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TIME TO REPLACEMENT OF A SYSTEM WITH PERMISSIVE AND OBLIGATORY THRESHOLDS

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Abstract: Shock exerts on the system is a common phenomenon in reliability theory. These shocks will create damage to the system due to its impact. The system receives shocks in two mutually exclusive ways, internally (circuit problem, a heavy supply of voltage, etc.) and externally (shocks by the circumstances). Adequate replacement of the system due to the damages is not realistic since it involves cost. A stochastic model is constructed with three different cases of shocks and the time to replacement of a system is obtained, when the cumulative damages cross its obligatory threshold. The numerical illustration has been made to the mean and variance of time to replacement and the realistic conclusion is presented.

Keywords and Phrases: Time to replacement, cumulative damages, permissive threshold, obligatory threshold, shock model approach.

2010 Mathematics Subject Classification: Primary: 90B25, Secondary: 60K05, 60K20.

1. Introduction

A shocks creating damages to the system placed in the environment is a common phenomenal in reliability theory. The system receives shocks in many different categories and it is classified into two mutual exclusive shocks (i) Internal power supply or voltage problem. (ii) Shocks due to circumstances. These shocks will create damage to the system due to its impact. The time at which the cumulative damages crosses the obligatory threshold, cannot be predicted. Defining a permissive threshold (less than obligatory threshold) fills the gap to expect the replacement of a system. If the cumulative damages cross permissive threshold, it gives alertness about the replacement, where the system may or may not be replaced, whereas the cumulative damages crosses the obligatory threshold, the system is replaced. In this context author [2] has studied many reliability models with various assumptions on damages. Authors in [1], [3] and [4] have studied the concept of manpower planning and the shock model approach. Considering the shock model approach, the replacement of a system is carried out whenever the cumulative damages crosses its obligatory threshold and the system may or may not be replaced if the cumulative damages crosses permissive threshold.

In this paper, the mean and variance of time to replacement of a system is determined for the three different cases of shocks. In case-I: it is assumed that the time between two consecutive shocks forms a sequence of independent and identically distributed random variables. There are some minute shocks that will never create damage to the system. In case-II: it is assumed that the system receives $n + r$ shocks. Of these, n shocks will create the damages with probability $0 < p < 1$ to the system. Some system receives more shocks internally than the shocks due to the external circumstances. In case-III: it is assumed that the shocks received by the system has been classified into two mutually exclusive types.
For these three cases, the mean and variance of time to replacement have been determined when the cumulative damages cross its obligatory threshold and the system may or may not be replaced if the cumulative damages cross the permissive threshold. The results are numerically illustrated and the findings in the illustration coincide with the realistic observation.

2. Model Description:

Consider the system in which its functioning gets affected due to the impact of the shocks. Let $B_i, (i = 1, 2, 3, ...)$ be a stochastic process which represents the damage due to the i^{th} shock with exponential distribution function $G_i(.)$ of parameter $\alpha > 0$. Let A_i be the stochastic process that represents the time between $i - 1^{th}$ and i^{th} shock. Let the probability $0 < q < 1$ represents the system is not replaced after the cumulative damages crosses the permissive threshold. The randomly indexed partial sum S_l represents the cumulative damages to the system by the first l shocks. Let $N(t)$ is the stochastic process that represents the number of shocks exerted to the system up to the time t. Let R is a random variable that represents time to replacement of the system, with distribution function $L(.)$, density function $l(.)$ with Laplace transform $\bar{l}(.)$. Random variable Y represents the permissive threshold for the cumulative damages that follows an exponential distribution with parameter $\gamma_1 > 0$. Let Z be an exponential obligatory threshold for the cumulative damages with parameter $\gamma_2 > 0$. It is assumed that damages created to the system, inter-shock times and the thresholds are stochastically independent.

3. Analytical Results

The analytical results for the mean and variance of time to replacement has been derived for the three different cases of inter-shock times.

Case - I.

In Reliability, there exits some system which receives shocks in a periodic manner. Hence, in this case, it is assumed that time between the two consecutive shocks forms a sequence of independent and identically distributed exponential random variables with parameter $\lambda > 0$. According to the policy, the replacement occurs before the time t is equivalent to the cumulative damages crosses its obligatory threshold and the system is replaced or the cumulative damages cross a permissive threshold and the system is not replaced and the cumulative damages cross the obligatory threshold before the time t. Hence the distribution function of time to replacement is determined as

$$P(R < t) = P(S_{N(t)} > Y)(1-q) + P(S_{N(t)} > Y)(q)P(S_{N(t)} > Z)$$

Using the law of total probability, the distribution function of time to replacement is given by

$$L(t) = 1 - e^{-\lambda t(1-\overline{g}(\gamma_1))} - qe^{-\lambda t(1-\overline{g}(\gamma_2))} + qe^{-\lambda t(2-\overline{g}(\gamma_1)-\overline{g}(\gamma_2))}$$

By differentiating with respect to t, Laplace transform for the probability density function of time to replacement is determined. Now, differentiating the Laplace transform of time to replacement with respect to s, the mean time to replacement is determined at $s = 0$.

