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FUZZY MATROIDS FROM FUZZY VECTOR SPACES

Shabna O. K. and Sameena K.

PG & Research Department of Mathematics,
MES Mampad College (Autonomous), Kerala - 676542, INDIA

E-mail : shabnanoushadok@gmail.com, sameena@mesmampadcollege.edu.in

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Abstract: In this paper, we have made an attempt to obtain a fuzzy matroid from a fuzzy vector space. As a result, the concept of representable fuzzy matroid is presented with some properties. It is proved through an example that a graphic fuzzy matroid cannot be representable over any field, in general. Also, we established a sufficient condition that a graphic fuzzy matroid occur representable over any given field.

Keywords and Phrases: Fuzzy matroid, fuzzy vector matroid, representable fuzzy matroid.

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1. Introduction

Theory of matroids has vast applications in combinatorial optimization problems, Operation research, system analysis as an abstract generalization of a graph and a matrix. Graphic and representable matroids form a fundamental class of crisp matroids. Goetschel and Voxman [4] characterised fuzzy matroids. In that paper they generalised matroids to fuzzy fields using the concept of fuzzy independent set.

In [15], we presented the idea of inducing fuzzy matroid from a fuzzy graph, namely graphic fuzzy matroid and dealt with some properties of this class of fuzzy matroids. Since vector space is one of the motivation and basic examples of crisp matroids, in this paper we procure the construction of fuzzy matroids from fuzzy vector space, and we obtain a connection between graphic and representable fuzzy matroids.

A crisp matroid can be defined as follows.

Definition 1.1. [9] *A Matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:*

i. $\phi \in \mathcal{I}$

ii. *If $A \in \mathcal{I}$ and $A' \subseteq A$, then $A' \in \mathcal{I}$*

iii. *If A_1 and A_2 are in $\mathcal{I}$ and $|A_1| < |A_2|$, then there is an element e of $A_2 - A_1$ such that $A_1 \cup \{e\} \in \mathcal{I}$*

The members of $\mathcal{I}$ are the independent sets of M, the subsets of E those are not members of $\mathcal{I}$ are called dependent sets of M and E is called ground set of M.

Definition 1.2. [9] *Let E_1 and E_2 be two finite sets. Suppose that $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ are two matroids. M_1 and M_2 are isomorphic if there exists a mapping $\psi : E_1 \longrightarrow E_2$ such that ψ satisfies the following conditions:*

i. *ψ is a one-to-one correspondence,*

ii. *For each $X \subseteq E_1, X \in \mathcal{I}_1$ if and only if $\psi(X) \in \mathcal{I}_2$,*

denoted by $M_1 \cong M_2$. The mapping ψ s called an isomorphic mapping from M_1 to M_2.

The matroid derived from a subset U of a vector space V is called the *vector matroid* [9] induced by U, symbolized by $M[U]$. Thus, a *representable matroid* [9] is a matroid which is isomorphic to a vector matroid.

To define a fuzzy matroid we adopt the following notation.
A *fuzzy subset* [14] ν on a set E is a function $\nu : E \longrightarrow [0, 1]$ The family of fuzzy sets on E is symbolised by $\mathscr{F}(E)$. If $\nu, \mu \in \mathscr{F}$, then

$$supp\, \nu = \{e \in E \mid \nu(e) > 0\},$$

$$m(\nu) = inf\{\ \nu(e) \mid e \in supp(\nu)\},$$

$$C_r(\nu) = \{e \in E \mid \nu(e) \geq r\}, where\ 0 \leq r \leq 1,$$

$$\nu \vee \mu = max\{\ \nu,\ \mu\},$$

$$\nu \wedge \mu = min\{\ \nu,\ \mu\}.$$

If $\nu, \mu \in \mathscr{F}$, then we take down $\nu < \mu$ if

i. $\nu(e) \leq \mu(e)$ for each e in E ,

ii. $\nu(e) < \mu(e)$ for some e in E

A fuzzy matroid can be defined as follows.

Definition 1.3. [4] *Let E be a finite set and $\mathscr{I} \subseteq \mathscr{F}(E)$ be a nonempty family of fuzzy sets satisfying:*

i. If $\nu(x) \in \mathscr{I}$, $\mu \in \mathscr{F}(E)$, and $\mu < \nu$, then $\mu \in \mathscr{I}$ (Hereditary property)

ii. If ν, $\mu \in \mathscr{I}$ and $|supp\ \nu| < |supp\ \mu|$, then there exists $\xi \in \mathscr{I}$ such that

a. $\nu < \xi < \nu \vee \mu$

b. $m(\xi) \geq min\ \{\ m(\nu),\ m(\mu)\}$. (Exchange property)

Then the structure $\mathscr{M} = (E, \mathscr{I})$ is a fuzzy matroid on E.

