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HEMI-SLANT SUBMANIFOLDS OF GENERALIZED D-CONFORMAL DEFORMED β-KENMOTSU MANIFOLD

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Abstract: We study some geometric properties such as integrability, geodesic foliation of hemi-slant submanifolds of generalized D-conformal deformed β-Kenmotsu manifold.

Keywords and Phrases: Hemi-slant submanifolds, generalized D-conformal deformation, integrability, geodesic foliation.

2010 Mathematics Subject Classification: 53C25, 53C40, 53C50.

1. Introduction

Study of slant submanifolds was initiated by Chen [8], as a generalization of both holomorphic and totally real submanifolds of a Kahler manifold. Slant submanifolds have been studied in different kind of structures: almost contact [9], neutral Kahler [2], Lorentzian Sasakian [3], and Sasakian [5] by several geometers. Papaghiuc [12] introduced semi-slant submanifolds of a Kahler manifold as a natural generalization of slant submanifold. Sari and Vanli [13] investigated semi-slant submanifolds of a Lorentz Kenmotsu manifold and obtained some curvature properties for semi-slant submanifold of a Lorentz Kenmotsu space form. Carriazo [6], introduced bi-slant submanifolds of an almost Hermitian manifold as a generalization of semi-slant submanifolds. One of the classes of bi-slant submanifolds is that of anti-slant submanifolds, which are studied by Carriazo [6].

In [1], generalized D-conformal deformations are applied to trans-Sasakian manifolds where the covariant derivatives of the deformed metric is evaluated under the condition that the functions used in deformation depend only on the direction of the characteristic vector field of the trans-Sasakian structure. In this current year 2019, Ozdemir, Aktay and Solgun [10], have derived a relation between the covariant derivatives of the ambient manifold and its generalized D-conformal deformed manifold i.e. ∇ and $\nabla^\star$ for an almost contact metric structure without choosing any condition on the positive functions a and b. The integrability conditions of the distributions of submanifolds have been studied since years. Geometers have proved, if a and b depend only in the direction of ξ then the generalized D-conformal deformed β-Kenmotsu manifold turns into $\beta^\star$-Kenmotsu manifold where $\beta^\star = \frac{1}{a}\beta + \frac{1}{2ab}\xi(b)$.

In the present paper, we study the integrability of hemi-slant submanifolds and totally geodesic foliation of generalized D-conformal deformed β-Kenmotsu manifold.

2. Preliminaries

Let M be an m dimensional almost contact metric manifold with the structure tensors (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ a vector field, η a 1-form and g is a Riemannian metric on M [2] satisfying

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \ \ \eta \cdot \phi = 0, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \end{aligned} \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$.

Let Φ denote the 2-form in M given by $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric manifold (M, ϕ, ξ, η, g) is called β-Kenmotsu manifold, if the relation

$$(\nabla_X \phi)Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.2}$$

is satisfied, where β is a smooth function on M.

For β-Kenmotsu manifold the following relations hold:

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \tag{2.3}$$

$$\nabla_X \xi = \beta(X - \eta(X)\xi). \tag{2.4}$$

If we take

$$\phi^\star = \phi, \ \ \xi^\star = \frac{1}{a}\xi, \ \ \eta^\star = a\eta, \ \ g^\star = bg + (a^2 - b)\eta \otimes \eta, \tag{2.5}$$

where a and b are positive functions on M, one can easily check that $(M, \phi^\star, \xi^\star, \eta^\star, g^\star)$ is an almost contact metric manifold too. This deformation is

called a generalized D-conformal deformation.
After this deformation, the derivation of new fundamental 2-form $\tilde{\Phi}$ is

$$d\Phi^\star = d\Phi(X,Y,Z) + X(b)\Phi(Y,Z) - Y(b)\Phi(X,Z) + Z(b)\Phi(X,Y). \tag{2.6}$$

Lemma [A]. [10] *Consider a generalized D-conformal deformation of an almost contact metric structure such that $g(\nabla_X\xi, Y) = g(\nabla_Y\xi, X)$, where a and b are positive functions. After a generalized D-conformal deformation, the new Levi-Civita covariant derivative is*

$$\begin{aligned}\nabla^\star_X Y = \nabla_X Y &+ \frac{(a^2-b)}{b} g(\nabla_X\xi, Y)\xi + \frac{1}{2b}(X(b)Y + Y(b)X)\\ &+ \frac{1}{a}(X(a)\eta(Y) + Y(a)\eta(X))\xi + \frac{a}{b}\eta(Y)\phi^2(grad\ a)\\ &- \frac{1}{a}\xi(a)\eta(X)\eta(Y)\xi - \frac{1}{2a^2}\xi(b)g(\phi X, \phi Y)\xi\\ &+ \frac{1}{2b} g(\phi X, \phi Y)\phi^2(grad\ a).\end{aligned} \tag{2.7}$$

