,y_0,z_0)),x_0))+d(F(y_0,x_0,y_0),y_0)\\
&\quad+\ d(F(z_0,y_0,x_0),z_0)].\\
&\quad=\ \frac{k^n-k^m}{3(1-k)}[d(F(x_0,y_0,z_0)),x_0))+d(F(y_0,x_0,y_0),y_0)\\
&\quad+\ d(F(z_0,y_0,x_0),z_0)].\\
&\quad\leq\ \frac{k^n}{3(1-k)}[d(F(x_0,y_0,z_0)),x_0))+d(F(y_0,x_0,y_0),y_0)\\
&\quad+\ d(F(z_0,y_0,x_0),z_0)].
\end{aligned}$$

for all $m>n\geq N_0$. Thus $\{F^n(x_0,y_0,z_0)\}$ is Cauchy sequence in X.

Similarly, we can prove that $\{F^n(y_0,x_0,y_0)\},\{F^n(z_0,y_0,x_0)\}$ are Cauchy Sequences in X. Since X is complete cone metric space, therefore there exist $x,y,z\in X$ such that $x^n=F^n(x_0,y_0,z_0)\to x,y^n=F^n(y_0,x_0,y_0)\to y_0$ and $z^n=F^n$

$(z_0, y_0, x_0) \to z$ as $n \to \infty$. Again, We shall prove that $F(x,y,z) = x, F(y,x,y) = y$ and $F(z,y,x) = z$. Let $0 << c$. Choose a natural number N, such that

$$d(x_{n+1}, x) = d(F^{n+1}(x_0, y_0, z_0), x) << \frac{c}{3}$$

for all $n \geq N_1$. Since F is continuous, there exists N_2 such that for all $n \geq N_2$, one has $(x_n, y_n, z_n) \to (x,y,z)$ implies that $d(F(x_n, y_n, z_n), F(x,y,z)) << \frac{c}{3}$ for all $n \geq N_2$. By triangle inequality

$$\begin{aligned} d(F(x,y,z),x) &\leq d(F(x,y,z),x_{n+1}) + d(x_{n+1},x) \\ &= d(F(x,y,z),F^{n+1}(x_0,y_0,z_0)) + d(F^{n+1}(x_0,y_0,z_0),x) \\ &= d(F(x,y,z),F(F^n(x_0,y_0,z_0),F^n(y_0,x_0,z_0),F^n(z_0,y_0,x_0) \\ &+ d(F^{n+1}(x_0,y_0,z_0),x). \end{aligned}$$

Hence, choose $N_0 = max.N_1, N_2$, for all $n > N_0$.

$$\begin{aligned} d(F(x,y,z),x) &\leq d(F(x,y,z),F(F^n(x_0,y_0,z_0),F^n(y_0,x_0,y_0),F^n(z_0,y_0,x_0))) \\ &+ d(F^{n+1}(x_0,y_0,z_0),x) << c. \end{aligned}$$

Thus, $d(F(x,y,z),x) << \frac{c}{l}$ for all $l > 1$. Hence, $\frac{c}{l} - d(F(x,y,z),x) \in P$ for all $l > 1$. $\frac{c}{l} \to 0$ as $l \to \infty$. Therefore, $-d(F(x,y,z),x) \in P$. Also, $d(F(x,y,z),x) \in P$ implies that $d(F(x,y,z),x) = 0$. $\therefore F(x,y,z) = x$. We can show that $F(y,x,y) = y, F(z,y,x) = z$.

Theorem 3.2. *Let (X,d) be a complete cone metric space. Let $F : X^3 \to X$ be a mapping having the fixed monotone property on X. Assume that X has the following properties*

1. *if a non-decreasing sequences $\{x_n\} \to x, \{z_n\} \to z$, then $x_n \leq x, z_n \leq z$ for all n.*

2. *if a non-decreasing sequences $\{y_n\} \to y$, then $y \leq y_n$ for all n.*

Assume that there exists a $k \in [0,1)$ with

$$d(F(x,y,z),F(u,v,w)) \leq \frac{k}{3}[d(x,u) + d(y,v) + d(z,w)]$$

for all $u \leq x, y \leq v, z \leq w$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0,y_0,z_0), F(y_0,x_0,y_0) \leq y_0$ and $z_0 \leq F(z_0,y_0,x_0)$,there exist $x,y,z \in X$ such that $x = F(x,y,z), y = F(y,x,y)$

and $z = F(z, y, x)$.

Proof. Let $0 << c$. Since $x_n = F^n(x_0, y_0, z_0) \to x, y_n = F^n(y_0, x_0, y_0) \to y$ and $z_n = F^n(z_0, y_0, x_0) \to z$, there exists N_0 such that $d(F^n(x_0, y_0, z_0), x << \frac{c}{3}$, $d(F^m(y_0, x_0, y_0), y << \frac{c}{3}$ and $d(F^l(z_0, y_0, x_0), z << \frac{c}{3}$ for all $m, n, l > N_0$.

