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CONGRUENCES FOR $(4,5)$-REGULAR BIPARTITIONS INTO DISTINCT PARTS

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Abstract: Let $B_{4,5}(n)$ denote the number of $(4,5)$-regular bipartitions of a positive integer n into distinct parts. In this paper, we establish many infinite families of congruences modulo powers of 2 for $B_{4,5}(n)$. For example,

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n + 2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma} - 1\right) q^n$$
$$\equiv 2f_1^3 \pmod 4, \text{ for all } \alpha,\beta,\gamma \geq 0.$$

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1. Introduction

Throughout this paper, we let $|q| < 1$. We use the standard notation

$$f_k := (q^k; q^k)_\infty.$$

Following Ramanujan, we define

$$f(-q) \;:=\; f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.1}$$

Ramanujan's general theta function $f(a,b)$ [1] is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \;\; |ab|<1. \tag{1.2}$$

In Ramanujan's notation, Jacobi's famous triple product identity becomes

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.3}$$

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. An ℓ-regular partition is a partition in which none of its parts are divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n with $b_\ell(0)=1$. The generating function for $b_\ell(n)$ is

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}.$$

Recently, arithmetic properties of ℓ-regular partition functions have been studied by a number of mathematicians. Calkin et al. [2] established congruences for 5-regular partitions modulo 2 and 13-regular partitions modulo 2 and 3 using the theory of modular forms. For more details, one can see [3], [5], [6] and [7].
Suppose ℓ, $m>0$ and $(\ell,m)=1$. A partition is an (ℓ,m)-regular partitions of the positive integer n if none of the parts are divisible by ℓ or m. Let $a_{\ell,m}(n)$ denote the number of such partitions of n into distinct parts with $a_{\ell,m}(0)=1$. The generating function is given by

$$\sum_{n=0}^{\infty} a_{\ell,m}(n)q^n = \frac{(-q;q)_\infty(-q^{\ell m};q^{\ell m})_\infty}{(-q^\ell;q^\ell)_\infty(-q^m;q^m)_\infty}. \tag{1.4}$$

For example, there are 3 partitions for $a_{3,5}(11)$, namely

$$11, \;\; 8+2+1, \;\; 7+4.$$

For more details, one can see [9] and [10].
Let $B_{\ell,m}(n)$ denote the number of (ℓ,m)-regular bipartitions of n into distinct parts with $B_{\ell,m}(0)=1$ and the generating function is given by

$$\sum_{n=0}^{\infty} B_{\ell,m}(n)q^n = \frac{(-q;q)_\infty^2(-q^{\ell m};q^{\ell m})_\infty^2}{(-q^\ell;q^\ell)_\infty^2(-q^m;q^m)_\infty^2} = \frac{f_2^2 f_{2\ell m}^2 f_\ell^2 f_m^2}{f_{2\ell}^2 f_{2m}^2 f_1^2 f_{\ell m}^2}. \tag{1.5}$$

For example, there are 12 bipartitions for $B_{4,5}(6)$, namely

(0, 6), (6, 0), (3, 3), (2+1, 2+1), (3+2+1, 0), (0, 3+2+1)
(1, 3+2), (3+2,1), (2, 3+1), (3+1, 2), (3, 2+1), (2+1, 3).

2. Preliminary Results

Lemma 2.1. *we have*

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.1}$$

For proof, see [1, p. 40].

Lemma 2.2. *The following 3-dissection holds :*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^2}. \tag{2.2}$$

For proof, see [1, p. 395].

Lemma 2.3. *The following 2-dissections hold :*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \tag{2.3}$$

and

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q \frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{2.4}$$

The equation (2.3) was proved by Hirschhorn and Sellers [5], see also [11]. Replacing q by $-q$ in (2.3) and using the fact that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4},$$

we obtain (2.4).