Theorem [A]. [10] *Let (M, ϕ, ξ, η, g) be a β-Kenmotsu manifold and consider a generalized D-conformal deformation with a and b positive functions. If $grad\ a = g(grad\ a, \xi)\xi$ and $grad\ b = g(grad\ b, \xi)\xi$, then $(M, \phi^\star, \xi^\star, \eta^\star, g^\star)$ is a $\beta^\star$-Kenmotsu manifold, where*

$$\beta^\star = \frac{1}{a}\beta + \frac{1}{2ab}\xi(b). \tag{2.8}$$

For a $\beta^\star$-Kenmotsu manifold, the following relations hold:

$$grad\ a = \xi(a)\xi \tag{2.9}$$

and

$$(\nabla^\star_X \phi^\star)Y = \beta^\star(g^\star(\phi^\star X, Y)\xi^\star - \eta^\star Y\phi^\star X). \tag{2.10}$$

Let N be n- dimensional immersed submanifold of M. Then the Gauss and Weingarten formulas [3] are respectively, given by

$$\nabla_X Y = \tilde{\nabla}_X Y + \sigma(X,Y) \tag{2.11}$$

and

$$\nabla_X V = -A_V X + \tilde{\nabla}^\perp_X V, \tag{2.12}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^\perp)$, where σ, $\tilde{\nabla}$, $\tilde{\nabla}^\perp$ and A denote respectively the second fundamental form, Levi-civita connection, the normal connection and the shape operator on the submanifold N. The second fundamental form and shape operator are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V), \tag{2.13}$$

where g denotes the induced metric on N as well as the Riemannian metric g on M.

Let N be a submanifold of an almost contact metric manifold $M(\phi, \xi, \eta, g)$. For $X \in \Gamma(TN)$, we write

$$\phi X = TX + FX, \tag{2.14}$$

where $TX \in \Gamma(TN)$ and $FX \in \Gamma(T^{\perp}N)$.

For $V \in \Gamma(T^{\perp}N)$, we have

$$\phi V = tV + fV, \tag{2.15}$$

where $tV \in \Gamma(TN)$ and $fV \in \Gamma(T^{\perp}N)$.

3. Hemi-slant Submanifolds of Generalized D-conformal Deformed β-Kenmotsu Manifold

Definition 3.1. *A submanifold N of M is said to be hemi-slant submanifold of an almost contact metric manifold M if there exists two orthogonal complementary distributions D_1 and D_2 on N such that*

1. $TN = D_1 \oplus D_2 \oplus \langle \xi \rangle$,
2. *the distribution D_2 is slant with slant angle $\theta \neq \frac{\pi}{2}$,*
3. *the distribution D_1 is anti-invariant, i.e. $\phi D_1 \subseteq T^{\perp}N$.*

Let a non-zero $X \in \Gamma(D_2)$, we derive

$$cos\theta_2 = \frac{g^{\star}(\phi^{\star}X, TX)}{\|\phi^{\star}X\|\|TX\|} = \frac{\|TX\|}{\|\phi X\|}. \tag{3.1}$$

Squaring on the both sides of (3.1), we get

$$cos^2\theta_2\|\phi X\|^2 = \|TX\|^2. \tag{3.2}$$

Replacing X by $X + Y$ for $Y \in \Gamma(D_2)$ and a simple calculation gives

$$g^{\star}((T^2 - cos^2\theta_2\phi^2)X, Y) = g^{\star}((T^2 - cos^2\theta_2\phi^2)Y, X). \tag{3.3}$$

But $T^2 - cos^2\theta_2\ \phi^2$ is symmetric and since $X, Y \in \Gamma(D_2)$ then from (3.3) we have

$$T^2X = -cos^2\theta_2 X. \tag{3.4}$$

Therefore we state the following

Proposition 3.1. *Let N be a hemi-slant submanifold of a generalized D-conformal deformed manifold M. Then for any $X \in \Gamma(TN)$,*

