Abstract: In this paper we give a complete characterization of essential ascent and essential descent of weighted composition operators on $l^p$ spaces.

Keywords and Phrases: Essential Ascent, Essential Descent, Weighted Composition Operator.

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1. Introduction
Let $X$ denote an arbitrary vector space and $T$ be a linear operator on $X$. Let $D(T)$, $N(T)$ and $R(T)$ denote domain, kernel and range of $T$ respectively. Let $\mathbb{N}$ denote the set of natural numbers. Let $l^p$, $(1 \leq p < \infty)$ be the Banach space of all $p$-summable sequences of complex numbers under the standard $p$-norm on it and let $u$ be a complex-valued function with domain $\mathbb{N}$. For $f \in l^p$ define

$$(uC_{\phi}) (f)(n) = u(n)f(\phi(n)), \text{ for each } n \in \mathbb{N}. $$

If $(uC_{\phi})(f) \in l^p$ whenever $f \in l^p$ then $uC_{\phi}$ is a linear transformation on $l^p$ and is called a weighted composition operator on $l^p$. When $u$ is identically equal to one
we get the composition operator $C_{\phi}$. In this paper $S(u)$ denotes the support of $u$. Weighted composition operators appear naturally in the study weighted shift operators due to Shield [24]. These operators have been subject matter of study by authors such as Kumar [14], Singh [12], Whitley [28] and others ([3], [11], [13]).

**Definition 1.1.** If there is some integer $n \geq 0$ such that $\dim (N(T^{n+1})/N(T^n))$ is finite, the smallest such integer is called the essential ascent of $T$ and is denoted by $a_e(T)$. If no such integer exists then $a_e(T) = \infty$; see [20].

**Definition 1.2.** If there is some integer $n \geq 0$ such that $\dim (R(T^n)/R(T^{n+1}))$ is finite, the smallest such integer is called the essential descent of $T$ and is denoted by $d_e(T)$. If no such integer exists then $d_e(T) = \infty$; see [20].

### 2. Essential Ascent and Essential Descent of Weighted Composition Operators On $l^p$ spaces

In this section we prove results about essential ascent and essential descent of weighted composition operators on $l^p$ spaces where $1 \leq p < \infty$.

**Theorem 2.1.** $a_e(uC_{\phi}) = \infty$ if and only if there exist a sequence $\{E_k\}_{k=1}^{\infty}$ of subsets of $\mathbb{N}$ such that each $E_k$ is infinite, $E_k \subseteq \phi^{k-1}(N_{k-1})$ and $\phi^k(N_k) \cap E_k = \phi$ for each $k \in \mathbb{N}$, where $N_k = \{n \in S(u) : \phi^i(n) \in S(u) ; \forall i , 1 \leq i \leq k - 1 \}$.

**Proof.** Suppose that $a_e(uC_{\phi}) = \infty$. Let $E_k = \{m : m \in \phi^{k-1}(N_{k-1}) - \phi^k(N_k)\}$. By construction of $E_k$, it is clear that $E_k \subseteq \phi^{k-1}(N_{k-1})$ and $\phi^k(N_k) \cap E_k = \phi$ for each $k \in \mathbb{N}$. We claim that $E_k$ is infinite set. Let $E_K$ be finite for some $K$. We make the following claims :

Claim-I : If $n \in E_K$, this implies that $n \in \phi^{K-1}(N_{K-1})$ and $n \notin \phi^K(N_K)$. Therefore $(\phi^{K-1})^{-1}(n) \cap N_{K-1} \neq \phi$ but $(\phi^K)^{-1}(n) \cap N_K = \phi$. There exists an $i \in (\phi^{K-1})^{-1}(n)$ such that $u(i)u(\phi(i)) \ldots u(\phi^{K-2}(i)) \neq 0$ but $u(j)u(\phi(j)) \ldots u(\phi^{K-1}(j)) = 0$ for each $j \in (\phi^K)^{-1}(n)$.

Thus $(uC_{\phi})^K(\chi_n) = \sum_{j \in (\phi^K)^{-1}(n)} u(j)u(\phi(j)) \ldots u(\phi^{K-1}(j))\chi_j = 0$ and $(uC_{\phi})^{K-1}(\chi_n) = \sum_{i \in (\phi^{K-1})^{-1}(n)} u(i)u(\phi(i)) \ldots u(\phi^{K-2}(i))\chi_i \neq 0$.