Lemma 2.4. [8] *We have*

$$f_1 f_5^3 = f_2^3 f_{10} - q \frac{f_2^2 f_{10}^2 f_{20}}{f_4} + 2q^2 f_4 f_{20}^3 - 2q^3 \frac{f_4^4 f_{10} f_{40}^2}{f_2 f_8^2} \tag{2.5}$$

and

$$\frac{1}{f_1 f_5^3} = \frac{f_4 f_{20}^3}{f_{10}^8} + q \frac{f_{20}^4}{f_2 f_{10}^7} + 2q^2 \frac{f_4^2 f_{20}^6}{f_2^3 f_{10}^9} + +2q^3 \frac{f_4^5 f_{20}^3 f_{40}^2}{f_2^4 f_8^2 f_{10}^8}. \tag{2.6}$$

Lemma 2.5. [1, p. 303, Entry 17 (v)] *We have*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.7}$$

where $A(q) = f(-q^3, -q^4)$, $B(q) = f(-q^2, -q^5)$ *and* $C(q) = f(-q, -q^6)$.
We shall prove the following Theorems :

Theorem 2.1. *Let* $r_1 \in \{62, 78\}$, $r_2 \in \{14, 46, 62, 78\}$, $r_3 \in \{14, 62, 158, 206\}$ *and* $r_4 \in \{46, 94, 142, 238\}$. *Then for all* $\alpha, \beta, \gamma \geq 0$, *we have for modulo 16,*

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 5^{2\alpha}n + 6 \cdot 5^{2\alpha} - 1\right) q^n \equiv 8f_4f_5 + 8qf_1^3f_{10}^3, \tag{2.8}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 5^{2\alpha+1}n + 14 \cdot 5^{2\alpha+1} - 1\right) q^n \equiv 8f_1f_{20} + 8f_2^3f_5^3, \tag{2.9}$$

$$B_{4,5}\left(16 \cdot 5^{2\alpha+2}n + r_1 \cdot 5^{2\alpha+1} - 1\right) \equiv 0, \tag{2.10}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + 6 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right) q^n \equiv 8f_1^9, \tag{2.11}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1}n + 2 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma+1} - 1\right) q^n \equiv 8q^2f_7^9, \tag{2.12}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}n + 2 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma+1} - 1\right) q^n \equiv 8qf_5^9, \tag{2.13}$$

$$B_{4,5}\left(16 \cdot 3^{4\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma}n + r_2 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.14}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + 22 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right) q^n \equiv 8f_2f_3^3, \tag{2.15}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}n + 2 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma+1} - 1\right) q^n \equiv 8q^2f_{10}f_{15}^3, \tag{2.16}$$

$$B_{4,5}\left(16\cdot 3^{4\alpha+1}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_3\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.17}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(16\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+38\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n\equiv 8f_1f_6^3, \tag{2.18}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(16\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+46\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)q^n\equiv 8q^3f_5f_{30}^3, \tag{2.19}$$

$$B_{4,5}\left(16\cdot 3^{4\alpha+1}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_4\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)\equiv 0. \tag{2.20}$$

Theorem 2.2. *For $\alpha\geq 0$, we have*

$$B_{4,5}(2^{2\alpha+3}n+2^{2\alpha+3}-1)\equiv B_{4,5}(8n+7)\pmod{16}. \tag{2.21}$$

Theorem 2.3. *For $\alpha,\beta\geq 0$, we have for modulo 8,*

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+3}\cdot 5^{2\beta}n+2^{2\alpha+2}\cdot 5^{2\beta}-1)q^n\equiv 4qf_1f_5^7-2f_1^2f_5^2, \tag{2.22}$$

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+3}\cdot 5^{2\beta+1}n+2^{2\alpha+2}\cdot 5^{2\beta+1}-1)q^n\equiv 2f_1^2f_5^2+4f_1^7f_5, \tag{2.23}$$

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+4}\cdot 5^{2\beta}n+2^{2\alpha+3}\cdot 5^{2\beta}-1)q^n\equiv 4qf_1f_5^7-2f_1^2f_5^2, \tag{2.24}$$

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+4}\cdot 5^{2\beta+1}n+2^{2\alpha+3}\cdot 5^{2\beta+1}-1)q^n\equiv 2f_1^2f_5^2+4f_1^7f_5, \tag{2.25}$$

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+5}\cdot 5^{2\beta}n+2^{2\alpha+4}\cdot 5^{2\beta}-1)q^n\equiv 4qf_1f_5^7-2f_1^2f_5^2, \tag{2.26}$$