1. *if θ_1 is the slant angle with respect to the distribution D_1 then $T^2X = 0$,*
2. *if θ_2 is the slant angle with respect to the distribution D_2 then*
   a. *for $\theta_2 = 0$, we have $T^2X = X$,*
   b. *for $\theta_2 \neq \frac{\pi}{2}$, we have $T^2X = -cos^2\theta_2 X$.*

Let $X, Y \in D_2 \oplus \langle\xi\rangle$ and $Z \in D_1$, then we have

$$g^\star([X,Y],Z) = g^\star(\phi[X,Y],\phi Z) + \eta^\star([X,Y])\eta^\star(Z). \tag{3.5}$$

Since the distribution $Z \in \Gamma(D_1)$, we have $\eta^\star(Z) = 0$ and therefore (3.5) reduces to

$$g^\star([X,Y],Z) = g^\star(-(\nabla^\star_X\phi)Y + (\nabla^\star_Y\phi)X + \nabla^\star_X\phi Y - \nabla^\star_Y\phi X, \phi Z). \tag{3.6}$$

From (2.7), (2.12) and (2.5) and using the definition of hemi-slant submanifold, we obtain

$$g^\star([X,Y],Z) = b\ g(-A_{\phi Y}X + A_{\phi X}Y - \tilde{\nabla}^\perp_Y\phi X + \tilde{\nabla}^\perp_X\phi Y, \phi Z). \tag{3.7}$$

Hence we state the following:

Theorem 3.1. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. Then the distribution $D_2 \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(-A_{\phi Y}X + A_{\phi X}Y - \tilde{\nabla}^\perp_Y\phi X + \tilde{\nabla}^\perp_X\phi Y, \phi Z) = 0.$$

In view of the Theorem 3.1 and the Theorem A we state the following:

Corollary 3.1. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. If $grad\ a = g(grad\ a, \xi)\xi$ and $grad\ b = g(grad\ b, \xi)\xi$, then the distribution $D_2 \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(-A_{\phi Y}X + A_{\phi X}Y - \tilde{\nabla}^\perp_Y\phi X + \tilde{\nabla}^\perp_X\phi Y, \phi Z) = 0.$$

Next let $X, Y \in D_1 \oplus \langle\xi\rangle$ and $Z \in D_2$, then we have

$$g^\star([X,Y],Z) = g^\star(\phi[X,Y],\phi Z) + \eta^\star([X,Y])\eta^\star(Z). \tag{3.8}$$

Since the distribution $Z \in \Gamma(D_2)$, we have $\eta^\star(Z) = 0$ and therefore (3.8) reduces to

$$g^\star([X,Y],Z) = g^\star(-(\tilde{\nabla}_X\phi)Y + (\tilde{\nabla}_Y\phi)X + \tilde{\nabla}_X\phi Y - \tilde{\nabla}_Y\phi X, \phi Z). \tag{3.9}$$

From (2.7), (2.12) and (2.14) and using the definition of hemi-slant submanifold, we obtain

$$g^\star([X,Y],Z) = b\ g(\sigma^\star(X,TY) - \sigma^\star(TX,Y) + \tilde{\nabla}^\perp_X FY - \tilde{\nabla}^\perp_Y FX, \phi Z). \tag{3.10}$$

Hence we state the following:

Theorem 3.2. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. Then the distribution $D_1 \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(\sigma^\star(X,TY) - \sigma^\star(TX,Y) + \tilde{\nabla}^\perp_X FY - \tilde{\nabla}^\perp_Y FX, \phi Z) = 0.$$

And similarly we can state the corollary:

Corollary 3.2. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. If $grad\ a = g(grad\ a,\xi)\xi$ and $grad\ b = g(grad\ b,\xi)\xi$, then the distribution $D_1 \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(\sigma^\star(X,TY) - \sigma^\star(TX,Y) + \tilde{\nabla}^\perp_X FY - \tilde{\nabla}^\perp_Y FX, \phi Z) = 0.$$

Now let $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, then we may write

$$g^\star(\tilde{\nabla}^\star_Y X, Z) = g^\star(\phi\nabla^\star_Y X, \phi Z). \tag{3.11}$$

Using (2.7) and (2.14) in (3.11), we obtain

$$\begin{aligned} g^\star(\tilde{\nabla}^\star_Y X, Z) &= -g^\star(\nabla_Y X, T^2Z + FTZ) + g^\star(\nabla_Y\phi X - (\nabla_Y\phi)X, FZ) \\ &\quad + g^\star(\frac{1}{2}(X(b)\phi Y + Y(b)\phi X) + \frac{a}{b}(\eta(X)\phi^3 grad\ a) \\ &\quad + \frac{1}{2b}(g(\phi X,\phi Y)\phi^3 grad\ a), \phi Z). \end{aligned} \tag{3.12}$$