Therefore $\chi_n \notin N((uC_{\phi})^{K-1})$ but $\chi_n \in N((uC_{\phi})^K)$.

Claim-II : Let $n \in \phi^K(N_K)$. This implies that $n = \phi^K(m)$ for some $m \in N_K$.

Since $N_K \cap (\phi^K)^{-1}(n) \neq \phi$, hence $(uC_{\phi})^K(\chi_n) = \sum_{i \in (\phi^K)^{-1}(n)} u(i)u(\phi(i)) \ldots u(\phi^{K-1}(i))\chi_i \neq 0$.

Therefore $\chi_n \notin N((uC_{\phi})^K)$.

Claim-III : suppose $n \notin \phi^{K-1}(N_{K-1})$. Then for each $i \in N_{K-1}$ satisfying $\phi^{K-1}(i) =
n and $u(i)u(\phi(i))\cdots u(\phi^{K-2}(i)) = 0$.

Therefore $(uC_\phi)^{K-1}(\chi_n) = \sum_{i \in (\phi^{K-1})^{-1}(n)} u(i)u(\phi(i))\cdots u(\phi^{K-2}(i))\chi_i = 0$. So $\chi_n \in N((uC_\phi)^{K-1})$. Now we show that $N((uC_\phi)^K)/N((uC_\phi)^{K-1})$ is spanned by $\{\chi_n + N((uC_\phi)^{K-1}) : n \in E_K\}$.

Let $f = g + N((uC_\phi)^{K-1})$, where $g \in N((uC_\phi)^K)$.

Let $g = \sum \alpha_n \chi_n$. Now we can expressed $g$ as follows:

$$g = \sum_{m \in E_K} \alpha_m \chi_m + \sum_{p \in (N-(\phi^K(N_K) \cup E_K))} \alpha_p \chi_p.$$ 

Clearly $\sum_{p \in (N-(\phi^K(N_K) \cup E_K))} \alpha_p \chi_p$ belongs to $N((uC_\phi)^{K-1})$.

Then $f = g + N((uC_\phi)^{K-1}) = \sum_{m \in E_K} \alpha_m \chi_m + N((uC_\phi)^{K-1})$

$= \sum_{m \in E_K} \alpha_m (\chi_m + N((uC_\phi)^{K-1})).$

This implies that $\{\chi_n + N((uC_\phi)^{K-1}) : n \in E_K\}$ spans $N((uC_\phi)^K)/N((uC_\phi)^{K-1})$.

Therefore $\dim N((uC_\phi)^K)/N((uC_\phi)^{K-1}) \leq \overline{E_K} < \infty$. Thus $a_e(uC_\phi) \leq (K - 1)$. This is a contradiction. Hence $E_k$ is infinite set.

Conversely, assume that there exist a sequence $\{E_k\}_{k=1}^\infty$ of subsets of $\mathbb{N}$ such that each $E_k$ is infinite, $E_k \subseteq \phi^{k-1}(N_{k-1})$ and $\phi^k(N_k) \cap E_k = \phi$ for each $k \in \mathbb{N}$, where $E_k = \{m : m \in \phi^{k-1}(N_{k-1}) - \phi^k(N_k)\}$.

Now we claim that $\{\chi_n + N((uC_\phi)^{k-1}) : n \in E_k\}$ are linearly independent sequence of $N((uC_\phi)^k)/N((uC_\phi)^{k-1})$. It is sufficient if we prove that every finite subset $\{\chi_n + N((uC_\phi)^{k-1}) : n \in E_k\}$ are linearly independent in $N((uC_\phi)^k)/N((uC_\phi)^{k-1})$.

Let $\beta_1(\chi_{n_1} + N((uC_\phi)^{k-1}) + \cdots + \beta_l(\chi_{n_l} + N((uC_\phi)^{k-1})) = N((uC_\phi)^{k-1})$.

This implies that $\beta_1(\chi_{n_1} + \cdots + \beta_l(\chi_{n_l}) \in N((uC_\phi)^{k-1})$.

