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SOME COMBINATORIAL PROPERTIES OF n-COLOUR COMPOSITIONS

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Dedicated to Prof. George E. Andrews on his 70th Birthday

Abstract: A connection between a special kind of walks in the XY-plane studied by J.H. van Lint and R.M. Wilson [7] and n-colour compositions introduced recently by A.K. Agarwal [1] is shown. This leads to several new combinatorial properties of the walks and also gives a new binomial identity with its combinatorial interpretation.

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1. Introduction

A partition of a positive integer n is a finite non-increasing sequence of positive integers whose sum is n. The Ferrers graph of a partition $t_1, \cdots, t_i$ of n is a set of i rows of equi-spaced dots aligned on the left where the j^{th} row has t_j dots. For example, the Ferrers graph of the partition 4+4+3+2+1 of 14 is

$$\begin{array}{llll} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \\ \cdot & & & \end{array}$$

If the graph is read vertically by columns then this represents the partition 5+4+3+2 of 14. This new partition is called the conjugate of the given partition.

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Definition 1.1. A partition is said to be self-conjugate if it is identical with its conjugate. For example, 3+2+1 and 3+3+3 are self conjugate partitions.

Definition 1.2(Andrews [4]). An "odd-even" partition of a positive integer ν is a partition in which the parts (arranged in ascending order) alternate in parity starting with the smallest part odd. Thus for example , the "odd-even" partitions of 7 are: 7, 1+6, 3+4.

Definition 1.3(MacMahon [6]). Ordered partitions are called compositions. For example, there are 8 compositions of 4, viz., 4, 31, 13, 22, 211, 121, 112, 1111. In the sequel we shall denote the number of compositions of n by $c(n)$.

Definition 1.4(Frobenius [5]). A two-rowed array of nonnegative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

where $a_1 > a_2 > \cdots > a_r \geq 0,\ b_1 > b_2 > \cdots > b_r \geq 0$ and $\nu = r + \Sigma_{i=1}^{r} a_i + \Sigma_{i=1}^{r} b_i$ is called a Frobenius partition of ν.

Definition 1.5(Agarwal and Andrews [3]). An n-colour partition (or, a partition with "n copies of n") is a partition in which a part of size $n \geq 1$ can come in n different colours denoted by subscripts : $n_1, n_2, \cdots, n_n$ and parts satisfy the order $1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < 5_1 < 5_2 \cdots$ For example, there are six n-colour partitions of 3, viz., $3_1, 3_2, 3_3, 2_1 + 1_1, 2_2 + 1_1, 1_1 + 1_1 + 1_1$.

Definition 1.6(Agarwal[1]). An n-colour ordered partition is called an n-colour composition. For example, there are eight n-colour compositions of 3, viz., $3_1, 3_2, 3_3, 2_2 + 1_1, 1_1 + 2_2, 2_1 + 1_1, 1_1 + 2_1, 1_1 + 1_1 + 1_1$. In the sequel we shall denote the number of n-colour compositions of ν by $C(\nu)$.

Definition 1.7. The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined as $F_0 = 0,\ F_1 = 1$, and $F_n = F_{n-1} + F_{n-2},\ n \geq 2$.

Definition 1.8(Van Lint and Wilson [7]). Consider walks in the XY-plane where each step is $S : (x, y) \to (x + 1, y)$ or $T_a : (x, y) \to (x, y + a)$ with 'a' a positive integer. For $\nu \geq 1$, let A_ν denote the number of walks from (0,0) that contain a point on the line $x + y = \nu$.

Example 1.1. For $\nu = 3$, there are 13 relevant walks viz., $SSS,\ ST_2,\ T_2S,\ ST_1T_1,\ T_1T_1S,\ T_1ST_1,\ SST_1,\ ST_1S, T_1SS,\ T_3,\ T_2T_1,\ T_1T_2,\ T_1T_1T_1$.

2. Explicit Formula

In this section we shall prove an explicit formula for A_ν.

Theorem 2.1. For $\nu \geq 1$, $A_\nu = 1+\sum_{l=0}^{\nu-1}\sum_{m=1}^{\nu-l}\binom{\nu-l-1}{m-1}\binom{l+m}{l}$. (2.1)

Proof of Theorem 2.1. We split the walks enumerated by A_ν into three classes:

(i) which contain S steps only,

(ii) which contain T steps only, and

(iii) which contain both S steps as well as T steps.

In the first class there is only one relevant walk, viz., the walk which contains ν number of S steps and in the second class the number of relevant walks is the number of compositions of ν, that is, $c(\nu)$. In the third class, let the number of S steps be l, $1 \leq l \leq \nu-1$ and the number of T steps be m, $1 \leq m \leq \nu-l$. Since the number of T steps can be m in $c(\nu-l,m)$ ways, where $c(\nu-l,m)$ is the number of compositions of $\nu-l$ into m parts which is equal to $\binom{\nu-l-1}{m-1}$ (cf, MacMahon [6],Vol.1, p.151). We see that the number of relevant walks in the third case is $c(\nu-l,m)\binom{l+m}{l}$, $1 \leq l \leq \nu-1,\ 1 \leq m \leq \nu-l$.