Also, We have

$$\begin{aligned} F^n(x_0, y_0, z_0) &= x_n \leq x \\ y \leq y_n &= F^n(y_0, x_0, y_0) \\ F^n(z_0, y_0, x_0) &= z_n \leq z. \end{aligned}$$

Now,

$$\begin{aligned} d(F(x, y, z), x) &\leq d(F(x, y, z), x_{n+1}) + d(x_{n+1}, x) \\ &= d(F(x, y, z), F^{n+1}(x_0, y_0, z_0)) + d(F^{n+1}(x_0, y_0, z_0), x) \\ &= d(F(x, y, z), F(F^n(x_0, y_0, z_0), F^n(y_0, x_0, z_0), F^n(z_0, y_0, x_0) \\ &+ d(F^{n+1}(x_0, y_0, z_0), x) \\ &\leq [d(x, F^n(x_0, y_0, z_0)) + d(y, F^n(y_0, x_0, y_0) \\ &+ d(z, F^n(z_0, y_0, x_0))] + d(F^{n+1}(x_0, y_0, z_0) \\ &\leq \frac{k}{3}[\frac{c}{3} + \frac{c}{3} + \frac{c}{3}] + \frac{c}{3} << c \end{aligned}$$

for all $n > N$. This gives $F(x, y, z) = x$. Similarly, we can get the other two, $y = F(y, x, y), z = F(z, y, x)$.

Theorem 3.3. *Let $F : X^3 \to X$ be a continuous mapping having the fixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y, z), F(u, v, w)) \leq \frac{k}{3}[d(x, u) + d(y, v) + d(z, w)]$$

for all $u \leq x, y \leq v, z \leq w$.

Suppose that each triplet elements of X has an upper bound or lower bound in X. Then $x = y = z$.

Proof. We know that there exists $(x, y, z) \in X \times X \times X$ such that $F^n(x_0, y_0, z_0) \to x, F^n(y_0, x_0, y_0) \to y$ and $F^n(z_0, y_0, x_0) \to z$.

Let $(u, v, w) \in X \times X \times X$ be another triple fixed point of F. We consider two cases.

Case(i)**:** Suppose that (x, y, z) and (u, v, w) are comparable with respect to X^3. Then for each $n = 0, 1, 2, \ldots$ the triplet

$$(F^n(x, y, z), F^n(y, x, y), F^n(z, y, x)) = (x, y, z)$$

is comparable to

$$(F^n(u,v,w), F^n(v,u,v), F^n(w,v,u)) = (u,v,w)$$

Now,

$$\begin{aligned}
\rho((x,y,z),(u,v,w)) &= d(x,u)+d(y,v)+d(z,w) \\
&= d(F^n(x,y,z),F^n(u,v,w))+d(F^n(y,x,y),F^n(v,u,v)) \\
&+ d(F^n(z,y,x),F^n(w,v,u)) \\
&\leq k^n[d(x,u)+d(y,v)+d(z,w)] \\
&= k^n\rho((x,y,z),(u,v,w)) \\
\Rightarrow \rho((x,y,z),(u,v,w)) &= 0.
\end{aligned}$$

Hence, $x = u, y = v, z = w$
Case(ii)**:** Suppose that $(x,y,z),(u,v,w)$ are not comparable with respect to X^3. Then there exists an upper bound or lower bound $(p,q,r) \in X^3$ of (x,y,z) and (u,v,w). Thus, for each $n = 0,1,2,\dots$ the triplet $(F^n(p,q,r), F^n(q,p,q), F^n(r,q,p)) = (x,y,z)$ is comparable with $(F^n(x,y,z), F^n(y,x,y), F^n(z,y,x))$ and $(F^n(u,v,w), F^n(v,u,v), F^n(w,v,u)) = (u,v,w)$.

Now,

$$\begin{aligned}
\rho((x,y,z),(u,v,w)) &\leq \rho((F^n(x,y,z),F^n(y,x,y),F^n(z,y,x) \\
&\quad F^n(u,v,w),F^n(v,u,v),F^n(w,v,u)) = (u,v,w)))). \\
&\leq \rho((F^n(x,y,z),F^n(y,x,y),F^n(z,y,x)), \\
&\quad (F^n(p,q,r),F^n(q,p,q),F^n(r,q,p)) \\
&+ \rho(F^n(p,q,r),F^n(q,p,q),F^n(r,q,p)), \\
&\quad (F^n(u,v,w),F^n(v,u,v),F^n(w,v,u)) = (u,v,w))) \\
&\leq k^n(d(x,p)+d(y,q)+d(z,r) \\
&+ d(p,u)+d(q,v)+d(r,w) \longrightarrow 0.
\end{aligned}$$

as $n \to \infty$. $\therefore \rho((x,y,z),(u,v,w)) = 0$.

Hence, $x = u, y = v, z = w$.

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