Therefore $(uC_\phi)^{k-1}(\beta_1(\chi_{n_1} + \cdots + \beta_l(\chi_{n_l})) = 0.$ Thus

$$\beta_j \sum_{i \in (\phi^{k-1})^{-1}(n_j)} u(i)u(\phi(i))u(\phi^{k-2}(i))\chi_i = 0$$

for each $j, 1 \leq j \leq l$.

Hence $(uC_\phi)^k(\chi_{n_j}) = 0$ and $(uC_\phi)^{k-1}(\chi_{n_j}) \neq 0$ for each $j, 1 \leq j \leq l$.

Since $i \in N_{k-1} \cap (\phi^{k-1})^{-1}(n_j) \neq \phi$ for $1 \leq j \leq l$. This implies that

$$\sum_{i \in (\phi^{k-1})^{-1}(n_j)} u(i)u(\phi(i))u(\phi^{k-2}(i))\chi_i \neq 0$$

for each $j, 1 \leq j \leq l$.

Hence $\beta_j = 0$ for each $j, 1 \leq j \leq l$. Thus $\{\chi_n + N((uC_\phi)^{k-1}) : n \in E_k\}$ are linearly independent sequence of $N((uC_\phi)^k)/N((uC_\phi)^{k-1})$. Since each $E_k$ is infinite set.

Therefore $\dim (N((uC_\phi)^k)/N((uC_\phi)^{k-1})) = \infty$ for each $k \geq 1$.

Hence $a_e(uC_\phi) = \infty$.

**Remark 2.1.** The following example shows that for each $n \in \mathbb{N}$ there exist a weighted composition operator $uC_\phi$ on $l^p$ such that $a_e(uC_\phi) = n - 1$. 

Example 2.1. Let \( n \) be any fixed natural number and \( \phi \) be a self-map on \( \mathbb{N} \) defined as :

\[
\phi(m) = \begin{cases} 
m, & \text{if } n/(m - 1) \\
\phantom{m, } m-1, & \text{otherwise.}
\end{cases}
\]

and

\[
u = \left\{ \frac{1}{n} \right\}_{n=1}^{\infty}
\]

Then \( a_e(uC_\phi) = n - 1 \) and \( d_e(uC_\phi) = n - 1 \).

Theorem 2.2. \( d_e(uC_\phi) = \infty \) if and only if for each \( k \geq 0; \frac{1}{\phi^{-1}(n)} > 1 \) for infinitely many \( n \in \phi^k(N_k) \), where \( N_k = \{ n \in S(u) : \phi^i(n) \in S(u) \forall i, 1 \leq i \leq k - 1 \} \).

Proof. If possible, suppose \( A = \left\{ n \in \phi^K(N_0) : \frac{1}{\phi^{-1}(n)} > 1 \right\} \) is finite for some natural number \( K \). We claim that \( \dim \left( R((uC_\phi)^K)/R((uC_\phi)^{K+1}) \right) \leq \overline{A} < \infty \). Let \( f \in R((uC_\phi)^K) \). Then \( f = (uC_\phi)^K(g) \), for some \( g \in l^p \). Let \( g = \sum \alpha_n \chi_n \). Then

\[
(uC_\phi)^K(g) = \sum_{n \in (\phi^K)^{-1}(N_k)} u(n)u(\phi(n)) \ldots . u(\phi^{K-1}(n))\alpha_{\phi^K(n)}\chi_n
\]

\[= \sum_{n' \in (\phi^K)^{-1}(N_k)} u(n')u(\phi(n')) \ldots . u(\phi^{K-1}(n'))\alpha_{\phi^K(n')}\chi_{n'} \]

\[+ \sum_{n'' \in (\phi^K)^{-1}(N_k)} u(n'')u(\phi(n'')) \ldots . u(\phi^{K-1}(n''))\alpha_{\phi^K(n'')}\chi_{n''} \]

i.e.

\[
(uC_\phi)^K(g) = h_1 + h_2 \text{(say)} \tag{1}
\]

