J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.1, No.1 (2012), pp.123-130

ON HYPERGEOMETRIC PROOF OF CERTAIN CONTINUED FRACTION RESULTS

Remy Y. Denis, S.N.Singh* and S.P. Singh*
Department of Mathematics, University of Gorakhpur,
Gorakhpur-273009, India
*Department of Mathematics, T.D.P.G. College,
Jaunpur-222002, India
(Corresponding author: Remy Y.Denis)
E-mail: ddry@rediffmail.com

(Received August 20, 2012)

Abstract :
In this paper, we provide hypergeometric proof of certain results and deduce a number of new and known results. This result is equivalent to Entry 12 of Chapter XVI of Ramanujan's second Notebook.
Keywords and Phrases : Basic hypergeometric functions, q-series, continued fractions, Ramanujan's theta functions,
2000 AMS Subject classification:33A30,33D15,33D20.

1. INTRODUCTION, NOTATION AND DEFINITION

Ramanujan's contribution to continued fractions associated with analytic functions is remarkable. His Notebooks contain a large number of beautiful results associated with hypergeometric functions (both, basic and ordinary) and continued fractions. Many of his continued fraction results can be provided with hypergeometric proof. In a recent publication Denis and Singh [2, 3] provided hypergeometric proof of Entries 25 and 33 of Chapter XII of Ramanujan's [5] Second Notebook, and also provided their basic analogues.

Motivated by the above results, we propose to provide hypergeometric proof of the following results.

$$\frac{[a^2q^3, b^2q^3; q^4]_\infty}{[a^2q, b^2q; q^4]_\infty} = \frac{1}{1-a^2q\ -}\,\frac{q(b^2-a^2q^2]}{1+q^2\ +}\,\frac{q(a^2-b^2q^2)}{(1-a^2q)(1+q^4)\ -}\,\frac{(q^5(b^2-a^2q^6)}{1+q^6\ +}$$

$$\frac{q(a^2-b^2q^6)}{+(1-a^2q)(1+q^8)}\ -\frac{q^9(b^2-a^2q^{10})}{1+q^{10}+.......} \tag{1.1}$$

where

$$[\alpha, \beta; p]_\infty = [\alpha; p]_\infty [\beta; p]_\infty$$

and $$[\alpha; p]_\infty = \prod_{r=0}^{\infty} (1 - \alpha p^r), \mid p \mid < 1.$$

Entry 12 of Chapter XVI of Ramanujan's second Notebook [5] mentions a different continued fraction representation for the left side of (1.1).The present result provides an equivalent representation for the infinite product on the left.

Before we proceed further, we introduce a basic hypergeometric function:
For any numbers a and q, real or complex and $\mid q \mid < 1$,let

$$[\alpha; q]_n \equiv [\alpha]_n = \left\{ \begin{array}{ll} (1-\alpha)(1-\alpha q)(1-\alpha q^2).........(1-\alpha q^{n-1}); n > 0 \\ 1 & ; n = 0 \end{array} \right\} \tag{1.2}$$

Accordingly, we have

$$[\alpha; q]_{-n} = \frac{(-)^n q^{n(n+1)/2}}{\alpha^n [q/\alpha; q]_n}.$$

Also,

$$[a_1, a_2, ..., a_r; q]_n \equiv [a_1; q]_n [a_2; q]_n ... [a_r; q]_n.$$

Now, we define a basic hypergeometric series,

$${}_r\phi_s \left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array} \right]$$

$$= \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n \{(-)^n q^{n(n-1)/2}\}^{1+s-r}}{[q, b_1, b_2, ..., b_s; q]_n}, \tag{1.3}$$

where $0 < \mid q \mid < 1$ and $r < s + 1$.

We define a basic bilateral hypergeometric function as,

$${}_r\psi_s \left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array} \right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n \{(-)^n q^{n(n-1)/2}\}^{s-r}}{[b_1, b_2, ..., b_s; q]_n}, \tag{1.4}$$

where $| b_1.b_2....b_s/a_1.a_2....a_r |<| z |< 1$.

