South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 3 (2020), pp. 51-60

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

ESSENTIAL ASCENT AND ESSENTIAL DESCENT OF WEIGHTED COMPOSITION OPERATORS ON l^p SPACES

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(**Received:** *Jun. 05, 2020* **Accepted:** *Sep. 05, 2020* **Published:** *Dec. 30, 2020)*

Abstract: In this paper we give a complete characterization of essential ascent and essential descent of weighted composition operators on l^p spaces.

Keywords and Phrases: Essential Ascent, Essential Descent, Weighted Composition Operator.

2010 Mathematics Subject Classification: Primary: 47B33.

1. Introduction

Let X denote an arbitrary vector space and T be a linear operator on X. Let $D(T)$, $N(T)$ and $R(T)$ denote domain, kernel and range of T respectively. Let $\mathbb{N}$ denote the set of natural numbers. Let $l^p, (1 \leq p < \infty)$ be the Banach space of all p-summable sequences of complex numbers under the standard p-norm on it and let u be a complex-valued function with domain $\mathbb{N}$. For $f \in l^p$ define

$$(uC_\phi)(f)(n) = u(n)f(\phi(n)), \text{ for each } n \in \mathbb{N}.$$

If $(uC_\phi)(f) \in l^p$ whenever $f \in l^p$ then uC_ϕ is a linear transformation on l^p and is called a weighted composition operator on l^p. When u is identically equal to one we get the composition operator C_ϕ. In this paper S(u) denotes the support of u. Weighted composition operators appear naturally in the study weighted shift operators due to Shield [24]. These operators have been subject matter of study by authors such as Kumar [14], Singh [12], Whitley [28] and others ([3], [11], [13]).

Definition 1.1. *If there is some integer $n \geq 0$ such that dim $(N(T^{n+1})/N(T^n))$ is finite, the smallest such integer is called the essential ascent of T and is denoted by $a_e(T)$. If no such integer exists then $a_e(T) = \infty$; see [20].*

Definition 1.2. *If there is some integer $n \geq 0$ such that dim $(R(T^n)/R(T^{n+1}))$ is finite, the smallest such integer is called the essential descent of T and is denoted by $d_e(T)$. If no such integer exists then $d_e(T) = \infty$; see [20].*

2. Essential Ascent and Essential Descent of Weighted Composition Operators On l^p spaces

In this section we prove results about essential ascent and essential descent of weighted composition operators on l^p spaces where $1 \leq p < \infty$.

Theorem 2.1. *$a_e(uC_\phi) = \infty$ if and only if there exist a sequence $\{E_k\}_{k=1}^{\infty}$ of subsets of $\mathbb{N}$ such that each E_k is infinite, $E_k \subseteq \phi^{k-1}(N_{k-1})$ and $\phi^k(N_k) \bigcap E_k = \phi$ for each $k \in \mathbb{N}$, where $N_k = \{n \in S(u) : \phi^i(n) \in S(u); \forall i, 1 \leq i \leq k-1\}$.*

Proof. Suppose that $a_e(uC_\phi) = \infty$. Let $E_k = \left\{m : m \in \phi^{k-1}(N_{k-1}) - \phi^k(N_k)\right\}$. By construction of E_k, it is clear that $E_k \subseteq \phi^{k-1}(N_{k-1})$ and $\phi^k(N_k) \bigcap E_k = \phi$ for each $k \in \mathbb{N}$. We claim that E_k is infinite set. Let E_K be finite for some K. We make the following claims :

Claim-I : If $n \in E_K$, this implies that $n \in \phi^{K-1}(N_{K-1})$ and $n \notin \phi^K(N_K)$.
Therefore $(\phi^{K-1})^{-1}(n) \cap N_{K-1} \neq \phi$ but $(\phi^K)^{-1}(n) \cap N_K = \phi$. There exists an $i \in (\phi^{K-1})^{-1}(n)$ such that $u(i)u(\phi(i)) \ldots\ldots u(\phi^{K-2}(i)) \neq 0$ but $u(j)u(\phi(j)) \ldots\ldots u(\phi^{K-1}(j)) = 0$ for each $j \in (\phi^K)^{-1}(n)$.
Thus $(uC_\phi)^K(\chi_n) = \sum\limits_{j \in (\phi^K)^{-1}(n)} u(j)u(\phi(j)) \ldots\ldots u(\phi^{K-1}(j))\chi_j = 0$ and

$$(uC_\phi)^{K-1}(\chi_n) = \sum_{i \in (\phi^{K-1})^{-1}(n)} u(i)u(\phi(i)) \ldots\ldots u(\phi^{K-2}(i))\chi_i \neq 0.$$

Therefore $\chi_n \notin N((uC_\phi)^{K-1})$ but $\chi_n \in N((uC_\phi)^K)$.