Since $Z \in \Gamma(D_2)$ therefore $T^2Z = 0$, then from (3.12) we get

$$\begin{aligned} g^\star(\tilde{\nabla}^\star_Y X, Z) &= -g^\star(\nabla_Y X, FTZ) + g^\star(\nabla_Y\phi X - (\nabla_Y\phi)X, FZ) \\ &\quad - g^\star(\frac{1}{2}(X(b)\phi^2 Y + Y(b)\phi^2 X) + \frac{a}{b}(\eta(X)\phi^4 grad\ a) \\ &\quad + \frac{1}{2b}(g(\phi X,\phi Y)\phi^4 grad\ a), Z). \end{aligned} \tag{3.13}$$

From (2.1) and (2.2) and using the property of orthogonal distribution we find

$$g^{\star}(\tilde{\nabla}^{\star}_{Y}X, Z) = b\ g(A_{FZ}\phi X - A_{FTZ}X, Y) + g^{\star}(\frac{a}{b}\eta(X)\phi^{2} grad\ a + \frac{1}{2b} g(\phi X, \phi Y)\phi^{2} grad\ a, Z). \tag{3.14}$$

Therefore we state the following

Theorem 3.3. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. If grad $a \perp X$ and grad $a \perp \xi$ then the distribution D_1 defines a totally geodesic foliation if and only if*
$g(A_{FZ}\phi X - A_{FTZ}X, Y) = 0$ or $b = 0$.
In view of above theorem we have the following corollary:

Corollary 3.3. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. If $grad\ a = g(grad\ a, \xi)\xi$ and $grad\ b = g(grad\ b, \xi)\xi$, then the distribution D_1 defines a totally geodesic foliation if and only if*
$g(A_{FZ}\phi X - A_{FTZ}X, Y) = 0$ or $b = 0$.
Next let $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, then we may write

$$g^{\star}(\tilde{\nabla}^{\star}_{Y}X, Z) = g^{\star}(\phi\nabla^{\star}_{Y}X, \phi Z). \tag{3.15}$$

Using (2.7) and (2.14) in (3.15), we obtain

$$\begin{aligned} g^{\star}(\tilde{\nabla}^{\star}_{Y}X, Z) &= -g^{\star}(\nabla_{Y}X, T^{2}Z + FTZ) + g^{\star}(\nabla_{Y}\phi X - (\nabla_{Y}\phi)X, FZ) \\ &+ g^{\star}(\frac{1}{2}(X(b)\phi Y + Y(b)\phi X) + \frac{a}{b}(\eta(X)\phi^{3} grad\ a) \\ &+ \frac{1}{2b}(g(\phi X, \phi Y)\phi^{3} grad\ a), \phi Z). \end{aligned} \tag{3.16}$$

Since $Z \in \Gamma(D_2)$ therefore $T^{2}Z = -cos^{2}\theta_{2}$, then from (3.12) we get

$$\begin{aligned} sin^{2}\theta_{2}\ g^{\star}(\tilde{\nabla}^{\star}_{Y}X, Z) &= -g^{\star}(\nabla_{Y}X, FTZ) + g^{\star}(\nabla_{Y}\phi X - (\nabla_{Y}\phi)X, FZ) \\ &- g^{\star}(\frac{1}{2}(X(b)\phi^{2}Y + Y(b)\phi^{2}X) + \frac{a}{b}(\eta(X)\phi^{4} grad\ a) \\ &+ \frac{1}{2b}(g(\phi X, \phi Y)\phi^{4} grad\ a), Z). \end{aligned} \tag{3.17}$$

From (2.1) and (2.2) and using the property of orthogonal distribution we find

$$sin^{2}\theta_{2}\ g^{\star}(\tilde{\nabla}^{\star}_{Y}X, Z) = b\ g(A_{FZ}\phi X - A_{FTZ}X, Y) + g^{\star}(\frac{a}{b}\eta(X)\phi^{2} grad\ a + \frac{1}{2b} g(\phi X, \phi Y)\phi^{2} grad\ a, Z). \tag{3.18}$$

Hence we state the following theorem:

Theorem 3.4. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. If grad $a \perp X$ and grad $a \perp \xi$ then the distribution D_2 defines a totally geodesic foliation if and only if*
$g(A_{FZ}\phi X - A_{FTZ}X, Y) = 0$ *or* $b = 0$.

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