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SOME REMARKS ON WIENER-HOPF EQUATIONS AND VARIATIONAL INEQUALITIES IN BANACH SPACES

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Abstract : In this paper, we consider a class of implicit variational inequalities in Banach spaces and prove its equivalence with a class of Wiener-Hopf equations. Further, using this equivalence, we suggest and analyze a Mann type iterative algorithm for finding the appropriate solution of the class of Wiener-Hopf equation and discuss its convergence criteria. The theorems in the paper extend and improve many known results in the literature.

Keywords : Implicit variational inequality, Wiener-Hopf equation, Mann type iterative algorithm, sunny retraction, η-strongly accretive mapping.

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1. Introduction :

Let B be a real Banach space and let $T, g: B \to B$ be two nonlinear mappings. Let K be a non-empty closed convex set in B, we consider an implicit variational inequality problem (IVIP) of finding $u \in B$ such that $g(u) \in K$ and

$$\langle Tu, J(v - g(u))\rangle \geq 0, \quad \forall v \in K, \tag{1.1}$$

where $J: B \to B^*$ is the normalized duality mapping defined by the condition:

$$\langle x, Jx\rangle = \|x\|^2 = \|Jx\|^2, \quad \forall x \in B,$$

where <.,.> denotes the normalized duality pairing. Some properties and examples of J can be found in [1].

Special Cases: If $B \equiv H$, a Hilbert space and if $g(B) \subseteq K$, for any $v \in B$, $g(v) \in K$, and then IVIP reduces to the general variational inequality of finding $u \in H$ such that $g(u) \in K$.

$$<Tu,\ g(v) - g(u)> \underline{>} 0, \quad \forall g(v) \in K.$$

This problem represents odd order boundary value problems, see Noor [5].

We note that for suitable choices of the mappings T and g, IVIP (1.1) reduces to the well known forms of variational inequalities studied by various authors in Hilbert spaces, see for example Noor [5, and the references therein].

2. Preliminaries :

We first define the following concepts:

Definition 2.1. Let B be a real Banach space and let $\eta: B \times B \to B$ be a continuous mapping. A mapping $T: B \to B$ is said to be

(i) □–η–strongly accretive if there exists □>0 such that

$$<Tu - Tv,\ J\eta(u,v)> \underline{>} \square ||u-v||^2, \ \forall u,v \in B;$$

(ii) □–Lipschitz continuous if there exists a constant □>0 such that

$$||Tu - Tv|| \underline{<} \square ||u-v||, \ \forall u,v \in B.$$

Definition 2.2[2]. Let K be a nonempty closed convex subset of B. A mapping $R_K: B \to K$ is said to be retraction on K, if $R^2_K = R^2_K$ .The mapping R_K is said to be a nonexpansive retraction if, in addition

$$||R_K(u) - R_K(v)|| \underline{<} ||u-v||, \ \forall u,v \in B,$$

and R_K is a sunny retraction if for all $u \in B$,

$$R_K(R_K u+t(u-R_K(u)) = R_K(u)\ ,\ \forall t\in B.$$

Now, we shall give the following characterization of a sunny nonexpansive retraction mapping which can be found, e.g., in [3].

Lemma 2.1. Let R_K be a retraction, then R_K is a sunny nonexpansive retraction if and only if for all $u,v\in$ B,

$$<u-R_K\, u, J(R_K\, u-v)> \ \underline{>}\ 0.$$

We also need the following result.

Lemma 2.2[2]. Let B be a Banach space. Then for all $u,v\in B$, we have

$$||u+v||^2 \leq ||u||^2 + 2<v,J(u+v)>.$$

Let R_K be the retraction mapping of B into K and let $Q_K=I-R_K$, where I is the identity operator. If g^{-1} exists, then we consider the problem of finding $z\in B$ such that

$$Tg^{-1}R_K z+\square^{-1}Q_K z=0, \tag{2.1}$$

where $\square>0$ is a constant. Equations of the type (2.1) are called implicit Wiener–Hopf equations. For the general treatment of Wiener–Hopf equations, see [6].