Claim-II : Let $n \in \phi^K(N_K)$. This implies that $n = \phi^K(m)$ for some $m \in N_K$.
Since $N_K \cap (\phi^K)^{-1}(n) \neq \phi$, hence

$$(uC_\phi)^K(\chi_n) = \sum_{i \in (\phi^K)^{-1}(n)} u(i)u(\phi(i)) \ldots\ldots u(\phi^{K-1}(i))\chi_i \neq 0.$$

Therefore $\chi_n \notin N((uC_\phi)^K)$.

Claim-III : suppose $n \notin \phi^{K-1}(N_{K-1})$. Then for each $i \in N_{K-1}$ satisfying $\phi^{K-1}(i) =$

n and $u(i)u(\phi(i))\ldots\ldots u(\phi^{K-2}(i))=0$.
Therefore $(uC_\phi)^{K-1}(\chi_n) = \sum\limits_{i\in(\phi^{K-1})^{-1}(n)} u(i)u(\phi(i))\ldots\ldots u(\phi^{K-2}(i))\chi_i = 0$. So $\chi_n \in N((uC_\phi)^{K-1})$. Now we show that $N((uC_\phi)^K)/N((uC_\phi)^{K-1})$ is spanned by $\{\chi_n + N((uC_\phi)^{K-1}) : n \in E_K\}$.
Let $f = g + N((uC_\phi)^{K-1})$, where $g \in N((uC_\phi)^K)$.
Let $g = \sum \alpha_n \chi_n$. Now we can expressed g as follows :
$g = \sum\limits_{m\in E_K} \alpha_m\chi_m + \sum\limits_{p\in(\mathbb{N}-(\phi^K(N_K)\cup E_K))} \alpha_p\chi_p$.
Clearly $\sum\limits_{p\in(\mathbb{N}-(\phi^K(N_K)\cup E_K))} \alpha_p\chi_p$ belongs to $N((uC_\phi)^{K-1})$.
Then $f = g + N((uC_\phi)^{K-1}) = \sum\limits_{m\in E_K} \alpha_m\chi_m + N((uC_\phi)^{K-1})$
$= \sum\limits_{m\in E_K} \alpha_m(\chi_m + N((uC_\phi)^{K-1}))$.
This implies that $\{\chi_n + N((uC_\phi)^{K-1}) : n \in E_K\}$ spans $N((uC_\phi)^K)/N((uC_\phi)^{K-1})$.
Therefore dim $N((uC_\phi)^K)/N((uC_\phi)^{K-1}) \le \overline{\overline{E_K}} < \infty$. Thus $a_e(uC_\phi) \le (K-1)$. This is a contradiction. Hence E_k is infinite set.
Conversely, assume that there exist a sequence $\{E_k\}_{k=1}^{\infty}$ of subsets of $\mathbb{N}$ such that each E_k is infinite, $E_k \subseteq \phi^{k-1}(N_{k-1})$ and $\phi^k(N_k)\cap E_k = \phi$ for each $k\in\mathbb{N}$, where $E_k = \{m : m \in \phi^{k-1}(N_{k-1}) - \phi^k(N_k)\}$.
Now we claim that $\{\chi_n + N((uC_\phi)^{k-1}) : n \in E_k\}$ are linearly independent sequence of $N((uC_\phi)^k)/N((uC_\phi)^{k-1})$. It is sufficient if we prove that every finite subset $\{\chi_n + N((uC_\phi)^{k-1}) : n \in E_k\}$ are linearly independent in $N((uC_\phi)^k)/N((uC_\phi)^{k-1})$.
Let $\beta_1(\chi_{n_1} + N((uC_\phi)^{k-1}) + \ldots\cdots + \beta_l(\chi_{n_l} + N((uC_\phi)^{k-1}) = N((uC_\phi)^{k-1})$.
This implies that $\beta_1\chi_{n_1} + \ldots\cdots + \beta_l\chi_{n_l} \in N((uC_\phi)^{k-1})$.
Therefore $(uC_\phi)^{k-1}(\beta_1\chi_{n_1} + \ldots\cdots + \beta_l\chi_{n_l}) = 0$. Thus
$\beta_j \sum\limits_{i\in(\phi^{k-1})^{-1}(n_j)} u(i)u(\phi(i))u(\phi^{k-2}(i))\chi_i = 0$ for each j, $1 \le j \le l$.
Hence $(uC_\phi)^k(\chi_{n_j}) = 0$ and $(uC_\phi)^{k-1}(\chi_{n_j}) \ne 0$ for each j, $1\le j\le l$.
Since $i \in N_{k-1}\cap(\phi^{k-1})^{-1}(n_j) \ne \phi$ for $1 \le j \le l$. This implies that
$\sum\limits_{i\in(\phi^{k-1})^{-1}(n_j)} u(i)u(\phi(i))u(\phi^{k-2}(i))\chi_i \ne 0$ for each j, $1\le j\le l$.
Hence $\beta_j = 0$ for each j, $1 \le j \le l$. Thus $\{\chi_n + N((uC_\phi)^{k-1}) : n \in E_k\}$ are linearly independent sequence of $N((uC_\phi)^k)/N((uC_\phi)^{k-1})$. Since each E_k is infinite set.
Therefore dim $(N((uC_\phi)^k)/N((uC_\phi)^{k-1})) = \infty$ for each $k \ge 1$.
Hence $a_e(uC_\phi) = \infty$.