$$E(R) = \frac{(\alpha+\gamma_1)}{\lambda\gamma_1} + \frac{q(\alpha+\gamma_2)}{\lambda\gamma_2} - \frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{\lambda(2(\alpha+\gamma_1)(\alpha+\gamma_2) - \alpha(\alpha+\gamma_1) - \alpha(\alpha+\gamma_2))}$$

The second moment of time to replacement is determined by differentiating twice the Laplace transform of time to replacement with respect to s and $s = 0$. From these results, the variance of time to replacement is determined and it is given by

$$V(R) = \frac{2(\alpha+\gamma_1)^2}{(\lambda\gamma_1)^2} + \frac{2q(\alpha+\gamma_2)^2}{(\lambda\gamma_2)^2} - \frac{2q((\alpha+\gamma_1)(\alpha+\gamma_2))^2}{(\lambda(2(\alpha+\gamma_1)(\alpha+\gamma_2) - \alpha(\alpha+\gamma_1) - \alpha(\alpha+\gamma_2)))^2}$$

$$- \left(\frac{(\alpha+\gamma_1)}{\lambda\gamma_1} + \frac{q(\alpha+\gamma_2)}{\lambda\gamma_2} - \frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{\lambda(2(\alpha+\gamma_1)(\alpha+\gamma_2) - \alpha(\alpha+\gamma_1) - \alpha(\alpha+\gamma_2))}\right)^2$$

Case - II.

Now the analytical results for the mean and variance of time to replacement are determined by assuming that the system receives $n+r$ shocks. Of these, n shocks will create the damages with probability $0 < p < 1$ to the system. By proceeding as in case - I, differentiating the Laplace transform of time to replacement with respect to s, the mean and variance of time to replacement are determined

$$E(R) = \frac{(\alpha+\gamma_1)}{p\lambda\gamma_1} + \frac{q(\alpha+\gamma_2)}{p\lambda\gamma_2} - \frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{\lambda(2p(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2))}$$

$$V(R) = \frac{2(\alpha+\gamma_1)^2}{(p\lambda\gamma_1)^2} + \frac{2q(\alpha+\gamma_2)^2}{(p\lambda\gamma_2)^2} - \frac{2q((\alpha+\gamma_1)(\alpha+\gamma_2))^2}{(\lambda(2p(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2)))^2}$$

$$-\left(\frac{(\alpha+\gamma_1)}{p\lambda\gamma_1} + \frac{q(\alpha+\gamma_2)}{p\lambda\gamma_2} - \frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{\lambda(2p(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2))}\right)^2$$

Case - III.

In this case, the Poisson process $N(t)$ that represents the number of shocks exerted to the system is considered as the sum of two (Internal and External) independent Poisson process with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$. Now, the probability density function of time to replacement is derived by taking derivative for the distribution function with respect to t. Taking Laplace transform for the probability density function of time to replacement and differentiating the Laplace transform, the moments of time to replacement are determined.

$$E(R) = \frac{(\alpha+\gamma_1)}{(\lambda_1+\lambda_2)\gamma_1} + \frac{q(\alpha+\gamma_2)}{(\lambda_1+\lambda_2)\gamma_2} - \frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{(\lambda_1+\lambda_2)(2(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2))}$$

The variance of time to replacement is derived by using the first two moments of time to replacement. It is given by

$$V(R) = \frac{2(\alpha+\gamma_1)^2}{((\lambda_1+\lambda_2)\gamma_1)^2} + \frac{2q(\alpha+\gamma_2)^2}{((\lambda_1+\lambda_2)\gamma_2)^2}$$
$$-\frac{2q((\alpha+\gamma_1)(\alpha+\gamma_2))^2}{((\lambda_1+\lambda_2)(2(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2)))^2}$$

$$-\left(\frac{(\alpha+\gamma_1)}{(\lambda_1+\lambda_2)\gamma_1} + \frac{q(\alpha+\gamma_2)}{(\lambda_1+\lambda_2)\gamma_2} - \frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{(\lambda_1+\lambda_2)(2(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2))}\right)^2$$

4. Numerical Illusion

The following tables are the numerical values of the mean and variance of time to replacement for the three cases. By fixing the thresholds (the limits for the system could get from the circumstances with SI units of power) $C_1 = 200$ and $C_2 = 300$ and varying the other parameters λ (in days) and α (damages created to the system due to SI units of power), numerical values of mean and variance of time to replacement (in days) are studied.