Theorem 1.1. [4] *Let $\mathscr{M} = (E, \mathscr{I})$ be any fuzzy matroid, and consider r, where, $0 < r \leqslant 1$. Let*

$$\mathcal{I}_r = \{C_r(\nu) \mid \nu \in \mathscr{I}\}.$$

Then $M_r = (E,\ \mathcal{I}_r)$ is a (crisp) matroid for each r, $0 < r \leqslant 1$, on E.

In [15] we defined a graphic fuzzy matroid as follows.

Definition 1.4. *Let $G = (V, \sigma, \nu)$ be a fuzzy graph with $G^* = (V, E)$ as the corresponding underlying graph. Then for each r, where $0 < r \leq 1$, let*

$$E_r = \{e \in E \mid \nu(e) \geq r\}$$
$$\mathscr{F}_r = \{F \mid F\ is\ a\ forest\ in\ the\ (crisp) graph\ (V,\ E_r)\}$$
$$\mathscr{E}_r = \{\mathcal{E}(F) \mid F\ \in \mathscr{F}_r\}, where\ \mathcal{E}(F)\ is\ the\ set\ of\ edges\ in F.\}$$

If

$$\mathscr{I} = \{\nu \in \mathscr{F}(E) \mid C_r(\nu) \in \mathscr{E}_r\ for\ each\ r,\ where\ 0 < r \leq 1\ \}$$

Thus, $(E, \mathscr{I})$ is a fuzzy matroid, named fuzzy cycle matroid of G, symbolised by $\mathscr{M}_F(G)$. Any fuzzy matroid isomorphic to $\mathscr{M}_F(G)$ is called graphic fuzzy matroid.

2. Representable Fuzzy Matroids

This section focused on the construction of a fuzzy matroid from fuzzy vector space, and then a sufficient condition for a graphic fuzzy matroid occur representable over any given field is discussed.

Definition 2.1. [12] *Let W be a vector space over the field of real numbers and let $\nu : W \longrightarrow [0,1]$. Then the pair $\widehat{W} = (W,\ \nu)$ is called a fuzzy vector space if it satisfies the property that for all $a,\ b \in \mathbb{R}$ and $w_1, w_2 \in W$, we have $\nu(aw_1 + bw_2) \geq \nu(w_1) \wedge \nu(w_2)$.*

Definition 2.2. *If $\widehat{W} = (W, \nu)$ is a fuzzy vector space. Then, a finite set of vectors $\{w_k\}_{k=1}^n$ is fuzzy linearly independent in $\widehat{W}$ if and only if $\{w_k\}_{k=1}^n$ is linearly independent in W and for all $\{a_i\}_{k=1}^n \subset \mathbb{R}$,*

$$\nu\left(\sum_{k=1}^{n} a_k w_k\right) = \wedge_{k=1}^{n} \nu(a_k w_k)$$

Proposition 2.1. [12] *Consider a fuzzy vector space $\widehat{W} = (W, \nu)$. Then any set of vectors $\{w_k\}_{k=1}^N \subset W \setminus \{0\}$ which has distinct ν-values is linearly and fuzzy linearly independent.*

In the following, we can obtain a fuzzy matroid from fuzzy vector space.

Theorem 2.1. *Let $\widehat{W} = (W, \nu)$ be a fuzzy vector space, where W is a vector space over $\mathbb{R}$ and X be a subset of W. For each r, $0 < r \leq 1$, let*

$$X_r = \{x \in X \mid \nu(x) \geq r\}$$
$$\mathcal{I}_r = \{I = \{x_i\}_{i=1}^N \subset X \setminus \{0\} \mid x_i \text{ has distinct } \nu \text{ values }\}$$

If

$$\mathscr{I}_X = \{\nu \in \mathscr{F}(X) \mid C_r(\nu) \in \mathcal{I}_r\ for\ each\ r,\ where\ 0 < r \leq 1\ \}$$

then, $(X, \mathscr{I}_X)$ is a fuzzy matroid.