2. Proof of (1.1).

In a publication Singh [4] established the following result,

$$\frac{{}_3\phi_2\left[\begin{matrix} a,b,c;qde/abc \\ d,e \end{matrix}\right]}{{}_3\phi_2\left[\begin{matrix} aq,b,c;q;de/abc \\ dq,e \end{matrix}\right]} =$$

$$= 1 - \frac{(de/abc)(a-d)(1-b)(1-c)/(1-e)(1-d)(1-dq)}{(1-e/aq/(1-e)+}$$

$$\frac{(e/aq)(1-aq)(1-dq/b)(1-dq/c)/(1-e)(1-dq)(1-aq^2)}{1\ \ -+}$$

$$\frac{(deq/abc)(a-dq)(1-bq)(1-cq)/(1-eq)(1-dq^2)(1-dq^3)}{(1-e/aq)/(1-eq)\ +}$$

$$\frac{(e/aq)(1-aq^2)(1-dq^2/b)(1-dq^2/c)/(1-eq)(1-dq^3)(1-dq^4)}{1\ -}$$

$$\frac{deq^2/abc)(a-dq^2)(1-bq^2)(1-cq^2)/(1-eq^2)(1-dq^4)(1-dq^5)}{(1-eaq)/(1-eq^2)\ +.........} \tag{2.1}$$

Now, setting $a = 1$ in (2.1) and then taking $d = 1$ in it we get,

$$\frac{[e/b, e/c; q]_\infty}{[e, e/bc]_\infty} =$$

$$= \frac{1}{1\ -}\frac{(e/bc)(1-b)(1-c)/(1-q)}{(1-e/q)\ +}$$

$$\frac{(e/q)(1-q)(1-q/b)(1-q/c)/(1-q)(1-q^2)}{1-}$$

$$\frac{(eq/bc)(1-q)(1-bq)(1-cq)/(1-q^2)(1-q^3)}{(1-e/q)\ +}$$

$$\frac{(e/q)(1-q^2)(1-q^2/b)(1-q^2/c)/(1-q^3)(1-q^4)}{1\ -}$$

$$\frac{(eq^2/bc)(1-q^2)(1-bq^2)(1-cq^2)/(1-q^4)(1-q^5)}{(1-e/q)\ +........} \tag{2.2}$$

The above can be simplified to

$$\frac{[e/b,e/c;q]_\infty}{[e,e/bc]_\infty}=$$

$$\frac{1}{(1-e/q))-}\ \frac{(e/bc)(1-b)(1-c)/(1-q)}{1+}\ \frac{(e/q)(1-q/b)(1-q/c)/(1-q^2)}{(1-e/q)\ -}$$

$$\frac{(eq/bc)(1-q)(1-bq)(1-cq)/(1-q^2)(1-q^3)}{1\ +}$$

$$\frac{(e/q)(1-q^2)(1-q^2/b)(1-q^2/c)/(1-q^3)(1-q^4)}{(1-e/q)\ -}$$

$$\frac{(eq^2/bc)(1-q^2)(1-bq^2)(1-cq^2)/(1-q^4)(1-q^5)}{1+\ ....} \tag{2.3}$$

Now, replacing q by q^4 and then replacing e,b and c by q^{x+n+5},q^{2n+2} and q^2, respectively, in (2.3),we get,

$$\frac{[q^{x-n+3},q^{x+n+3};q^4]_\infty}{[q^{x+n+1},q^{x-n+1};q^4]_\infty}=$$

$$\frac{1}{(1-q^{x+n+1})}\ -\ \frac{q^{x-n+1}(1-q^{2n+2})(1-q^2)/(1-q^4)}{1\ +}$$

$$\frac{q^{x+n+1}(1-q^{2-2n})(1-q^2)/(1-q^8)}{(1-q^{x+n+1})\ -}$$

$$\frac{q^{x-n+5}(1-q^4)(1-q^{2n+6}(1-q^6)/(1-q^8)(1-q^{12})}{1\ +}$$

$$\frac{q^{x+n+1}(1-q^8)(1-q^{6-2n})(1-q^6)/(1-q^{12})(1-q^{16})}{(1-q^{x+n+1})\ -\ ....} \tag{2.4}$$

Now , setting $q^{n+x}=a^2$,$q^{x-n}=b^2$ and $q^{2n}=a^2/b^2$ in (2.4), we get

$$\frac{[a^2q^3,b^2q^3;q^4]_\infty}{[a^2q,b^2q;q^4]_\infty}=\frac{1}{(1-a^2q)\ -}\ \ \frac{q(b^2-a^2q^2)/(1+q^2)}{1\ +}$$

$$\frac{q(a^2-b^2q^2)/(1+q^2)(1+q^4)}{(1-a^2q)\ -}\ \frac{q^5(b^2-a^2q^6)/(1+q^4)(1+q^6)}{1\ +}$$

$$\frac{q(a^2-b^2q^6)/(1+q^6)(1+q^8)}{(1-a^2q)\ -\ \ldots\ldots\ldots\ldots} \tag{2.5}$$

which can be put in the form,

$$\frac{[a^2q^3,b^2q^3;q^4]_\infty}{[a^2q,b^2q;q^4]_\infty}=\frac{1}{(1-a^2q)\ -}\ \frac{q(b^2-a^2q^2)}{(1+q^2)\ +}\ \frac{q(a^2-b^2q^2)}{(1-a^2q)(1+q^4)\ -}$$

$$\frac{q^5(b^2-a^2q^6)}{(1+q^6)\ +}\ \frac{q(a^2-b^2q^6)}{(1-a^2q)(1+q^8)\ -\ \ldots\ldots} \tag{2.6}$$