We claim that \( h_2 \in R((uC_\phi)^{K+1}) \). Let \( g' = \sum \beta_n \chi_n \), where

\[
\beta_n = \begin{cases} 
0, & \text{when } n \notin \phi^{K+1}(N_{k+1}) \text{ or } \frac{1}{\phi^{-1}(n)} > 1 \\
\alpha_{\phi^{-1}(n)}/u(\phi^K(n)), & \text{when } n \in \phi^{K+1}(N_{k+1}) \text{ and } \phi^{-1}(n) = 1.
\end{cases}
\]
Then, clearly \( g' \in l^p \). Now
\[
(uC\phi)^{K+1}(g') = \sum_{n \in (\phi^{K+1})^{-1}(N_{k+1}) \text{ and } \phi^{-1}(n)=1} u(n)u(\phi(n)) \ldots u(\phi^n(n)) \beta_{\phi^{K+1}(n)} \chi_n 
+ \sum_{n \notin (\phi^{K+1})^{-1}(N_{k+1}) \text{ or } \phi^{-1}(n)>1} u(n)u(\phi(n)) \ldots u(\phi^n(n)) \beta_{\phi^{K+1}(n)} \chi_n 
= \sum_{n \in (\phi^{K+1})^{-1}(N_{k+1}) \text{ and } \phi^{-1}(n)=1} u(n)u(\phi(n)) \ldots u(\phi^n(n)) \beta_{\phi^{K+1}(n)} \chi_n
\]

Now put \( n'' = \phi^{-1}(n) \), then \( n'' \in \phi^k(N_k) \) and by our assumption we get \( \phi^{-1}(n) = 1 \iff \phi^{-1}(n'') = 1 \). Therefore
\[
(uC\phi)^{K+1}(g') = \sum_{n'' \in \phi^k(N_k) \text{ and } \phi^{-1}(n'')=1} u(n'')u(\phi(n'')) \ldots u(\phi^n(n'')) \beta_{\phi^n(n'')} \chi_{n''} = h_2
\]

Thus \( h_2 \in R((uC\phi)^{K+1}) \).

Combining equation (1) and (2), we get \( \dim (R((uC\phi)^K)/R((uC\phi)^{K+1})) \) is finite. Thus \( d_e(uC\phi) \leq K \).

Conversely, assume that \( \phi^{-1}(n) > 1 \) for infinitely many \( n \in \phi^k(N_0) \). Let \( \{n_m\}_{m=1}^\infty \subset \phi^k(N_0) \) such that \( \phi^{-1}(n_m) > 1 \) for each \( m \geq 1 \). Let \( \{\alpha_{n_m}, \beta_{n_m}\} \subset \phi^{-1}(n_m) \).

Define a sequence \( \{f_m\}_{m=1}^\infty \) as follows:
\[
f_m(n) = \begin{cases} 1, & \text{if } \phi^{k-1}(n) = \alpha_{n_m} \\ -1, & \text{if } \phi^{k-1}(n) = \beta_{n_m} \\ 0, & \text{otherwise.} \end{cases}
\]

Clearly \( \{f_m\}_{m=1}^\infty \in l^p \) and also define a sequence \( \{h_m\}_{m=1}^\infty \) as follows:
\[
h_m(n) = \begin{cases} 1/u(n)\phi(n) \ldots \phi^{k-2}(n), & \text{if } n = \alpha_{n_m} \\ -1/u(n)\phi(n) \ldots \phi^{k-2}(n), & \text{if } n = \beta_{n_m} \\ 0, & \text{otherwise.} \end{cases}
\]

Clearly \( \{h_m\}_{m=1}^\infty \in l^p \). We claim that \( \{f_m\}_{m=1}^\infty \in R((uC\phi)^{k-1}) \) and \( \{f_m\}_{m=1}^\infty \notin R((uC\phi)^k) \). Now
\[(uC_\phi)^{k-1}(h_m)(n) = \begin{cases} 
1, & \text{if } \phi^{k-1}(n) = \alpha_{nm} \\
-1, & \text{if } \phi^{k-1}(n) = \beta_{nm} \\
0, & \text{otherwise.} \end{cases} \]