Remark 2.1. *The following example shows that for each $n \in \mathbb{N}$ there exist a weighted composition operator uC_ϕ on l^p such that $a_e(uC_\phi) = n-1$.*

Example 2.1. Let n be any fixed natural number and ϕ be a self-map on $\mathbb{N}$ defined as :

$$\phi(m) = \begin{cases} \text{m}, & \text{if } n/(m-1) \\ \text{m-1}, & \text{otherwise.} \end{cases}$$

and

$$u = \{\tfrac{1}{n}\}_{n=1}^{\infty}$$

Then $a_e(uC_\phi) = n-1$ and $d_e(uC_\phi) = n-1$.

Theorem 2.2. *$d_e(uC_\phi) = \infty$ if and only if for each $k \geq 0$; $\overline{\overline{\phi^{-1}(n)}} > 1$ for infinitely many $n \in \ \phi^k(N_k)$, where $N_k = \{n \in S(u) : \phi^i(n) \in S(u); \forall i, 1 \leq i \geq k-1\}$.*

Proof. If possible, suppose $A = \left\{ n \in \phi^K(N_0) : \overline{\overline{\phi^{-1}(n)}} > 1) \right\}$ is finite for some natural number K. We claim that dim $(R((uC_\phi)^K)/R((uC_\phi)^{K+1})) \leq \overline{\overline{A}} < \infty$. Let $f \in \ R((uC_\phi)^K)$. Then $f = (uC_\phi)^K(g)$, for some $g \in l^p$. Let $g = \sum \alpha_n \chi_n$. Then

$$\begin{aligned} (uC_\phi)^K(g) &= \sum_{n \in \ (\phi^K)^{-1}(N_k)} u(n)u(\phi(n))\ldots\ldots u(\phi^{K-1}(n))\alpha_{\phi^K(n)}\chi_n \\ &= \sum_{\substack{n' \in \ (\phi^K)^{-1}(N_k) \\ and\ \overline{\overline{\phi^{-1}(n')}}>1}} u(n')u(\phi(n'))\ldots\ldots u(\phi^{K-1}(n'))\alpha_{\phi^K(n')}\chi_{n'} \\ &+ \sum_{\substack{n'' \in \ (\phi^K)^{-1}(N_k) \\ and\ \overline{\overline{\phi^{-1}(n'')}}=1}} u(n'')u(\phi(n''))\ldots\ldots u(\phi^{K-1}(n''))\alpha_{\phi^K(n'')}\chi_{n''} \end{aligned}$$

i.e.

$$(uC_\phi)^K(g) = h_1 + h_2(say) \tag{1}$$

We claim that $h_2 \in R((uC_\phi)^{K+1})$. Let $g' = \sum \beta_n \chi_n$, where

$$\beta_n = \begin{cases} 0, & \text{when } n \notin \phi^{K+1}(N_{k+1}) \text{ or } \overline{\overline{\phi^{-1}(n)}} > 1 \\ \alpha_{\phi^{-1}(n)}/u(\phi^K(n)), & \text{when } n \in \phi^{K+1}(N_{k+1}) \text{ and } \overline{\overline{\phi^{-1}(n)}} = 1. \end{cases}$$