By an appeal to analytic continuation the result holds for general values of the parameters. This proves (1.1)

In Chapter 16 of the second notebook of Ramanujan [5], entry 12 states that,

$$\frac{[a^2q^3,b^2q^3;q^4]_\infty}{[a^2q,b^2q;q^4]_\infty}=\frac{1}{(1-ab)\ +}\,\frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)\ +}\,\frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)\ +}$$

$$\frac{(a-bq^5)(b-aq^5)}{(1-ab)(1+q^6)\ +\ \ldots\ldots} \tag{2.7}$$

The result (2.6) provides an equivalent continued fraction representation for the function on the left.

3. Special cases

In this section we shall discuss certain interesting special cases of our result (1.1).

Taking $a \to 0$ and $b \to 1$ in (1.1),we get the following known result (cf.Andrews and Berndt [1;p.156]),

$$\frac{[q^3;q^4]_\infty}{[q;q^4]_\infty}=\frac{1}{1\ -}\ \frac{q}{1+q^2\ -}\ \frac{q^3}{1+q^4\ -}\ \frac{q^5}{1+q^6\ -}\ \frac{q^7}{1+q^8\ -\ \ldots\ldots} \tag{3.1}$$

Also, $a \to 1$ and $b \to 0$ in (1.1) leads to

$$\frac{[q^3;q^4]_\infty}{[q;q^4]_\infty}=\frac{1}{1-q\ +}\ \frac{q^3}{1+q^2\ +}\ \frac{q}{(1-q)(1+q^4)\ +}\ \frac{q^{11}}{1+q^6\ +}$$

$$\frac{q}{+(1-q)(1+q^8)\ +}\ \frac{q^{19}}{1+q^{10}\ +}\ \frac{q}{(1-q)(1+q^{12}\ +\ldots\ldots} \tag{3.2}$$

Next, if we take $a = 0$ and $b = i$ and also $a = i$ and $b = 0$ in (1.1), we get the following two equivalent continued fractions,

$$\frac{[-q^3;q^4]_\infty}{[-q;q^4]_\infty} = \frac{1}{1\ +}\,\frac{q}{1+q\ +}\,\frac{q^3}{1+q^4\ +}\,\frac{q^5}{1+q^6\ +\ \ldots..} \tag{3.3}$$
$$(a=0, b=i)$$

$$= \frac{1}{1+q\ -}\,\frac{q^3}{1+q^2\ -}\,\frac{q}{(1+q)(1+q^4)}\ -\ \frac{q^{11}}{1+q^6\ -}$$
$$\frac{q}{-(1+q)(1+q^8)\ -}\,\frac{q^{19}}{1+q^{10}\ -\ \ldots..} \tag{3.4}$$
$$(a=i, b=0)$$

Next, with $a=b=i$, (1.1) yields,

$$\frac{[-q^3;q^4]^2_\infty}{[-q;q^4]^2_\infty} = \frac{1}{1+q\ +}\,\frac{q(1-q^2)}{1+q^2\ -}\,\frac{q(1-q^2)}{(1+q)(1+q^4)\ +}\,\frac{q^5(1-q^6)}{1+q^6\ -}$$
$$\frac{q(1-q^6)}{-(1+q)(1+q^8)\ +}\,\frac{q^9(1-q^{10})}{1+q^{10}\ -\ \ldots...} \tag{3.5}.$$

Further, setting $a=b=\sqrt{q}$ in n(1.1),we get the following interesting result involving Ramanujan's ψ- theta function,

$$\psi^2(q^2) = \frac{[q^4;q^4]^2_\infty}{[q^2;q^4]^2_\infty} = \frac{1}{1-q^2\ -}\,\frac{q^2(1-q^2)}{1+q^2\ +}\,\frac{q^2(1-q^2)}{(1-q^2)(1+q^4)\ -}\,\frac{q^6(1-q^6)}{1+q^6\ +}$$
$$\frac{q^2(1-q^6)}{+\,(1-q^2)(1+q^8)\ +}\,\frac{q^{10}(1-q^{10})}{1+q^{10}\ -\ldots....} \tag{3.6}$$

where

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{[q^2;q^2]_\infty}{[q;q^2]_\infty}$$