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+5}\cdot 5^{2\beta+1}n+2^{2\alpha+4}\cdot 5^{2\beta+1}-1)q^n\equiv 2f_1^2f_5^2+4f_1^7f_5. \tag{2.27}$$

Theorem 2.4. *Let $r_5\in\{22,38\}$, $r_6\in\{34,66\}$, $r_7\in\{26,42,58,74\}$, $r_8\in\{88,152\}$, $r_9\in\{136,264\}$, $r_{10}\in\{104,168,232,296\}$, $r_{11}\in\{176,304\}$, $r_{12}\in$*

$\{272, 528\}$ *and* $r_{13} \in \{208, 336, 464, 592\}$. *Then for all* $\alpha, \beta, \gamma \geq 0$, *we have for modulo 4,*

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2f_1^3, \tag{2.28}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} - 1\right) q^n \equiv 2f_7^3, \tag{2.29}$$

$$\begin{aligned} & B_{4,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \\ & \equiv \quad \begin{cases} 2 & \text{if } \ n = k(3k+1)/2 \ \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{2.30}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2f_3^3, \tag{2.31}$$

$$B_{4,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 34 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.32}$$

$$B_{4,5}\left(16 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + r_5 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.33}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2f_5^3, \tag{2.34}$$

$$B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_6 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.35}$$

$$B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_7 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.36}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2f_1^3, \tag{2.37}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} - 1\right) q^n \equiv 2f_7^3, \tag{2.38}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right) \equiv \begin{cases} 2 & \text{if } \; n=k(3k+1)/2 \; \text{ for some } k\in\mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.39}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n \equiv 2f_3^3, \tag{2.40}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+34\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.41}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_5\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.42}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}-1\right)q^n \equiv 2f_5^3, \tag{2.43}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_6\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.44}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_7\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.45}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1)q^n \equiv 2f_1^3, \tag{2.46}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1)q^n \equiv 2f_7^3, \tag{2.47}$$

$$B_{4,5}(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1) \equiv \begin{cases} 2 & \text{if } \; n=k(3k+1)/2 \; \text{ for some } k\in\mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.48}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1)q^n \equiv 2f_3^3, \tag{2.49}$$

$$B_{4,5}(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+136\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1)\equiv 0, \tag{2.50}$$

$$B_{4,5}(64\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+r_8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1)\equiv 0, \tag{2.51}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1) q^n \equiv 2f_5^3, \tag{2.52}$$

$$B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_9 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.53}$$

$$B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{10} \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.54}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) q^n \equiv 2f_1^3, \tag{2.55}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} - 1) q^n \equiv 2f_7^3, \tag{2.56}$$

$$\begin{aligned} &B_{4,5}(64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) \\ &\equiv \begin{cases} 2 & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{2.57}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) q^n \equiv 2f_3^3, \tag{2.58}$$

$$B_{4,5}(64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 136 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.59}$$

$$B_{4,5}(64 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.60}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma} - 1) q^n \equiv 2f_5^3, \tag{2.61}$$

$$B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_9 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.62}$$

$$B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma} n + r_{10} \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.63}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2f_1^3, \tag{2.64}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1\right)q^n\equiv 2f_7^3, \tag{2.65}$$

$$\begin{aligned}&B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\\ &\equiv \begin{cases}2 & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{2.66}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_3^3, \tag{2.67}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+272\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.68}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+r_{11}\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.69}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_5^3, \tag{2.70}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_{12}\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.71}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_{13}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.72}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_1^3, \tag{2.73}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}-1\right)q^n\equiv 2f_7^3, \tag{2.74}$$

$$\begin{aligned}&B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\\ &\equiv \begin{cases}2 & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{2.75}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_3^3, \tag{2.76}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+272\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.77}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_{11}\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.78}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_5^3, \tag{2.79}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_{12}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.80}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_{13}\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)\equiv 0. \tag{2.81}$$