Then, clearly $g^{'} \in l^p$. Now

$$(uC_\phi)^{K+1}(g^{'}) = \sum_{\substack{n\in\ (\phi^{K+1})^{-1}(N_{k+1}) \\ and\ \overline{\overline{\phi^{-1}(n)}}=1}} u(n)u(\phi(n))\ldots\ldots u(\phi^K(n))\beta_{\phi^{K+1}(n)}\chi_n$$
$$+ \sum_{\substack{n\notin\ (\phi^{K+1})^{-1}(N_{k+1}) \\ or\ \overline{\overline{\phi^{-1}(n)}}>1}} u(n)u(\phi(n))\ldots\ldots u(\phi^K(n))\beta_{\phi^{K+1}(n)}\chi_n$$
$$= \sum_{\substack{n\in\ (\phi^{K+1})^{-1}(N_{k+1}) \\ and\ \overline{\overline{\phi^{-1}(n)}}=1}} u(n)u(\phi(n))\ldots\ldots u(\phi^K(n))\beta_{\phi^{K+1}(n)}\chi_n$$

Now put $n^{''} = \phi^{-1}(n)$, then $n^{''} \in \phi^K(N_k)$ and by our assumption we get $\overline{\overline{\phi^{-1}(n)}} = 1 \Leftrightarrow \overline{\overline{\phi^{-1}(n'')}} = 1$. Therefore

$$(uC_\phi)^{K+1}(g^{'}) = \sum_{\substack{n^{''}\in\ \phi^K(N_k) \\ and\ \overline{\overline{\phi^{-1}(n'')}}=1}} u(n^{''})u(\phi(n^{''}))\ldots\ldots u(\phi^{K-1}(n^{''}))\alpha_{\phi^K(n'')}\chi_{n''} = h_2 \tag{2}$$

Thus $h_2 \in R((uC_\phi)^{K+1})$.
Combining equation (1) and (2), we get dim $(R((uC_\phi)^K)/R((uC_\phi)^{K+1}))$ is finite. Thus $d_e(uC_\phi) \leq K$.
Conversely, assume that $\overline{\overline{\phi^{-1}(n)}} > 1$ for infinitely many $n \in\ \phi^k(N_0)$. Let $\{n_m\}_{m=1}^{\infty} \in \phi^k(N_0)$ such that $\overline{\overline{\phi^{-1}(n_m)}} > 1$ for each $m \geq 1$. Let $\{\alpha_{n_m}, \beta_{n_m}\} \subseteq \phi^{-1}(n_m)$.
Define a sequence $\{f_m\}_{m=1}^{\infty}$ as follows :

$$f_m(n) = \begin{cases} 1, & \text{if } \phi^{k-1}(n) = \alpha_{n_m} \\ -1, & \text{if } \phi^{k-1}(n) = \beta_{n_m} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\{f_m\}_{m=1}^{\infty} \in l^p$ and also define a sequence $\{h_m\}_{m=1}^{\infty}$ as follows :

$$h_m(n) = \begin{cases} 1/u(n)u(\phi(n))\ldots\ldots u(\phi^{k-2}(n)), & \text{if } n = \alpha_{n_m} \\ -1/u(n)u(\phi(n))\ldots\ldots u(\phi^{k-2}(n)), & \text{if } n = \beta_{n_m} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\{h_m\}_{m=1}^{\infty} \in l^p$. We claim that $\{f_m\}_{m=1}^{\infty} \in R((uC_\phi)^{k-1})$ and $\{f_m\}_{m=1}^{\infty} \notin R((uC_\phi)^k)$. Now

$$((uC_\phi)^{k-1}h_m)(n) = \begin{cases} 1, & \text{if } \phi^{k-1}(n) = \alpha_{n_m} \\ -1, & \text{if } \phi^{k-1}(n) = \beta_{n_m} \\ 0, & \text{otherwise.} \end{cases}$$

