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ON A THREE VARIABLE RECIPROCITY THEOREM

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Dedicated to Professor G. E. Andrews on his seventieth birthday

Abstract: In this note, we give a proof of three variable reciprocity theorem using q-binomial theorem and Gauss summation formula.

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1. Introduction

In his "lost" notebook [11] Ramanujan stated several results related to q-series and one of them is the following beautiful reciprocity theorem:

$$\rho(a,b) - \rho(b,a) = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(aq/b)_\infty \, (bq/a)_\infty \, (q)_\infty}{(-aq)_\infty \, (-bq)_\infty} \tag{1.1}$$

where

$$\rho(a,b) := \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(-1)^n \, q^{n(n+1)/2} \, a^n \, b^{-n}}{(-aq)_n}, \quad a \neq -q^{-n} \text{ and } |q| < 1,$$

$$(a)_0 := (a;q)_0 = 1,$$

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$$

and

$$(a)_n := (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad -\infty < n < \infty.$$

The first proof of (1.1) was given by Andrews [2] using four free-variable identity and Jacobi's triple product identity. Further, in his paper [3], Andrews proved two nice entries from "lost" notebook of Ramanujan related to Euler's partition identity stating that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts, which uses (1.1). Bhargava, Somashekara and Fathima [7] proved (1.1) using Ramanujan's ${}_1\psi_1$ summation formula and Heine's transformation for ${}_2\phi_1$-series. Using the same above mentioned two transformations, Berndt, Chan, Yeap and Yee [6] also proved (1.1). In fact, Berndt *et al.* [6] in the same paper have given two more proofs of (1.1) one using an identity of N. J. Fine and the other is purely combinatorial. Also, Adiga and Anitha [1] proved (1.1) which uses only Heine's transformation for ${}_2\phi_1$-series and further, they showed that the reciprocity Theorem 1.1 leads to a q-integral extension of the classical gamma function. Recently, Guruprasad and Pradeep [8] proved (1.1) using only q-binomial theorem.

Somashekara and Fathima [12] derived a generalization of Jacobi's triple product identity which is equivalent to (1.1) using Ramanujan's ${}_1\psi_1$ summation formula and the same was derived from q-binomial theorem by Kim, Somashekara and Fathima [10].

In [9], Kang generalized (1.1) as follows:

If $|c| < |a| < 1$ and $|c| < |b| < 1$, then

$$\rho_3(a,b,c) - \rho_3(b,a,c) \ = \ \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c)_\infty\,(aq/b)_\infty\,(bq/a)_\infty\,(q)_\infty}{(-c/a)_\infty\,(-c/b)_\infty\,(-aq)_\infty\,(-bq)_\infty} \tag{1.2}$$

where

$$\rho_3(a,b,c) := \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(c)_n\ (-1)^n\ q^{n(n+1)/2}\ a^n\ b^{-n}}{(-aq)_n\ (-c/b)_{n+1}}, \qquad a, c/b \neq -q^{-n}$$

for $n \in \mathbb{Z}^+$ and $|q| < 1$.

Kang [9] established (1.2) on employing Ramanujan's ${}_1\psi_1$ summation formula and Jackson's transformation of ${}_2\phi_1$ and ${}_2\phi_2$-series [4]. In fact, in the same paper she also generalized reciprocity theorem (1.1) for four variables.

The main purpose of this paper is to provide a new proof of (1.2) using q-binomial theorem [4],[5]

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n}\, t^n \ = \ \frac{(at)_\infty}{(t)_\infty}, \qquad |t| < 1, \quad |q| < 1, \tag{1.3}$$

and the Gauss summation formula [4], [5]

$$\sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n}{(c)_n\ (q)_n}\,(c/ab)^n \ = \ \frac{(c/a)_\infty\,(c/b)_\infty}{(c)_\infty\,(c/ab)_\infty}, \quad |c/ab| < 1, \ |q| < 1. \tag{1.4}$$

Note that Gauss summation formula (1.4) can be easily derived using only q-binomial theorem (1.3).

Changing a to a/b, t to bt, and letting $b \to 0$ in (1.3), we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n \ a^n \ q^{\frac{n(n-1)}{2}}}{(q)_n} t^n \ = \ (at)_\infty, \quad |q| < 1. \tag{1.5}$$

Putting $a = -1$ in the above identity, we deduce

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q)_n} t^n \ = \ (-t)_\infty, \quad |q| < 1. \tag{1.6}$$

2. Proof of the Three Variable Reciprocity Theorem

Before proving the main result, we prove a lemma.

Lemma 2.1. We have

$$\text{(i)} \quad \sum_{n=0}^{\infty} \frac{(c)_n \ q^{n(n+1)/2} \ z^n}{(dz)_{n+1} \ (-c/d)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-q/d)_n}{(-c/d)_{n+1}} (dz)^n, \ \ |dz| < 1 \ \text{and} \ |q| < 1, \tag{2.1}$$

and

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \frac{(c)_{n-1} \, q^{n(n-1)/2} \, z^{-n}}{(-d)_n \, (c/zd)_n} = \frac{1}{(1+c/d)} \sum_{n=1}^{\infty} \frac{(-d/c)_n}{(-d)_n} (c/zd)^n, \tag{2.2}$$

$|c/zd| < 1$ and $|q| < 1$.