This implies that \((uC_\phi)^{k-1}(h_m) = f_m\). Therefore \(\{f_m\}_{m=1}^\infty \in R((uC_\phi)^{k-1})\). We claim that \(\{f_m\}_{m=1}^\infty \notin R((uC_\phi)^k)\). If possible, assume that \(\{f_m\}_{m=1}^\infty \in R((uC_\phi)^k)\), for some \(m_0 \geq 1\). This implies that \(f_{m_0} = (uC_\phi)^k(h_0)\), for some \(h_0 \in l^p\). Let \(n_m^{(1)}\) and \(n_m^{(2)}\) be such that \(\phi^{k-1}(n_m^{(1)}) = \alpha_{nm}\) and \(\phi^{k-1}(n_m^{(2)}) = \beta_{nm}\), where \(\phi(\alpha_{nm}) = \phi(\beta_{nm}) = n_m\). A simple computation shows that \(\{f_m\}_{m=1}^\infty \notin R((uC_\phi)^k)\). Thus sequence \(\{f_m/R((uC_\phi)^k)\}_{m=1}^\infty\) are linearly independent in \(R((uC_\phi)^{k-1})/R((uC_\phi)^k))\). Therefore \(\dim (R((uC_\phi)^{k-1})/R((uC_\phi)^k))\) is not finite. Since \(k \geq 1\) is arbitrary it follows that \(d_e(uC_\phi) = \infty\).

**Remark 2.2.** From Example (2.1) it follows that for each \(n \in \mathbb{N}\) there exist a weighted composition operator \(uC_\phi\) on \(l^p\) such that \(d_e(uC_\phi) = n - 1\).

### 3. Example

Note that a linear operator \(T\) belongs to exactly one of the following cases:

1. \(a_e(T) = d_e(T) = \text{finite.}\)
2. \(a_e(T) = \infty\) but \(d_e(T)\) is finite.
3. \(d_e(T) = \infty\) but \(a_e(T)\) is finite.
4. \(a_e(T) = \infty\) and \(d_e(T) = \infty\).

We give examples of weighted composition operators, exactly one for each of the above type, as follows:

**Example 3.1.** Let \(\phi\) be a self-map on \(\mathbb{N}\) defined as:

\[\phi(n) = \begin{cases} 
n, & \text{if } n \text{ is odd} \\
n-1, & \text{if } n \text{ is even.} \end{cases} \]

and

\[u = \{\frac{1}{n}\}_{n=1}^\infty\]

Then \(a_e(uC_\phi) = 1\) and \(d_e(uC_\phi) = 1\).

**Example 3.2.** Let \(\phi\) be the self-map on \(\mathbb{N}\) defined as:

\[\phi(p^n_k) = p^n_{k+1} \text{ for all } k \in \mathbb{N}.

Where \(\{p_k : k \in \mathbb{N}\}\) denote the enumeration of primes.
and $\phi(n) = n$ when $n \in \left( \mathbb{N} - \bigcup_{k \in \mathbb{N}} E_k \right)$
where $E_k = \{ p_k^n : n \geq 1 \}$ for each $k \in \mathbb{N}$.

also

$$u = \left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$$

Then $a_e(uC_{\phi}) = \infty$ and $d_e(uC_{\phi}) = 0$.

**Example 3.3.** Let $\phi$ be the self-map on $\mathbb{N}$ defined as:

$$\phi(n) = n + 2, \text{ if } n \text{ is odd}$$

and

$$\phi(2n - 2) = \phi(2n) = n, \text{ if } n \text{ is even}.$$  

also

$$u(n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ -1, & \text{if } n \text{ is even.} \end{cases}$$

Then $a_e(uC_{\phi}) = 0$ and $d_e(uC_{\phi}) = \infty$.

**Example 3.4.** Let $P = \bigcup_{k \in \mathbb{N}} \{ p_k^n : n \in \mathbb{N} \}$ where $p_k$ denote the $k$-th prime and $\mathbb{N} - P = \{ q_k : k \geq 1 \} = \{ 1, 6, 10, 12, \ldots \}$. Clearly $\mathbb{N} - P$ is an infinite subset of $\mathbb{N}$ and $\phi$ be the self-map on $\mathbb{N}$ defined as:

$$\phi(p_k^n) = p_{k+1}^n \text{ for all } k \in \mathbb{N}.$$  

$$\phi(q_1) = \phi(q_2) = q_1$$

and

$$\phi(q_{2k-1}) = \phi(q_{2k}) = q_{2k-2} \text{ for each } k \geq 2.$$  

also

$$u = \left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$$

Then it is easy to show that $a_e(uC_{\phi}) = \infty$ and $d_e(uC_{\phi}) = \infty$. 
References