Combining all the cases, we get

$$\begin{aligned} A_\nu &= 1+c(\nu)+\sum_{l=1}^{\nu-1}\sum_{m=1}^{\nu-l}c(\nu-l,m)\binom{l+m}{l} \\ &= 1+\sum_{m=1}^{\nu}\binom{\nu-1}{m-1}+\sum_{l=1}^{\nu-1}\sum_{m=1}^{\nu-l}\binom{\nu-l-1}{m-1}\binom{l+m}{l} \\ &= 1+\sum_{l=0}^{\nu-1}\sum_{m=1}^{\nu-l}\binom{\nu-l-1}{m-1}\binom{l+m}{l}. \end{aligned}$$

3. Recurrence Formula

Theorem 3.1. $A_1 = 2,\ A_2 = 5$, and $A_\nu = 3A_{\nu-1} - A_{\nu-2}$ for $\nu > 2$.

Proof of Theorem 3.1. We have

$$
\begin{aligned}
A_\nu &= 1+\sum_{l=0}^{\nu-1}\sum_{m=1}^{\nu-l}\binom{\nu-l-1}{m-1}\binom{l+m}{l}\\
&= 1+\sum_{l=0}^{\nu-2}\sum_{m=1}^{\nu-l-1}\binom{\nu-l-1}{m-1}\binom{l+m}{l}+\nu+\sum_{l=0}^{\nu-2}\binom{\nu}{l}\\
&= 1+\sum_{l=0}^{\nu-2}\sum_{m=1}^{\nu-l-1}\left\{\binom{\nu-l-2}{m-1}+\binom{\nu-l-2}{m-2}\right\}\binom{l+m}{l}+\nu+\sum_{l=0}^{\nu-2}\binom{\nu}{l}\\
&= 1+\{A_{\nu-1}-1\}+\sum_{l=0}^{\nu-2}\sum_{m=1}^{\nu-l-1}\binom{\nu-l-2}{m-2}\binom{l+m-1}{l}\\
&\quad+\sum_{l=0}^{\nu-2}\sum_{m=1}^{\nu-l-1}\binom{\nu-l-2}{m-2}\binom{l+m-1}{l-1}+\nu+\sum_{l=0}^{\nu-2}\binom{\nu}{l}\\
&= A_{\nu-1}+\sum_{l=0}^{\nu-2}\sum_{m=0}^{\nu-l-2}\binom{\nu-l-2}{m-1}\binom{l+m}{l}\\
&\quad+\sum_{l=0}^{\nu-2}\sum_{m=1}^{\nu-l-1}\left\{\binom{\nu-l-1}{m-1}-\binom{\nu-l-2}{m-1}\right\}\binom{l+m-1}{l-1}+\nu+\sum_{l=0}^{\nu-2}\binom{\nu}{l}\\
&= A_{\nu-1}+\left\{A_{\nu-1}-1-\sum_{l=0}^{\nu-2}\binom{\nu-1}{l}\right\}\\
&\quad+\sum_{l=-1}^{\nu-3}\sum_{m=1}^{\nu-l-2}\left\{\binom{\nu-l-2}{m-1}-\binom{\nu-l-3}{m-1}\right\}\binom{l+m}{l}+\nu+\sum_{l=0}^{\nu-2}\binom{\nu}{l}\\
&= \ 2A_{\nu-1}-1-\sum_{l=0}^{\nu-2}\binom{\nu-1}{l}+\sum_{l=0}^{\nu-3}\sum_{m=1}^{\nu-l-2}\binom{\nu-l-2}{m-1}\binom{l+m}{l}\\
&\quad-\sum_{l=0}^{\nu-3}\sum_{m=1}^{\nu-l-2}\binom{\nu-l-3}{m-1}\binom{l+m}{l}+\nu+\sum_{l=0}^{\nu-2}\binom{\nu}{l}\\
&= 2A_{\nu-1}-1-\sum_{l=0}^{\nu-2}\binom{\nu-1}{l}+\left\{A_{\nu-1}-1-\sum_{l=0}^{\nu-3}\binom{\nu-1}{l}-(\nu-1)\right\}-\{A_{\nu-2}-1\}\\
&\quad+\nu+\sum_{l=0}^{\nu-2}\binom{\nu}{l}\\
&= 3A_{\nu-1}-A_{\nu-2}.
\end{aligned}
$$

4. Generating Function

Theorem 4.1. Let $M(q)$ be the enumerative generating function for A_ν. Then

$$M(q) \;=\; \frac{q(2-q)}{q^2-3q+1}.$$

Proof of Theorem 4.1.

