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HYPOTHESIS OF VALUE DISTRIBUTION AND ITS ASSOCIATED PROBLEMS OF INFINITE DIMENSION

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Abstract: In this current paper, we introduced the overture of the subsequent field given by the span of a finite number of vectors as follows: **(1)** The complete normed inner product space of Nevanlinna theory. **(2)** A complete normed vector space of Nevanlinna theory over the real or complex field.

Keywords and Phrases: Nevanlinna theory, infinite-dimensional space, E-valued function, Hilbert space, Banach space.

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1. Introduction and Results

Ever since first and second fundamental theorem of Nevanlinna appeared in 1925 [2]. Many authors have researched on this theorem taking the range of the function in infinite-dimensional Banach space and Hilbert space. In this paper, we are examined initial and subsequent theorem of Nevanlinna E-value in Hilbert space as well as Banach space via representation sections by the concept and properties of meromorphic maps.

2. Theory of Value Distribution of Infinite-dimension

2.1. Theory of Value distribution in Hilbert space

Before establishing Nevanlinna formula of $Poisson - Jensen$, the vector-valued function of finite dimension was extended from the classical Nevanlinna theory of meromorphic function by Ziggler [13] in 1982. Followed by this in 1997, with complete orthogonal basis $\{e_j\}_i^\infty$ from the theory of Nevanlinna of infinite-dimension in Hilbert space E was extended by Hu and Yang [7]. Using the results of [4], the hypothesis that a function of a complex variable is analytic in a sphere and along its boundary attains its maximum absolute numerical quantity and a method for assigning values to certain improper integrals in the sense of Bochner and the Laurent's extension of a vector valued holo-morphic function are along the span in E were established the first fundamental theorem of E-valued Nevanlinna (see [7] for details).

Let $H(z) = (h_1(z), ...h_j(z), ...)$ be a meromorphic E-valued function on $\mathbb{C}$, if $\Sigma_{j=1}^{\infty}|h_j(z)|$ is uniform convergence in any dense subset D in $\mathbb{C}$ then $H(z)$ is said to be of dense coordinate convergence.

Let E_1 be a space that is wholly contained in another space of E, which comprise of all the components of $H(z)$ in C. Without careful proof (cf. [7]), E_1 satisfies the second main theorem of Nevanlinna E-value if $H(z)$ is compact coordinate convergence on $\mathbb{C}$.

On the disk $|z| < R$, $H(z)$ is meromorphic E-Valued function to $E - \{0\}$, the volume component of the curve $\sim H$ defined as $\frac{1}{2} \triangle \;\; log||H(z)|| \; dx \wedge dy$, using correspondent results of [13], alongside the meromorphic vector operator $H(z)$, the volume function can be represented by

$$V(r, b) = \int_0^v \frac{v(t, b)}{t} dt,$$

hence

$$V(t, b) = \frac{1}{2\pi} \int_{S(0,t)} \triangle \; log||F(z) - b|| \;\; dx \wedge dy, \quad S(0, t) = Z : |Z| < t.$$

In 1998, by using the corresponding outcome of [4], Liu and Hu [12] extracted the Hilbert space of first main theorem of Nevanlinna and gave a attentive verification as follows:

2.2. E-valued Nevanlinna's first fundamental theorem

A meromorphic E-valued mapping denoted by $h(z)$ in C_R, where $0 < r < R$, $b \in E$ and b,

$$\mathtt{T}(\mathtt{r}, h) = \mathtt{V}(\mathtt{r}, b) + \mathtt{N}(\mathtt{r}, b) + \mathtt{m}(\mathtt{r}, b) + \mathtt{log}||c_q(b)|| + \epsilon(\mathtt{r}, b).$$

a function $\epsilon(\mathtt{r}, b)$ in such a way that

$$|\epsilon(\mathtt{r}, b)| \le \mathtt{log}^+||b|| + \mathtt{log}2, \quad \epsilon(\mathtt{r}, 0) \equiv \mathtt{0}.$$

$$\mathtt{m}(\mathtt{r}, b) = \frac{1}{2\pi} \int_0^{2\pi} log^+ \frac{1}{||\mathtt{f}(re^{i\theta}) - b||} d\theta,$$

$$\mathtt{N}(\mathtt{r}, b) = \mathtt{n}(0, b) \ \mathtt{log} \ \mathtt{r} + \int_0^r \frac{n(\mathtt{t}, b) - \mathtt{n}(0, b)}{\mathtt{t}} dt$$

$$T(r, h) = m(r, h) + N(r, h)$$

On the closed disk $|z| < R$, we denote h a unstable meromorphic E-valued function, then