3. Proof of the Theorem (2.1).

From (1.5), we find that

$$\sum_{n=0}^{\infty}B_{4,5}(n)q^n=\frac{f_2^2f_4^2f_{40}^2}{f_8^2f_{10}^2f_{20}^2}\times\frac{f_5^2}{f_1^2}. \tag{3.1}$$

Using (2.3) in (3.1) and extracting the terms involving q^{2n+1} from both sides, we arrive at

$$\sum_{n=0}^{\infty}B_{4,5}(2n+1)q^n=2\frac{f_2^5f_{20}^2}{f_4^2f_{10}f_1^3f_5}. \tag{3.2}$$

Using (2.1) and (2.4) in (3.2), we get

$$\sum_{n=0}^{\infty}B_{4,5}(4n+1)q^n=2\frac{f_2^{11}f_{10}^5}{f_1^8f_4^3f_5^4f_{20}}-8q\frac{f_2^2f_4^3f_{10}^2f_{20}}{f_1^5f_5^3} \tag{3.3}$$

and

$$\sum_{n=0}^{\infty}B_{4,5}(4n+3)q^n=8\frac{f_4^5f_{10}^5}{f_1^4f_2f_5^4f_{20}}-2\frac{f_2^{14}f_{10}^2f_{20}}{f_1^9f_4^5f_5^3}. \tag{3.4}$$

From the binomial theorem, it is easy to see that for any positive integers k and m,

$$f_k^{2m}\equiv f_{2k}^m \pmod 2, \tag{3.5}$$

$$f_k^{4m}\equiv f_{2k}^{2m} \pmod{2^2}, \tag{3.6}$$

$$f_k^{8m}\equiv f_{2k}^{4m} \pmod{2^3}. \tag{3.7}$$

From (3.7) along with (2.1) and (2.6) in (3.3), we get, modulo 16,

$$\sum_{n=0}^{\infty} B_{4,5}(8n+1)q^n \equiv 2\frac{f_2 f_{10}}{f_1 f_5} + 8q f_2^2 f_{10} f_1 f_5^3 \tag{3.8}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(8n+5)q^n \equiv 8 f_2^4 f_{10} + 8q^2 f_{10}^5 f_1 f_5^3. \tag{3.9}$$

Employing (2.5) in (3.9), we find that

$$\sum_{n=0}^{\infty} B_{4,5}(16n+5)q^n \equiv 8 f_4 f_5 + 8q f_1^3 f_{10}^3 \tag{3.10}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(16n+13)q^n \equiv 8q f_5^9. \tag{3.11}$$

The equation (3.10) is $\alpha = 0$ case of (2.8). Suppose the result (2.8) is true for $\alpha \geq 0$. Ramanujan recorded the following identity in his notebooks without proof:

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2 R(q^5)), \tag{3.12}$$

where $R(q) = \dfrac{f(-q,-q^4)}{f(-q^2,-q^3)}$.
For a proof of (3.12), one can see [4], [12].
Using (3.12) in (2.8) and then extracting the coefficients of q^{5n+4} from both sides, we see that

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 5^{2\alpha+1} n + 14\cdot 5^{2\alpha+1} - 1\right) q^n \equiv 8 f_1 f_{20} + 8 f_2^3 f_5^3, \tag{3.13}$$

which is (2.9). Again using (3.12) in (3.13) and extracting the terms involving q^{5n+1} from both sides, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 5^{2\alpha+2} n + 6\cdot 5^{2\alpha+2} - 1\right) q^n \equiv 8 f_4 f_5 + 8q f_1^3 f_{10}^3, \tag{3.14}$$

which implies that the congruence (2.8) is true for $\alpha+1$. Hence, by induction, the congruence (2.8) is true for non-negative integer α.