3. Main Results :

Firstly, we shall prove the following result.

Theorem 3.1. The IVIP (1.1) has a solution $u\in B$ such that $g(u)\in K$, if and only if the implicit Wiener–Hopf equation (2.1) has a solution $z\in$ B, where

$$z = g(u)-\square Tu,$$

$$g(u) = R_K z, \tag{3.1}$$

where R_K is the retraction of B onto K and $\square>0$ is a constant.

Proof. Let $u \in B$ be the solution of (1.1). Then by Lemma 2.1, it follows that

$$g(u) = R_K [g(u) - \rho Tu]. \tag{3.2}$$

Using $Q_K = I - R_K$ and applying (3.1) repeatedly, we obtain

$$\begin{aligned} Q_K [g(u) - \rho Tu] &= g(u) - \rho Tu - R_K[g(u) - \rho Tu] \\ &= -\rho Tu \\ &= -\rho T g^{-1} R_K[g(u) - \rho Tu], \end{aligned}$$

from which it follows that

$$T g^{-1} R_K z + \rho^{-1} Q_K z = 0,$$

where

$$z = g(u) - \rho Tu,$$

and g^{-1} is the inverse of the operator g.

Conversely, suppose that $z \in B$ is a solution of (2.1). Then we have,

$$T g^{-1} R_K z = -\rho^{-1} Q_K z$$

or,

$$\rho T g^{-1} R_K z = -Q_K z = R_K z - z. \tag{3.3}$$

Now, from (3.3) and Lemma 2.1, for all $g(v) \in K$, we obtain

$$0 \leq \langle R_K z - z, J(v - R_K z)\rangle = \langle -\rho T g^{-1} R_K z, J(v - R_K z)\rangle.$$

Thus, $g(u) = R_K z$ is a solution of (1.1), and from (3.3), we have

$$\rho Tu = g(u) - z,$$

$$z = g(u) - \rho Tu.$$

Remark 3.1. It is obvious that IVIP (1.1) and Wiener–Hopf equations are equivalent. Using this equivalence and by some suitable rearrangement, one can suggest a number of new iterative algorithms for solving IVIP (1.1).

The implicit Wiener–Hopf equation (2.1) can be written as

$$Q_K z = -\square T g^{-1} R_K z,$$

which implies by using (3.1),

$$\begin{aligned} z &= R_K z - \square T g^{-1} R_K z \\ &= g(u) - \square T u. \end{aligned} \tag{3.4}$$

On the basis of this formulation, we shall propose the following iterative algorithm for solving IVIP (1.1).

Mann Type Iterative Algorithm (MTIA) 3.1. For a given $z_0 \in B$, compute z_{n+1} by the iterative scheme

$$g(u_n) = R_K z_n, \tag{3.5}$$

$$z_{n+1} = (1-\square_n)\, z_n + \square_n\, [g(u_n) - \square T u_n] \tag{3.5}$$

for $n=0,1,2,\ldots$, where $\{\square_n\}$ is the sequence in [0,1] satisfying the following conditions:

(i) $\square_0 = 1$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. .

We now study those conditions under which the approximate solution z_{n+1} obtained from MTIA 3.1 converges to the exact solution z of implicit Wiener–Hopf equation (2.1).

Theorem 3.2. Let B be a real Banach space. Let $T: B \to B$ be $\square$–η–strongly accretive and $\square$–Lipschitz continuous; $g: B \to B$ be

$\square$–η–strongly accretive and $\square$–Lipschitz continuous and let $\eta : B \times B \to B$ be $\square$– Lipschitz continuous. If $z_{n+1} \in B$ is the solution obtained from MTIA 3.1 and is the exact solution of the implicit Wiener–Hopf equation (2.1), then $z_{n+1} \to z$, strongly in B, for $\square > 0$ such that

$$\left| \rho - \frac{\alpha - \beta(\delta + \lambda)}{2\beta^2} \right| < \frac{\sqrt{l^2[\alpha - \beta(\delta+\lambda)]^2 - 2\beta^2(l^2\delta^2 - 1)}}{2\beta^2 l}, \tag{3.7}$$

$$l\alpha > l\beta(\delta + \lambda) + \beta\sqrt{2(l^2\delta^2 - 1)}, \quad l\delta > 1,$$

where $l = \dfrac{\lambda}{\nu}$.