1. For $\mathtt{R} > \mathtt{r} > 0$, $\mathtt{T}(\mathtt{r}, h)$ is an increasing function of $\mathtt{r}$.
2. For $\mathtt{R} > \mathtt{r} > 0$, $\mathsf{T}(\mathsf{r}, h)$ is a convex function of $log\, r$.
3. $\mathsf{n}(\mathtt{r}, h_j) \leq \mathtt{n}(\mathtt{r}, h)$.
4. If $\mathtt{N}(\mathtt{r}, h_j) \leq \mathtt{N}(\mathtt{r}, h)$ and $\mathtt{T}(\mathtt{r}, h_j) \leq \mathtt{T}(\mathtt{r}, h)$, then $\mathtt{z} = 0$ is not a pole of h.
5. If $\mathtt{N}(\mathtt{r}, h_j) + O(1) \leq \mathtt{N}(\mathtt{r}, h)$ and $\mathtt{T}(\mathtt{r}, h_j) \leq \mathtt{T}(\mathtt{r}, h) + \mathtt{O}(1)$, then $z = 0$ is a pole of h.

Specifically, Hilbert space of the Nevanlinna second main theorem is obtained by Hu and Yang [8] in 1992, and is given as follows:

2.3. E-valued Nevanlinna's second fundamental theorem

A non-constant meromorphic E-valued mappings of dense projection in C_R is given by $h(z)$ and $b^{[\mathtt{k}]} \in \mathtt{E} \bigcup \{\infty\} (\mathtt{k} = 1, 2, ..., \mathtt{q})$ be $\mathtt{q} \geq 3$ well defined finite points or infinite points. Subsequently

$$\sum_{k=1}^{q} m(\mathtt{r}, b^{[\mathtt{k}]}) + J(\mathtt{r}, h) \leq \mathrm{T}(\mathsf{r}, h) - \mathtt{N}_1(\mathtt{r}) + \mathtt{S}(\mathtt{r}),$$

given

$$\mathtt{N}_1(\mathtt{r}) = \mathtt{N}(\mathtt{r}, 0, h') + 2\mathsf{N}(\mathsf{r}h) - \mathtt{N}(\mathtt{r}, h')$$

in addition

$$\mathtt{J}(r, h) = \int_0^{\mathtt{r}} \frac{dt}{2\pi} \int_{C_r} \triangle log||h'(\chi)|| \ d\sigma \wedge dr.$$

If $\mathtt{R}$ is plus infinity, then $\mathtt{S}(\mathtt{r})$ persuade $S(\mathtt{r}) = \mathtt{O}\{logT(\mathtt{r}, \mathtt{f})\} + \mathtt{O}(logr)$ as $\mathtt{r} \longrightarrow +\infty$ outward a set L of atypical intervals of specific quantity

$$\int_L dr < +\infty.$$

Further fascinating structure of the E-valued Nevanlinna second main theorem is expressed as,

$$(q-1)T(r,h)+J(r,h)+N_1(r)\leq\sum_{k=1}^{q+1}[V(r,b^{[k]})+N(r,b^{[k]})]+S(r).$$

Or

$$(q-2)T(r,h)+J(r,h)\leq\sum_{k=1}^{q}[V(r,b^{[k]})+N(r,b^{[k]})]+S(r),$$

accompanied by

$$\overline{N}(r,b)=\overline{n}(0,b)\log\ r+\int_0^r\frac{\overline{n}(t,b)-\overline{n}(t,b)}{t}dt,$$

where the solution $\overline{n}(t,b)$ is counted only once, which indicates the solutions of $h(z)-b$ in $|Z|\leq t$.

3. Nevanlinna Theory in Banach Spaces

3.1. Nevanlinna Theory on one complex variable in Banach spaces

A holo-morphic E-valued function $h(z)$ on Ω is forenamed to be of dense projection if and only if $||P_nh(z)-h(z)||<\epsilon$, as adequately huge n in any stable dense subset $\Omega_1\subset\Omega$.
Following we define the meromorphic E-valued functions of Borel exceptional values. Suppose E be a composite Banach space and $\mathbb{C}$ is the Argand plane. Enable $\Omega=C_r=\{Z:|Z|<r\}$.