Proof. We prove the properties (i) and (ii) of definition 1.3

i. Suppose $\mu \in \mathscr{I}_X, \nu \in \mathscr{F}(E)$, and $\nu \leq \mu$. Then $C_r(\mu) \in \mathcal{I}_r$ for each $0 < r \leq 1$.
   Also, for each r, $C_r(\nu) \subseteq C_r(\mu)$ and $(X, \mathcal{I}_r)$ is a crisp matroid by Theorem 1.1.

$$\Rightarrow C_r(\nu) \in \mathcal{I}_r\ for\ each\ r.$$
$$\Rightarrow \nu \in \mathscr{I}_X$$

   Thus, $(X, \mathscr{I}_X)$ satisfies the first property.

ii. Suppose that $\alpha, \beta \in \mathscr{I}_X$ and $|supp\ \alpha| < |supp\ \beta|$. Let $\delta = min\{m(\alpha),\ m(\beta)\}$. Then we have, $supp\ \alpha \in I_\delta$ and $supp\ \beta \in I_\delta$.
Thus, $\exists$ a set $Y \in I_\delta$ such that $supp\ \alpha \subseteq Y \subseteq supp\ \alpha \cup supp\ \beta$, as $(X,\ \mathcal{I}_\delta)$ is a crisp matroid,
Let

$$\xi(w) = \begin{cases} \alpha(w), & \text{if } w \in supp\ \alpha \\ \delta, & \text{if } w \in Y \setminus supp\ \alpha \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Then verily $\xi \in \mathscr{I}_X$, $\alpha < \xi \leq \ \alpha \vee \beta$ and $m(\xi) \geq min\{\ m(\alpha),\ m(\beta)\}$. Thus, $(X, \mathscr{I}_X)$ is a fuzzy matroid.

In the above theorem $(\widehat{E}, \mathscr{I}_X)$ is called the *fuzzy vector matroid* induced by X, denoted by $\mathscr{M}_F\,[X]$.

Example 2.1. Consider $\widehat{W} = (\mathbb{R}^3, \nu)$, where $\nu\,[(0,0,0)] = 1,\ \nu\,[(0,0,\mathbb{R}\setminus\{0\})] = 1/2,\ \nu\,[(0,\mathbb{R}\setminus\{0\},0)] = 1/4$ and $\nu\,[(\mathbb{R}\setminus\{0\},\mathbb{R},\mathbb{R})] = 1/3$.
Now, consider the following matrix

$$\begin{array}{ccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 \end{array}$$
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 4 & 2 & 2 \\ 0 & 0 & 0 & 4 & 8 & 4 & 0 \end{bmatrix}$$

Let $X \subseteq \mathbb{R}^3$ be the set $\{w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6,\ w_7\}$ of column labels of the matrix. Then, clearly $\nu\,(w_1) = 1,\ \nu\,(w_2) = 1/3,\ \nu\,(w_3) = 1/4,\ \nu\,(w_4) = 1/2,\ \nu\,(w_5) = 1/4,\ \nu\,(w_6) = 1/4,\ \nu\,(w_7) = 1/4$.
Let $\mathscr{I}_X$ be the family of fuzzy subsets of X under ν such that each w_i has distinct ν values. Then $\mathscr{I}_X$ consists of all fuzzy subsets of $X \setminus \{w_1\}$ with atmost three elements except for $\{w_2,\ w_3,\ w_7\}$, $\{w_4,\ w_6,\ w_7\}$, $\{w_4,\ w_5,\ w_7\}$ and any fuzzy subset containing $\{w_5,\ w_6\}$.
Then, $(X, \mathscr{I}_X)$ is a fuzzy vector matroid.

Definition 2.3. *Let $\mathscr{M} = (E, \mathscr{I})$ be a fuzzy matroid. If we can find a subset X of some fuzzy vector space $\widehat{W} = (W, \mu)$ such that $\mathscr{M} \cong \mathscr{M}_F[X]$, then $\mathscr{M}$ is called a representable fuzzy matroid, we also say that $\mathscr{M}$ is representable.*

Note. *In crisp matroid theory, we have the result that a graphic matroid is representable over any field. However, through the next example we show that not all fuzzy matroids are representable over any field. This statement is vindicated below as an example.*

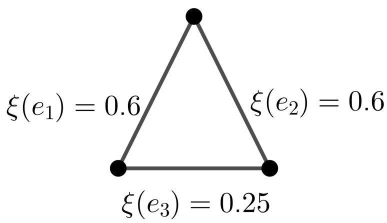
Example 2.2. Let $X = \{x_1,\ x_2,\ x_3\}$, $\mathcal{I}_1 = \{\Phi\}$, $\mathcal{I}_{0.6} = \{\Phi,\ \{x_1\},\ \{x_2\},$ $\{x_1,\ x_2\}\}$, $\mathcal{I}_{0.25} = \{\Phi,\ \{x_1\},\ \{x_2\},\ \{x_3\},\ \{x_1,\ x_2\},\ \{x_1,\ x_3\},\ \{x_2,\ x_3\}\}$.
Then clearly, $(X,\ \mathcal{I}_1)$, $(X,\ \mathcal{I}_{0.25})$ and $(X,\ \mathcal{I}_{0.6})$ are crisp matroids, and $\mathcal{I}_1 \subset \mathcal{I}_{0.6} \subset \mathcal{I}_{0.25}$.
Let

$$\mathscr{I}_r = \begin{cases} \mathcal{I}_{0.25}, & r \in (0,\ 0.25] \\ \mathcal{I}_{0.6}, & r \in (0.25,\ 0.6] \\ \mathcal{I}_1, & r \in (0.6,\ 1] \end{cases}$$

and let

$$\mathscr{I} = \{\mu \in \mathscr{F}(X) |\ C_r(\mu) \in \mathscr{I}_r,\ r \in (0,1]\ \}.$$

Then, $(X, \mathscr{I})$ is a fuzzy matroid.
Let $G = (G^*, \xi)$ be a fuzzy graph of following figure.