Extracting the coefficients of q^{5n+3} and q^{5n+4} in (3.13) along with (3.12), we obtain (2.10). Extracting the coefficients of q^{5n+1} from both sides of (3.11), we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(80n+29\right)q^n \equiv 8f_1^9, \tag{3.15}$$

which is $\alpha=\beta=\gamma=0$ case of (2.11). Let us consider the case $\beta=\gamma=0$. Suppose that the congruence (2.11) holds for some integer $\alpha \geq 0$. Employing the equation (2.2) in (2.11) with $\beta=\gamma=0$ and then extracting the coefficients of q^{3n} from both sides, we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(80\cdot 3^{4\alpha+1}n+10\cdot 3^{4\alpha}-1\right)q^n \equiv 8f_1^3+8qf_3^9 \equiv 8f_3+8qf_3^9+8qf_9^3, \tag{3.16}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(80\cdot 3^{4\alpha+2}n+10\cdot 3^{4\alpha+2}-1\right)q^n \equiv 8f_1^9+8f_3^3 \equiv 8qf_3^2f_9^3+8q^2f_3f_9^6+8q^3f_9^9. \tag{3.17}$$

Collecting the coefficients of q^{3n} from both sides, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(80\cdot 3^{4\alpha+3}n+10\cdot 3^{4\alpha+2}-1\right)q^n \equiv 8qf_3^9, \tag{3.18}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(80\cdot 3^{4\alpha+4}n+10\cdot 3^{4\alpha+4}-1\right)q^n \equiv 8f_1^9, \tag{3.19}$$

which implies that the congruence (2.11) is true for $\alpha+1$. By induction, the congruence (2.11) holds for all $\alpha \geq 0$ with $\beta=\gamma=0$.
Now, suppose the congruence (2.11) is true for $\alpha,\beta \geq 0$ with $\gamma=0$. Utilizing (3.12) in (2.11) and then extracting the terms involving q^{5n+4}, we deduce that

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+2}n+14\cdot 3^{4\alpha}\cdot 5^{2\beta+2}-1\right)q^n \equiv 8qf_5^9, \tag{3.20}$$

Extracting the coefficient of q^{5n+1} in (3.20), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+3}n+2\cdot 3^{4\alpha+1}\cdot 5^{2\beta+3}-1\right)q^n \equiv 8f_1^9. \tag{3.21}$$

Thus, the congruence (2.11) is true for $\beta+1$. Hence, by mathematical induction, the congruence (2.11) holds for all $\alpha, \beta \geq 0$ with $\gamma = 0$. Suppose the congruence (2.11) is true for $\alpha, \beta, \gamma \geq 0$. Employing (2.7) in (2.11) and then extracting the coefficients of q^{7n+4} from the resultant equation, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma+1} - 1\right) q^n \equiv 8q^2 f_7^9, \quad (3.22)$$

which is (2.12). Extracting the coefficients of q^{7n+2} in (3.22), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} n + 6 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} - 1\right) q^n \equiv 8f_1^9, \quad (3.23)$$

which implies that the congruence (2.11) is true for $\gamma+1$.
Hence, by induction, the congruence (2.11) holds for all integers $\alpha, \beta, \gamma \geq 0$. Employing (3.12) in (2.11) and then collecting the coefficients of q^{5n+4}, we get (2.13). From (2.13), we arrive at (2.14). Utilizing (2.2) in (2.11) and then collecting the coefficients of q^{3n+1} and q^{3n+2}, we obtain (2.15) and (2.18) respectively. From the equations (2.15) and (2.18) along with (3.12), we get (2.16) and (2.19) respectively. From the equations (2.16) and (2.19), we obtain (2.17) and (2.20) respectively.

4. Proof of the Theorem (2.2).

Employing (3.5) and (3.7) in (3.4), we find that, modulo 16,

$$\sum_{n=0}^{\infty} B_{4,5}(4n+3) q^n \equiv 8 f_2^7 f_{10} - 2\frac{f_2^2 f_{10}^2 f_{20}}{f_4 f_1 f_5^3}. \quad (4.1)$$

Using (2.6) in (4.1), we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}(8n+3) q^n \equiv 8 f_2^2 f_1^3 f_5 - 2 f_1^2 f_5^2 - 4q \frac{f_2 f_{10}^5}{f_1 f_5^3} \quad (4.2)$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(8n+7) q^n \equiv -2\frac{f_{10} f_1 f_5^3}{f_2} - 4q \frac{f_{20}^2 f_5^2}{f_1^2}. \quad (4.3)$$