Definition 3.1. *An E-valued Picard exceptional value (evp) in a non-constant meromorphic E-valued function h of extremity $b\in E\bigcup\{\infty\}$ is defined by $V(r,b)+N(r,b)=O\{log\ r\}$. We sound b an E-valued evp of h for zeros of order $\leq k$ provided*

$$V(r,b)+\overline{N}_k(r,b,f)=O\{log\ r\}.$$

3.2. Deficiency relation of Nevanlinna E-valued function

Acceptable with the features of dense projection we define an E-valued function which is meromorphic of $h(z)$. Summing over all points for the countable set $\{b\in E\bigcup\{\infty\}:\Theta(b)>0\}$, we have

$$\sum_b[\delta(b)+\theta(b)]+\delta_J\leq\sum_b\Theta(b)+\delta_J\leq 2,$$

where

$$\delta(b) = \underline{lim}_{r \to R} \frac{m(\mathtt{r}, \mathtt{b})}{\mathtt{T}(\mathtt{r}, \mathtt{b})}$$

is the *deficiency of the point* b,

$$\theta(b) = \underline{lim}_{r \to R} \frac{\mathtt{N}(\mathtt{r}, b) - \overline{N}(\mathtt{r}, b)}{T(\mathtt{r}, h)}$$

is the *index of multiplicity of* b.

$$\Theta(b) = \underline{lim}_{r \to R} \frac{\mathtt{m}(\mathtt{r}, b) + \mathtt{N}(\mathtt{r}, b) - \overline{\mathtt{N}}(\mathtt{r}, b)}{\mathtt{T}(\mathtt{r}, h)},$$

and

$$\delta_J = \underline{lim}_{r \to R} \frac{J(r, h)}{T(r, h)}$$

is the *Ricci index of* h.

On the contrary, by utilizing the *Green's residue theorem* (see [10]) for the instance of holo-morphic mapping on complex manifolds in $\mathbb{C}^m$ to infinite dimension(cf. [9]) the Nevanlinna first main theorem can established as follows:

Theorem (Holomorphic Hermitian line bundles in the first main theorem of Nevanlinna): *Enable a parabolic manifold* (M, Υ) *of size* m, *and a Riemann sphere* $\hat{N}$ *on* H, *as well meromorphic map* $h : M \longrightarrow \hat{N}$ *independent of* $\boldsymbol{u}$ *(i.e.,* $h(\mathtt{M} - I_J)$*) is not hold in the zero set* $Z(u)$ *with features of dense projection. Presume a holo-morphic line bundle* $\overline{W} = L$. *Subsequently*

$$T_h(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{L}, \boldsymbol{k}) = N_{h,u}(r, s, L) + M_{h,u}(r, L, k) - Mh, u(s, L, k), \quad 0 < s < r.$$

Where $T_h(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{L}, \boldsymbol{k})$ *is the characteristic function,* $m_{f,\boldsymbol{u}}(\boldsymbol{r}, \boldsymbol{L}, \boldsymbol{k})$ *is the compensation function and* $N_{f,u}(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{L})$ *is the valence function.*

Theorem (The first main theorem of Nevanlinna for an operation): *Suppose that* $h = (h_1, h_2, ..., h_k)$ *is independent of* θ. *After that the compensation function*

$$m_{\dot{\theta} f}(\boldsymbol{r}) = m_{h_1 \dot{\theta} \ldots \dot{\theta} h_k}(a) = \int_{M_0(r)} log \frac{||\theta||}{||\dot{\theta} f||}, \quad \sigma \geq 0.$$

subsist $\forall \;\; \mathrm{r} \in R_T$ *and enlarge to a continuous function on* $[\mathit{0}, +\infty)$ *alike that for* $0 > \boldsymbol{s} > \boldsymbol{r}$,

$$\sum_{s=1}^{k} T_{fj}(\boldsymbol{r}, \boldsymbol{s}) = N_{\dot{\theta} f}(\boldsymbol{r}, \boldsymbol{s}) + M_{\dot{\theta} f}(\boldsymbol{r}) - M_{\dot{\theta} f}(s) + T_{\theta f}(r, s)$$