It is easy to see that $(X, \mathscr{I}) \cong M_F(G)$. Thus, $(X, \mathscr{I})$ is a graphic fuzzy matroid. Suppose that there exist a fuzzy vector space $\widehat{W} = (W, \nu)$ and $Y \subseteq W$ such that $M_F(G) \cong M_F[Y]$. By definition of graphic and representable fuzzy matroids, we obtain that Y consists of three linearly dependent elements, in which any two are linearly independent. Let $Y = \{y_1,\ y_2,\ y_3\}$ by isomorphism between two fuzzy matroids, there should exist a vector such that its value in ν is exactly 0.25. Let us assume that, $\nu(u_1) = 0.25$. Since y_1 is the linear combination of y_2 *and* u_3, it follows that $0.25 = \nu(y_1) \geq \nu(y_2) \wedge \nu(y_3) = 0.6$, a contradiction.
Thus, $(W, \mathscr{I})$ is not a representable fuzzy matroid.

Now, we think about the condition which makes a graphic fuzzy matroid occur representable over any field.

Theorem 2.2. [12] *Given a vector space E with basis $B = \{v_\alpha\}_{\alpha \in A}$, constant $\mu_0 \in (0,1]$ and any set of constants $\{\mu_\alpha\}_{\alpha \in A} \subset (0,1]$ such that $\mu_0 \geq \alpha$ for all $\alpha \in A$. Let us construct a function $\mu : E \longrightarrow [0,1]$ in the following way. Any $z \neq 0, z \in E$ can be uniquely written as $z = \sum_{i=1}^{N} a_i \nu_{\alpha_i}$, with $a_i \neq 0$. Define, $\mu(z) = \wedge_{i=1}^{N} \mu_{\alpha_i}$ and $\mu(0) = \mu_0$.*

Clearly μ is defined for all $z \in E$ and is well-defined. Thus, $\widehat{E} = (E,\ \mu)$ is a fuzzy vector space with fuzzy basis B.

Theorem 2.3. *Let $\mathscr{M} = (E, \mathscr{I})$ be a fuzzy matroid. If $\mathscr{M} \cong \mathscr{M}_F(G)$ where G is a fuzzy tree, then $\mathscr{M}$ is representable fuzzy matroid over any field F.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$ be the edge set of G, where $n > 0$, and let m be the number of edges in G. Let F be any field and suppose 0 and 1 are respectively the additive and multiplicative identity elements of F. Let $W_n(F)$ be an n dimensional vector space over F.
Now consider the collection of vectors
$\mathcal{E} = \{e_i \in W_n(F) |$ ith component of e_i is 1 and all other components are $0\}$
Now, we have to obtain a fuzzy vector space $\widehat{W}$ such that the fuzzy matroid induced by $\widehat{E}$ is isomorphic to $\mathscr{M}$.
For that, set a map ψ from $E(G)$ to $W_n(F)$ such that $\psi(e) = e_j - e_i$, where e is the edge connecting the vertices v_i and v_j.
Now, consider the collection

$$B = \{\psi(e) \mid e \in E(G)\}$$

and let $W =< B >$, that is, W is the subspace spanned by B.
In crisp case, we have, for any subset Y of $E(G)$, Y does not contain any cycles in G if and only if the vector family of $\psi(Y)$ is independent on W.
Obviously, the vectors in B are linearly independent, since G is a fuzzy tree. By Theorem 2.2, we can get a fuzzy vector space $\widehat{W}$ with B as fuzzy basis.
Thus by definition of graphic and representable fuzzy matroids $\mathscr{M}_F(G) \cong \mathscr{M}_F[W]$.
Therefore, $\mathscr{M}$ is representable over F.

3. Conclusion

This study describes the construction of a new class of fuzzy matroids which is from fuzzy vector space, and introduced the concept of representable fuzzy matroids. Even though, in crisp matroid theory, a graphic matroid is representable over any field, through an example we have shown that not all graphic fuzzy matroids can be representable over any field. But, when a fuzzy matroid is isomorphic to a fuzzy cycle matroid which is induced from a fuzzy tree, it is representable over any field.

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