Proof (i). Clearly the left side summation of (2.1) can be written as

$$\frac{(c)_\infty (q)_\infty}{(dz)_\infty \ (-c/d)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} \ z^n}{(q)_n} \ \frac{(dzq^{n+1})_\infty \, (-cq^{n+1}/d)_\infty}{q^{n+1})_\infty \ (cq^n)_\infty}.$$

Now using q-binomial theorem (1.3) and then changing the order of the summation in the above identity we obtain

$$\frac{(c)_\infty \, (q)_\infty}{(dz)_\infty \, (-c/d)_\infty} \sum_{t=0}^{\infty} \frac{(-q/d)_t}{(q)_t} c^t \sum_{m=0}^{\infty} \frac{(dz)_m}{(q)_m} q^m \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q)_n} \ (zq^{t+m+1})^n.$$

Employing (1.3), (1.5) and (1.6) in the above identity and after some simple manipulations, we obtain right side summation of (2.1).

(ii). Proof of (2.2) is similar.

Now we prove the main result (1.2).

We have

$$\sum_{n=-m}^{\infty} \frac{(-q/d)_n (cq^m/dz)_n (zd)^n}{(q^{1+m})_n (-cq/d)_n} = \sum_{n=0}^{\infty} \frac{(-q/d)_{n-m} (cq^m/dz)_{n-m} (zd)^{n-m}}{(q^{1+m})_{n-m} (-cq/d)_{n-m}}$$

$$= \frac{(zd)^{-m} (-q/d)_{-m} (cq^m/dz)_{-m}}{(q^{1+m})_{-m} (-cq/d)_{-m}} \sum_{n=0}^{\infty} \frac{\left(-q^{1-m}/d\right)_n (c/dz)_n (zd)^n}{(q)_n \left(-cq^{1-m}/d\right)_n}$$

$$= \frac{(zd)^{-m} (-q/d)_{-m} (cq^m/dz)_{-m}}{(q^{1+m})_{-m} (-cq/d)_{-m}} \frac{(c)_\infty \left(-zq^{1-m}\right)_\infty}{(-cq^{1-m}/d)_\infty (zd)_\infty}, \quad \text{on using (1.4)}$$

$$= \frac{q^m (1/q^m)_m (-d/c)_m}{(-d)_m (dz/cq^{m-1})_m} \frac{(c)_\infty \left(-zq^{1-m}\right)_\infty}{(-cq^{1-m}/d)_\infty (zd)_\infty}$$

$$= \frac{(q)_m (-1/z)_m (c)_\infty (-zq)_\infty}{(-d)_m (c/dz)_m (-cq/d)_\infty (zd)_\infty}. \tag{2.3}$$

Letting $m \to \infty$ in (2.3) and then multiplying by $\frac{1}{(1+c/d)}$ we obtain

$$\sum_{n=0}^{\infty} \frac{(-q/d)_n}{(-c/d)_{n+1}} (dz)^n + \frac{1}{(1+c/d)} \sum_{n=1}^{\infty} \frac{(-d/c)_n}{(-d)_n} (c/zd)^n$$

$$= \frac{(c)_\infty (-zq)_\infty (-1/z)_\infty (q)_\infty}{(c/dz)_\infty (-c/d)_\infty (zd)_\infty (-d)_\infty}.$$

Using (2.1) and (2.2) in the above identity we obtain

$$\sum_{n=0}^{\infty} \frac{(c)_n q^{n(n+1)/2} z^n}{(dz)_{n+1} (-c/d)_{n+1}} + \sum_{n=1}^{\infty} \frac{(c)_{n-1} q^{n(n-1)/2} z^{-n}}{(-d)_n (c/zd)_n}$$

$$= \frac{(c)_\infty (-zq)_\infty (-1/z)_\infty (q)_\infty}{(c/dz)_\infty (-c/d)_\infty (zd)_\infty (-d)_\infty}.$$

Multiplying both sides by $z(1-zd)(1+d)$ we get

$$(1+d) \sum_{n=0}^{\infty} \frac{(c)_n q^{n(n+1)/2} z^{n+1}}{(dzq)_n (-c/d)_{n+1}} + (1-zd) \sum_{n=0}^{\infty} \frac{(c)_n q^{n(n+1)/2} z^{-n}}{(-dq)_n (c/zd)_{n+1}}$$

$$= (1+z) \frac{(c)_\infty (-zq)_\infty (-q/z)_\infty (q)_\infty}{(c/dz)_\infty (-c/d)_\infty (zdq)_\infty (-dq)_\infty}.$$

Now changing $d \to b$ and $z \to -a/b$; then multiplying the resulting equation by $-1/a$ we obtain (1.2).

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