Employing (2.3) and (2.5) in (4.3), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}(16n+7) q^n \equiv 8q f_{10}^2 f_1 f_5^3 - 2 f_1^2 f_5^2 - 4q \frac{f_{10}^3 f_1^3 f_5}{f_2} \quad (4.4)$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(16n+15)q^n \equiv 2\frac{f_{10}f_1f_5^3}{f_2} - 4f_2^2f_{10}^2. \tag{4.5}$$

Using (2.5) in (4.5), we get

$$\sum_{n=0}^{\infty} B_{4,5}(32n+15)q^n \equiv 4q\frac{f_{10}^3f_1^3f_5}{f_2} - 2f_1^2f_5^2 \tag{4.6}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(32n+31)q^n \equiv -2\frac{f_{10}f_1f_5^3}{f_2} - 4q\frac{f_{20}^2f_5^2}{f_1^2}. \tag{4.7}$$

From the equations (4.3) and (4.7), we have

$$B_{4,5}(32n+31) \equiv B_{4,5}(8n+7). \tag{4.8}$$

Hence, by mathematical induction on α, we obtain (2.21).

5. Proof of the Theorem (2.3)

From the equation (4.2), we get, modulo 8,

$$\sum_{n=0}^{\infty} B_{4,5}(8n+3)q^n \equiv 4qf_1f_5^7 - 2f_1^2f_5^2 \tag{5.1}$$

which is $\alpha \geq 0$ and $\beta = 0$ case of (2.22). Suppose the congruence (2.22) is true for $\alpha, \beta \geq 0$. Using (3.12) in (2.22) and then collecting the coefficients of q^{5n+2}, we get

$$\sum_{n=0}^{\infty} B_{4,5}(2^{2\alpha+3}\cdot 5^{2\beta+1}n + 2^{2\alpha+2}\cdot 5^{2\beta+1} - 1)q^n \equiv 2f_1^2f_5^2 + 4f_1^7f_5, \tag{5.2}$$

which proves (2.23). Again collecting the coefficients of q^{5n+2} from (5.2) along with (3.12), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}(2^{2\alpha+3}\cdot 5^{2\beta+2}n + 2^{2\alpha+2}\cdot 5^{2\beta+2} - 1)q^n \equiv 4qf_1f_5^7 - 2f_1^2f_5^2, \tag{5.3}$$

which implies that the congruence (2.22) is true for $\beta + 1$. By mathematical induction, the congruence (2.22) is true for all integers $\alpha, \beta \geq 0$. From the equation (4.4), we find that

$$\sum_{n=0}^{\infty} B_{4,5}(2^{2\alpha+4}n + 2^{2\alpha+3} - 1)q^n \equiv 4qf_1f_5^7 - 2f_1^2f_5^2, \tag{5.4}$$

which is $\alpha \geq 0$ and $\beta = 0$ case of (2.24). The rest of the proofs of the identities (2.24) and (2.25) are similar to the proofs of the identities (2.22) and (2.23). So, we omit the details. From the congruence (4.6), we get,

$$\sum_{n=0}^{\infty} B_{4,5}(32n+15)q^n \equiv 4qf_1f_5^7 - 2f_1^2f_5^2, \tag{5.5}$$

which is $\alpha \geq 0$ and $\beta = 0$ case of (2.26). The rest of the proofs of the identities (2.26) and (2.27) are similar to the proofs of the identities (2.22) and (2.23). So, we omit the details.

6. Proof of the Theorem (2.4)

From the equation (3.8), we get, modulo 4,

$$\sum_{n=0}^{\infty} B_{4,5}(8n+1)q^n \equiv 2\frac{f_1f_{10}}{f_5} \tag{6.1}$$

Using (2.4) in (6.1), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}(16n+1)q^n \equiv 2f_1^3 \tag{6.2}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(16n+9)q^n \equiv 2f_5^3. \tag{6.3}$$

The equation (6.2) is $\alpha = \beta = \gamma = 0$ case of (2.28). Suppose that the congruence (2.28) holds for some integer $\alpha \geq 0$ with $\beta = \gamma = 0$. Employing the equation (2.2) in (2.28) with $\beta = \gamma = 0$, we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha}n + 2 \cdot 3^{2\alpha} - 1\right) q^n \equiv 2(f_3 + qf_9^3). \tag{6.4}$$