Proof. Let $z \in B$ satisfy the implicit Wiener–Hopf equation. Now, equation (2.1) can be written as (3.1) and (3.4). Hence, from (3.4) and (3.6), we have

$$\|z_{n+1} - z\| = \|(1-\square_n)z_n + \square_n[g(u_n) - \square T(u_n)] - (1-\square_n)z - \square_n[g(u) - \square Tu]\|$$

$$\leq (1-\square_n)\|z_n - z\| + \square_n\|g(u_n) - g(u) - \square(Tu_n - Tu)\|. \tag{3.8}$$

Now, using Lemma 2.2, we have

$$\|g(u_n) - g(u) - \square(Tu_n - Tu)\|^2$$

$$\leq \|g(u_n) - g(u)\|^2 - 2\square\langle Tu_n - Tu, J(g(u_n) - g(u) - \square(Tu_n - Tu))\rangle$$

$$\leq \square^2\|u_n - u\|^2 - 2\square\langle Tu_n - Tu, J\eta(u_n, u)\rangle$$

$$- 2\square\langle Tu_n - Tu, J(g(u_n) - g(u) - \square(Tu_n - Tu)) - J\eta(u_n, u)\rangle$$

$$\leq (\square^2 - 2\square\square)\|u_n - u\|^2 + 2\square\|Tu_n - Tu\|$$

$$\times [\|g(u_n) - g(u)\| + \square\|Tu_n - Tu\| + \lambda\|u_n - u\|]$$

$$\leq [\square^2 - 2\square\square + 2\square\square(\square + \square\square + \square)]\|u_n - u\|^2.$$

Hence, (3.8) becomes

$$||z_{n+1}-z|| \leq (1-\square_n)||z_n-z||+\square_n\square_1||u_n-u|| \tag{3.9}$$

where $\square_1 := [\square^2-2\square(\square-\square(\square+\square\square+\square))]^{1/2}$.

Now, since g is □–η–strongly accretive mapping and η is □–Lipschitz continuous mapping, the we have

$$\square||u_n-u||\ ||g(u_n)-g(u)|| > \ < g(u_n)-g(u), \eta(u_n,u)> \ \geq \nu||u_n-u||^2$$

i.e.,
$$||u_n-u|| \leq \frac{\lambda}{\nu}||g(u_n)-g(u)||$$

$$= \frac{\lambda}{\nu}||R_K z_n-R_K z|| \leq \frac{\lambda}{\nu}||z_n-z||,$$

where we have used (3.1).

Using (3.9), we get

$$||z_{n+1}-z|| \leq (1-\square_n)||z_n-z||+\square_n\square||z_n-z||$$

$$= [(1-\square_n(1-\square)]||z_n-z||,$$

where $\square := \dfrac{\lambda}{\nu}\square_1$.

Now, by condition (3.7), it follows that $0<\square<1$, and hence by iteration, we have

$$||z_{n+1}-z|| \leq \prod_{i=0}^{n} (1-\square_i(1-\square))||z_0-z||. \tag{3.10}$$

Since $\sum_{n=0}^{\infty} \square_n=\infty$, $\square_0=1$, $\square_n\leq 1$ and $0 \leq \square<1$, then $\lim_{n\to\infty}\prod_{i=0}^{n} (1-\square_i(1-\square))=0$, see Kazmi [4], and hence (3.10) implies that $z_{n+1}\to z$ strongly in B that completes the proof.

Remark 3.2. It is clear that □ ≤ □ and □ ≤ □. Further, condition (3.7) is true for suitable values of constants, for example □=□=1; □=□=0.1; □=0.01; □=1.

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