Case-I

$1/\lambda$	$1/\alpha$	q	E(R)	V(R)
0.033	0.017	0.4	36.366	1065.4
0.040	0.017	0.4	30.002	725.11
0.050	0.017	0.4	24.002	464.07
0.066	0.017	0.4	18.183	266.34
0.10	0.017	0.4	12.001	116.02
0.028	0.020	0.4	42.861	1479.9
0.028	0.025	0.4	42.862	1479.9
0.028	0.033	0.4	42.865	1480.0
0.028	0.050	0.4	42.867	1480.3
0.028	0.100	0.4	42.877	1480.9
0.028	0.017	0.5	44.646	1514.9
0.028	0.017	0.6	46.432	1543.9
0.028	0.017	0.7	48.218	1565.9
0.028	0.017	0.8	50.004	1581.9
0.028	0.017	0.9	51.789	1591.4

Case-II

p	q	$1/\lambda$	$1/\alpha$	E(R)	V(R)
0.1	0.4	0.033	0.017	363.66	106536
0.1	0.4	0.040	0.017	300.02	72511.1
0.1	0.4	0.050	0.017	240.02	46407.1
0.1	0.4	0.066	0.017	181.88	26634
0.1	0.4	0.10	0.017	120.01	11601
0.1	0.4	0.028	0.020	428.61	147985.8
0.1	0.4	0.028	0.025	428.62	147992.5
0.1	0.4	0.028	0.033	428.64	148003.2
0.1	0.4	0.028	0.050	428.67	148025.9
0.1	0.4	0.028	0.100	428.77	148092.7
0.2	0.4	0.028	0.017	214.32	36995.4
0.3	0.4	0.028	0.017	142.87	16442.4
0.4	0.4	0.028	0.017	107.15	9248.9
0.5	0.4	0.028	0.017	85.721	5919.3
0.6	0.4	0.028	0.017	71.434	4110.6
0.1	0.5	0.028	0.017	446.46	151489.6
0.1	0.6	0.028	0.017	464.32	154359.6
0.1	0.7	0.028	0.017	482.18	156591.7
0.1	0.8	0.028	0.017	500.04	158186.1
0.1	0.9	0.028	0.017	517.89	159142.6

Case-III

$1/\lambda_1$	$1/\lambda_2$	$1/\alpha$	q	E(R)	V(R)
0.033	0.033	0.017	0.4	18.183	266.34
0.040	0.033	0.017	0.4	16.440	217.71
0.050	0.033	0.017	0.4	14.459	168.41
0.066	0.033	0.017	0.4	12.122	118.37
0.10	0.033	0.017	0.4	9.0233	65.588
0.028	0.040	0.017	0.4	17.648	250.90
0.028	0.050	0.017	0.4	15.386	190.69
0.028	0.066	0.017	0.4	12.767	131.30
0.028	0.10	0.017	0.4	9.375	70.812
0.028	0.20	0.017	0.4	5.2636	22.318
0.028	0.033	0.020	0.4	19.673	311.80
0.028	0.033	0.025	0.4	19.674	311.81
0.028	0.033	0.033	0.4	19.675	311.84
0.028	0.033	0.050	0.4	19.677	311.88
0.028	0.033	0.100	0.4	19.681	312.03
0.028	0.033	0.017	0.5	20.493	319.18
0.028	0.033	0.017	0.6	21.313	325.23
0.028	0.033	0.017	0.7	22.133	329.93
0.028	0.033	0.017	0.8	22.952	333.29
0.028	0.033	0.017	0.9	23.772	335.31

5. Numerical Results

In all the three cases, if the probability of not replacing the system ($q > 0$) increases, it elongates the time of replacement. Hence, the mean time to replacement increases. In Case II, if the probability of shocks producing damages to the system ($p > 0$) increases, then the system receives more damages. Hence, the cumulative damages takes less time to cross the breakdown threshold. Thus, the mean time to replacement of a system decreases. For the Cases I and II, if the average inter shocks time increases, then the epoch of occurrences of shocks decreases. Thus, the occurrences of shocks increases, that creates more damages to the system. Hence, the mean time to replacement declines. In Case - III, if the average inter external shocks and inter internal shocks increases, then the epoch of the occurrence of shocks (external and internal shocks) decreases. Thus, the number of shocks exerted to the system increases which creates more damages to the system. This reduces the mean time to replacement.

In Cases I, II and III, if the average damages due to the shocks increases, then the damages exerted to the system decreases. Hence, the cumulative damages takes more time to cross the breakdown threshold. Thus, the mean time to replacement of a system increases. Comparing all the three cases, Case II gives more time to replacement of a system. Perceiving the results for the three cases from the tables, the numerical results coincides with the realistic surveillance.

6. Conclusion:

The general observation of hike in damages leads to the reduction of time to replacement and the time between the shock (inter-shock times) elongates, the replacement time of a system extended. This observation coincides with realistic scrutiny. The concept of considering the independence in the inter-shock times and the damages can be dropped in the future to study the dependence nature of shock and its damages.

References

[1] Bartholomew, D. J., Renewal theory models in manpower planning, Symposium Proceedings Series, No. 8, The Institute of Mathematics and its Applications, 57-73, (1976).

[2] Barlow R. E. and Proschan F., Statistical theory of reliability and life testing, Holt, Rinehart and Winston, Inc., New York, (1975).

[3] Grinold R.C., and Marshall K. J., Man Power Planning, North-Holland, New York (1977).

[4] Esary, J. D., A. W. Marshall and F. Proschan, Shock models and wear processes. Ann. Prob. 1, 627-649,(1973).

[5] Medhi J., Stochastic processes, New Age International Publishers, Third edition, (2009).