Again, taking $a=\omega i$ and $b=\omega^2 i$ ($\omega = e^{2\pi i/3}$) in (1.1),we get the following interesting result

$$\prod_{n=1}^{\infty}\frac{(1-q^{4n-1}+q^{8n-2})}{(1-q^{4n-3}+q^{8n-6})} = \frac{1}{1+\omega^2 q\ +}\,\frac{\omega q(1-\omega q^2)}{1+q^4)\ -}\,\frac{\omega^2 q(1-\omega^2 q^2)}{(1+\omega^2 q)(1+q^4)\ +}$$

$$\frac{\omega q^5(1-\omega q^6)}{+\ 1+q^6\ -}\,\frac{\omega^2 q(1-\omega^2 q^6)}{(1+\omega^2 q)(1+q^8)\ +}\,\frac{\omega q^9(1-\omega q^{10})}{1+q^{10}\ -}\,\frac{\omega q^2(1-\omega^2 q^{10})}{(1+\omega^2 q)(1+q^{12})\ +\ \ldots.} \tag{3.7}$$

If we put $a = b = q$ in (1.1), we get

$$\frac{[q;q^4]^2_\infty}{[q^3;q^4]^2_\infty} = \frac{(1-q^2)^2}{1-q^3\ -}\ \frac{q^3(1-q^2)}{1+q^2\ +}\ \frac{q^3(1-q^2)}{(1-q^3)(1+q^4)\ -}\ \frac{q^7(1-q^6)}{1+q^6\ +}$$
$$\frac{q^3(1-q^6)}{+(1-q^3)(1+q^8)\ -}\ \frac{q^{11}(1-q^{10})}{1+q^{10}\ +\ .....} \tag{3.8}$$

Further,for $a = 0$ and $b = q$ in (2.1),we get,

$$\frac{[q;q^4]_\infty}{[q^3;q^4]_\infty} = \frac{1-q}{1\ -}\ \frac{q^3}{1+q^2\ -}\ \frac{q^5}{1+q^4\ -}\ \frac{q^7}{1+q^6\ -\ ...} \tag{3.9}$$

Lastly, if we replace q by q^2 in (1.1) and then set $a = q^{3/2}$ and $b = \sqrt{q}$ in it,we get

$$\frac{[q,q^7;q^8]_\infty}{[q^3,q^5;q^8]_\infty} = \frac{1-q}{1-q^5\ -}\ \frac{q^3(1-q^6)}{1+q^4\ +}\ \frac{q^5(1-q^2)}{(1-q^5)(1+q^8)\ -}\ \frac{q^{11}(1-q^{14})}{1+q^{12}\ +}$$
$$\frac{q^5(1-q^{10})}{+\ (1-q^2)(1+q^{16})\ -\ ...} \tag{3.10}$$

A number of other special cases could easily be deduced.

Acknowledgement

The authors are thankful to the Department of Science and Technology, Govt. of India, New Delhi, for support under major research projects No. SR/S4/MS-461/07 dtd.13.2.2008 entitled" A study of basic hypergeometric functions with special reference to Ramanujan mathematics"; No.SR/S4/MS:524 dtd.10.2.2008 entitled "Glimpses of Ramanujan's mathematics in the field of $q-series$" and No.SR/S4/MS:561/08,entitled "Investigations of Ramanujan's work in the field of basic hypergeometric series " sanctioned to them, respectively. Third author is also thankful to the authorities of UGC for sanctioning a minor research project no. F. No. 8-3 (115)/2011(MRP/NRCB) to him

References

[1] Andrews, G.E.and Berndt,B.C., Ramanujan's Lost Notebook Part I, Springer (2005

[2] Denis,R.Y. and Singh,S.N., Hypergeometric functions and continued fractions,Far East J.MathSc.2(3)(2000),385-400.

[3] Denis,R.Y. and Singh,S.N., On certain results involving q-series and continued fractions,Far East J.MathSc.3(5)(2001),731-737.

[4] Singh,S.N., On basic hypergeometric series and continued fractionsm,Math.Student ,56(1988),91-96.

[5] Ramanujan,S.,Notebook of Srinivasa Ramanujan Vol.II,Tata Institute of Fundamental Research, Bombay (1957).