Extracting the coefficients of q^{3n+1} from (6.4), we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha+1}n + 2 \cdot 3^{2\alpha+2} - 1\right) q^n \equiv 2f_3^3, \tag{6.5}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha+2}n + 2 \cdot 3^{2\alpha+2} - 1\right) q^n \equiv 2f_1^3, \tag{6.6}$$

which implies that the congruence (2.28) is true for $\alpha+1$. Hence, by induction, the congruence (2.28) holds for any non-negative integer α with $\beta=\gamma=0$. Now, suppose that the congruence (2.28) holds for some integers $\alpha, \beta \geq 0$ and $\gamma=0$. Employing the equation (3.12) in the equation (2.28), we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} n+2 \cdot 3^{2\alpha} \cdot 5^{2\beta}-1\right) q^n \equiv 2 f_{25}^3\left(R(q^5)^{-1}-q-q^2 R(q^5)\right)^3 . \tag{6.7}$$

Extracting the coefficients of q^{5n+3} in (6.7), we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} n+2 \cdot 3^{2\alpha} \cdot 5^{2\beta+2}-1\right) q^n \equiv 2 f_5^3 . \tag{6.8}$$

Extracting the coefficients of q^{5n} in (6.8), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} n+2 \cdot 3^{2\alpha} \cdot 5^{2\beta+2}-1\right) q^n \equiv 2 f_1^3, \tag{6.9}$$

which implies that the congruence (2.28) is true for $\beta+1$. Hence, by induction, the congruence (2.28) holds for any non-negative integers α and β with $\gamma=0$. Suppose that the congruence (2.28) holds for some integers $\alpha, \beta, \gamma \geq 0$. Employing (2.7) in (2.28) and then collecting the coefficients of q^{7n+6} from the resultant equation, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n+2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}-1\right) q^n \equiv 2 f_7^3, \tag{6.10}$$

which proves (2.29) and extracting the coefficients of q^{7n} in (6.10), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} n+2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}-1\right) q^n \equiv 2 f_1^3, \tag{6.11}$$

which implies that the congruence (2.28) is true for $\gamma+1$. Hence, by induction, the congruence (2.28) holds for any non-negative integers α, β and γ. Employing the equation (2.2) in the equation (2.28) and then extracting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we obtain (2.30), (2.31) and (2.32) respectively. Collecting the coefficients of q^{3n+1} and q^{3n+3} from (2.31), we get (2.33). Using the equation

(2.28) along with the equation (3.12), we obtain (2.34) and (2.35). From the equation (2.34), we get (2.36). Extracting the coefficients of q^{5n} in (6.3), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(80n+9\right) q^{n} \equiv 2f_1^3, \tag{6.12}$$

which is $\alpha=\beta=\gamma=0$ case of (2.37). The rest of the proofs of the identities (2.37)- (2.45) are similar to the proofs of the identities (2.28)- (2.36). So, we omit the details. From the equation (4.4), we find that

$$\sum_{n=0}^{\infty} B_{4,5}(32n+7)q^{n} \equiv \frac{2f_1f_5^3}{f_{10}} \equiv 2f_2^3+2qf_{10}^3, \tag{6.13}$$

which yields

$$\sum_{n=0}^{\infty} B_{4,5}(64n+7)q^{n} \equiv 2f_1^3 \tag{6.14}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(64n+39)q^{n} \equiv 2f_5^3. \tag{6.15}$$

The rest of the proofs of the identities (2.46)- (2.63) are similar to the proofs of the identities (2.28)- (2.36). So, we omit the details.
From the congruence (5.5), we have

$$\sum_{n=0}^{\infty} B_{4,5}(64n+15)q^{n} \equiv \frac{2f_1f_5^3}{f_{10}} \equiv 2f_2^3+2qf_{10}^3, \tag{6.16}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}(128n+15)q^{n} \equiv 2f_1^3 \tag{6.17}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(128n+79)q^{n} \equiv 2f_5^3. \tag{6.18}$$

The rest of the proofs of the identities (2.64)-(2.81) are similar to the proofs of the identities (2.28)- (2.36).
So, we omit the details.

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