where

$$T_{fj}(r,s) = \int_s^r A_{fj}(t)(\frac{dt}{t})$$

also

$$A_{fj}(t) = (\frac{1}{t^{2m-2}}) \int_{M_0[t]} h_j^* \wedge \nu^{m-1}$$

3.3. Second main theorem of Nevanlinna in Banach spaces for small function

Here, we undertake E_0 is a Banach algebra, that is to say $h(\not\equiv 0)$ is a meromorphic function of E_0-value with its derivative $h^j (j = 1, ..., q)$, and that one of its differential polynomials can be represented as

$$P(z) = P(h, h', ..., h^q) = \sum_{k=1}^{p} a_k(z) \prod_{j=0}^{q} \{h^{(j)}(z)\}^{s_{kj}}$$

given $a_k(z)(k = 1, ..., p)$ are meromorphic E_0-valued functions.

Let us take non-linear, linearly independent, E_0-valued meromorphic function $\psi_k(z)$ for $(k = 1, 2, ..., t)$, and assume EMF, a set is made up of all meromorphic E_0-valued functions on $\mathbb{C}$. For non-linear meromorphic E_0-valued functions $\psi_k(z)(k = 1, 2, ..., t; t \geq 1)$ and a meromorphic E_0-valued function $h(z)$, define $A_0 = W(\psi_1, ...\psi_t)$ mean the Wronskian of $\psi_k(z)(k = 1, 2, ..., t)$ also $W(\psi_1, ..., \psi_t, f)$ mean the Wronskian of $\psi_k(z)(k = 1, 2, .., t)$ as well h, where

$$A_0 = \begin{bmatrix} \psi_1 & \psi_2 & \dots \psi_t \\ \psi_1' & \psi_2' & \dots \psi_t' \\ \vdots & \vdots & \vdots \\ \psi_1^{t-1} & \psi_2^{t-1} & \dots \psi_t^{t-1} \end{bmatrix}$$

Assume that $I = J\{E_r(\psi_1, ..., \psi_t)\}$ with dimension EMF_ν of basis $\{\alpha_j(z)\}_{j=1}^{I}$ and $I' = J\{E_{\nu+1}(\psi_1, ..., \psi_t)\}$ with dimension $EMF_{(\nu+1)}$ of basis $\{\beta_j(z)\}_{j=1}^{I'}$.

Suppose ψ_k^{-1} exists. Describe a "linear operator" $L(h) = W\ (\psi_1, ..., \psi_t, h)\ W^{-1}$ $(\psi_1, ..., \psi_t)$. Conclude that the inverse function of $F(z) = L(h)$ is proportionate as $h(z) = \sum_{k=1}^{t} c_k(z)\psi_k(z)$, where $c_k(k = 1, 2, ..., t)$ are meromorphic E_0-valued function persuading the succeeding relation

$$c_k' = (-1)^{t-k} \phi_k A_0^{-1} H,$$

for $k = 1, 2, ..., p$ and after the cancellation of horizontal p and the vertical line k in t_0, we define a vector-valued determinant ϕ_k.

Assume that

$$\phi_j(z) = \sum_{k=1}^{t} c_{jk}\psi_k(z), \ \ j = 1, ..., s(s \geq 2)$$

where c_{jk} are constants.

By using the results of chapter 1 in [11] we state an E_0-valued small function [8] of second fundamental theorem of Nevanlinna is as follows:

Theorem: *If* $h - \phi_j^{-1}(j = 1, ...s)$ *with* $P(h)^{-1}$ *exists. then*

$$\sum_{j=1}^{s} m(r, (h - \phi_j)^{-1}) \leq T(r, h) + \frac{I'}{I} N(r, h) - \frac{1}{I} N(r, (P(h)^{-1})) + S(r, h),$$

and

$$\sum_{j=1}^{s} m(r, (h - \phi_j)^{-1}) \leq T(r, h) + (I + \frac{I - 1}{2})\overline{N}(r, h) - \frac{1}{I} N(r, (P(h)^{-1})) + S(r, h),$$

where $S(r, h)$ *satisfies* $lim_{r\to\infty} \frac{S(r,h)}{T(r,h)} = 0$, *outward a set* σ *of specific quantity of the atypical intervals ,* I *also* I' *are positive integer as well* $P(h) = W(\beta_1, ..., \beta_r, h_{\alpha_1}, ...h_{\alpha_I})$ *is the vector-valued Wronskian of* β_i, *also* $h_{\alpha k}$, *being* $i = 1, ..., I'; k = 1, ..., I$.

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