$$\begin{aligned}
M(q) &= \sum_{\nu=1}^{\infty} A_\nu q^\nu \\
&= A_1 q + A_2 q^2 + \sum_{\nu=3}^{\infty} A_\nu q^\nu \\
&= 2q + 5q^2 + \sum_{\nu=3}^{\infty} (3A_{\nu-1} - A_{\nu-2}) q^\nu \\
&= 2q + 5q^2 + 3\sum_{\nu=3}^{\infty} A_{\nu-1} q^\nu - \sum_{\nu=3}^{\infty} A_{\nu-2} q^\nu \\
&= 2q + 5q^2 + 3\sum_{\nu=2}^{\infty} A_\nu q^{\nu+1} - \sum_{\nu=1}^{\infty} A_\nu q^{\nu+2} \\
&= 2q + 5q^2 + 3M(q)q - 6q^2 - M(q)q^2
\end{aligned}$$

$$\begin{aligned}
(q^2-3q+1)M(q) &= 2q - q^2 \\
M(q) &= \frac{q(2-q)}{q^2-3q+1}.
\end{aligned}$$

5. Connection of A_ν with n-Colour Compositions

From Theorem 2, we have $A_1 = 2$, $A_2 = 5$, and $A_\nu = 3A_{\nu-1} - A_{\nu-2}$ for $\nu > 2$. But $F_{2\nu+1}$ also satisfies the same initial conditions and the same recurrence relation. $\Rightarrow A_\nu = F_{2\nu+1}$.

Also $F_{2\nu+1} = F_{2\nu+2} - F_{2\nu}$ and $F_{2\nu} = C(\nu)$, (cf. Agarwal [1]).

Therefore,

$$A_\nu \;=\; C(\nu+1) - C(\nu). \tag{5.1}$$

6. New Combinatorial Properties of A_ν

In this section we give six new combinatorial properties of A_ν without proofs as they follow easily by using the methods of proofs of Section 3 of [2] for the first five properties and using the fact that the number of compositions of n into parts ≤ 2 equals F_{n+1} for the sixth.

Theorem 6.1. A_ν is equal to the number of "odd-even" partitions with largest part $2\nu+1$.

Example 6.1. For $\nu=2$, there are five relevant "odd-even" partitions, viz., 5, 5+4+3+2+1, 5+4+3, 5+4+1, 5+2+1. We have earlier seen that $A_2=5$.

Theorem 6.2. A_ν is equal to the number of compositions of $2\nu+1$ into odd parts.

Example 6.2. For $\nu=2$, there are five relevant compositions of 5, viz., 5, 3+1+1, 1+3+1, 1+1+3, 1+1+1+1+1.

Theorem 6.3. A_ν equals the number of "odd-even" partitions with largest part 1 or even and $\leq 2\nu$.

Example 6.3. For $\nu=2$, there are five relevant "odd-even" partitions, viz., 1, 4+3+2+1, 4+3, 4+1, 2+1.

Theorem 6.4. A_ν is equal to the number of self-conjugate partitions with largest part $2\nu+1$ such that in the Frobenius notation $\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$, r is odd and a_i alternate in parity.

Example 6.4. For $\nu=2$, there are five relevant self-conjugate partitions, viz.,
$5^5,\ 5^33^2,\ 5^232^2,$
$53^21^2,\ 51^4$.

Theorem 6.5. A_ν equals the number of partitions into an odd number of odd parts with largest part $4\nu+1$ such that the parts are alternately $\equiv$1 and 3 (mod 4).

Example 6.5. For $\nu=2$, there are five relevant partitions, viz., 9, 9+3+1, 9+7+1, 9+7+5, 9+7+5+3+1.

Theorem 6.6. A_ν equals the number of compositions of 2ν where each part is less than or equal to 2.

Example 6.6. For $\nu=2$, there are five relevant compositions, viz., 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1.

7. A New Binomial Identity With its Combinatorial Interpretation

It is known that

$$C(\nu) = \sum_{m=1}^{\nu} \binom{\nu+m-1}{2m-1} \quad \text{(cf. Agarwal [1]).} \tag{7.1}$$

Therefore, (2.1),(5.1) and (7.1) give rise to a new binomial identity stated in the following theorem.

Theorem 7.1. For $\nu \geq 1$,

$$1+\sum_{l=0}^{\nu-1}\sum_{m=1}^{\nu-l}\binom{\nu-l-1}{m-1}\binom{l+m}{l}=\sum_{m=1}^{\nu+1}\binom{\nu+m-1}{2m-2}.$$

The following theorem provides a combinatorial interpretation of Theorem 7.1.

Theorem 7.2. For $\nu \geq 1$, the number of walks in the XY-plane starting from (0,0) that contain a point on the line $x+y=\nu$ taking S or T_a steps is equal to the number of n-colour compositions of $\nu+1$ without 1_1 on left(right) extreme.

Proof of Theorem 7.2. It is an immediate consequence of (2.1),(5.1) and the fact that the n-colour compositions of ν are in 1-1 correspondence with the n-colour compositions of $\nu+1$ with 1_1 on left(right) extreme.

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