This implies that $(uC_\phi)^{k-1}(h_m) = f_m$. Therefore $\{f_m\}_{m=1}^{\infty} \in R((uC_\phi)^{k-1})$. We claim that $\{f_m\}_{m=1}^{\infty} \notin R((uC_\phi)^k)$. If possible, assume that $\{f_{m_0}\}_{m_0=1}^{\infty} \in R((uC_\phi)^k)$, for some $m_0 \geq 1$. This implies that $f_{m_0} = (uC_\phi)^k(h_0)$, for some $h_0 \in l^p$. Let $n_m^{(1)}$ and $n_m^{(2)}$ be such that $\phi^{k-1}(n_m^{(1)}) = \alpha_{n_m}$ and $\phi^{k-1}(n_m^{(2)}) = \beta_{n_m}$, where $\phi(\alpha_{n_m}) = \phi(\beta_{n_m}) = n_m$. A simple computation shows that $\{f_m\}_{m=1}^{\infty} \notin R((uC_\phi)^k)$. Thus sequence $\{f_m/R((uC_\phi)^k)\}_{m=1}^{\infty}$ are linearly independent in $R((uC_\phi)^{k-1})/R((uC_\phi)^k)$. Therefore dim $(R((uC_\phi)^{k-1})/R((uC_\phi)^k))$ is not finite. Since $k \geq 1$ is arbitrary it follows that $d_e(uC_\phi) = \infty$

Remark 2.2. *From Example* (2.1) *it follows that for each $n \in \mathbb{N}$ there exist a weighted composition operator uC_ϕ on l^p such that $d_e(uC_\phi) = n - 1$.*

3. Example

Note that a linear operator T belongs to exactly one of the following cases:
1. $a_e(T) = d_e(T) =$ finite.
2. $a_e(T) = \infty$ but $d_e(T)$ is finite.
3. $d_e(T) = \infty$ but $a_e(T)$ is finite.
4. $a_e(T) = \infty$ and $d_e(T) = \infty$.

We give examples of weighted composition operators, exactly one for each of the above type, as follows:

Example 3.1. Let ϕ be a self-map on $\mathbb{N}$ defined as:

$$\phi(n) = \begin{cases} \text{n}, & \text{if } n \text{ is odd} \\ \text{n-1}, & \text{if } n \text{ is even .} \end{cases}$$

and

$$u = \left\{\tfrac{1}{n}\right\}_{n=1}^{\infty}$$

Then $a_e(uC_\phi) = 1$ and $d_e(uC_\phi) = 1$.

Example 3.2. Let ϕ be the self-map on $\mathbb{N}$ defined as :

$$\phi(p_k^n) = p_{k+1}^n \text{ for all } k \in \mathbb{N}.$$

Where $\{p_k : k \in \mathbb{N}\}$ denote the enumeration of primes.

$$\text{and } \phi(n) = n \text{ when } n \in \left(\mathbb{N} - \bigcup_{k\in\mathbb{N}} E_k\right)$$
$$\text{where } E_k = \{p_k^n : n \geq 1\} \text{ for each } k \in \mathbb{N}.$$

also

$$u = \left\{\tfrac{1}{n}\right\}_{n=1}^{\infty}$$

Then $a_e(uC_\phi) = \infty$ and $d_e(uC_\phi) = 0$.

Example 3.3. Let ϕ be the self-map on $\mathbb{N}$ defined as:

$$\phi(n) = n + 2, \text{ if } n \text{ is odd}$$

and

$$\phi(2n - 2) = \phi(2n) = n, \text{ if } n \text{ is even.}$$

also

$$u(n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ \text{-1}, & \text{if } n \text{ is even.} \end{cases}$$

Then $a_e(uC_\phi) = 0$ and $d_e(uC_\phi) = \infty$.

Example 3.4. Let $P = \bigcup_{k\in\mathbb{N}} \{p_k^n : n \in \mathbb{N}\}$ where p_k denote the k-th prime and $\mathbb{N} - P = \{q_k : k \geq 1\} = \{1, 6, 10, 12, \ldots\ldots\}$. Clearly $\mathbb{N} - P$ is an infinite subset of $\mathbb{N}$ and ϕ be the self-map on $\mathbb{N}$ defined as :

$$\phi(p_k^n) = p_{k+1}^n \text{ for all } k \in \mathbb{N}.$$

$$\phi(q_1) = \phi(q_2) = q_1$$

and

$$\phi(q_{2k-1}) = \phi(q_{2k}) = q_{2k-2} \text{ for each } k \geq 2.$$

also

$$u = \left\{\tfrac{1}{n}\right\}_{n=1}^{\infty}$$

Then it is easy to show that $a_e(uC_\phi) = \infty$ and $d_e(uC_\phi) = \infty$.

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