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UNIQUENESS AND ITS GENERALIZATION OF MEROMORPHIC FUNCTIONS CONCERNING DIFFERENTIAL POLYNOMIALS

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Abstract: Considering the generalization of uniqueness of meromorphic functions of differential monomials, we obtain that if two non-constant meromorphic functions $f(z)$ and $g(z)$ satisfy $E_k(1, f^n f^{(k)}) = E_k(1, g^n g^{(k)})$, where k and n are two positive integers satisfying $k \geq 3$ and $n \geq 2k+9$, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2, c are three constants, satisfying $(-1)^k (c_1 c_2)^n c^{2k} = 1$.

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1. Introduction and Main Results

In this paper, we use the standard notations and terms in the value distribution theory [1].

Let $f(z)$ be a non constant meromorphic function on the complex plane C. Define $E(a, f) = \{z/f(z) - a = 0\}$, where a zero point with multiplicity m is counted m times in the set. If there zero points are only counted once, then we denote the set by $\overline{E}(a, f)$. Let k be a positive integer. Define

$E_k(a,f) = \{z/f(z) - a = 0, \exists\ i, 1 \leq i \leq k, f^{(i)}(z) \neq 0\}$, where a zero point with multiplicity m is counted m times in the set.

Let $f(z)$ and $g(z)$ be two non constant meromorphic funtions. If $E(a,f) = E(a,g)$, then we say that $f(z)$ and $g(z)$ share the value CM; if $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that $f(z)$ and $g(z)$ share the value IM.

Additional, we denote by $N_{k)}(r,f)$ the counting function for poles of $f(z)$ with multiplicity $\leq k$, and by $\overline{N}_{k)}$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,f)$ be the counting function for poles of $f(z)$ with multiplicity $\geq k$, and by $\overline{N}_{(k}(r,f)$ the corresponding one for which multiplicity is not counted. Set

$$N_k(r,f) = \overline{N}(r,f) + \overline{N}_{(2}(r,f) + - - - + \overline{N}_{(k}(r,f).$$

Similarly, we have the notation: $N_{k)}(r,\frac{1}{f}), \overline{N}_{k)}(r,\frac{1}{f}), N_{(k}(r,\frac{1}{f}), \overline{N}_{(k}(r,\frac{1}{f}), N_k(r,\frac{1}{f})$. If $\bar{E}(1,f) = \bar{E}(1,g)$, we denote by $N_{11}(r,\frac{1}{f-1})$ the counting function for common simple 1-points of both $f(z)$ and $g(z)$ where multiplicity is not counted.

In 2011, H. Huang and B. Huang [10] extend the above result as follows.

Theorem A. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, $k(\geq 3)$, $n(\geq 11)$ be two positive integers. If $E_k(1, f^n f') = E_k(1, g^n g')$, then either $f_{(z)} = c_1 e^{cz}, g_{(z)} = c_2 e^{-cz}$, where c_1, c_2, c are three constants, satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

Theorem B. *Let $f(z)$ and $g(z)$ be two non constant meromorphic functions, $n(\geq 13)$ be a positive integer. If $E_2(1, f^n f') = E_2(1, g^n g')$, then the conclusion of Theorem A holds.*

Theorem C. *Let $f(z)$ and $g(z)$ be two non constant meromorphic functions, $n(\geq 19)$ be a positive integer. If $E_1(1, f^n f') = E_1(1, g^n g')$, then the conclusion of Theorem A holds.*

In this paper, we will extend the above results as follows.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, $k(\geq 3)$, $n(\geq 2k+9)$ be two positive integers. If $E_k(1, f^n f^{(k)}) = E_k(1, g^n g^{(k)})$, then either $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where c_1, c_2, c are three constants, satisfying $(-1)^k (c_1c_2)^n c^{2k} = 1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two non constant meromorphic functions, $n(\geq 2k+11)$ be a positive integer. If $E_2(1, f^n f^{(k)}) = E_2(1, g^n g^{(k)})$, then the conclusion of Theorem 1.1 holds.*

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two non constant meromorphic functions, $n(\geq 4k+15)$ be a positive integer. If $E_1(1, f^n f^{(k)}) = E_1(1, g^n g^{(k)})$, then the conclusion of Theorem 1.1 holds.*

2. Some Lemmas

For the proof of our results, we need the following lemmas.

Lemma 2.1 ([7]). *Let f be a non-constant meromorphic function and let $a_0, a_1, ..., a_n$ be finite complex numbers such that $a_n \neq 0$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f' + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2 (see[4]). *Let f be a non-constant meromorphic functions and a_1, a_2, a_3 be three distinct small meromorphic functions of f, then*

$$T(r, f) \leq \sum_{j=1}^{3} \bar{N}(r, \frac{1}{f - a_j}) + S(r, f).$$

Lemma 2.3. *Let $f, g \in A, n \geq 2$ and k be a positive integer. If $f^n f^{(k)} g^n g^{(k)} = 1$, then $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ where c_1, c_2 and c are constants such that $(-1)^k (c_1 c_2)^{n+1} c^{2k} = 1$.*

Proof. From

$$f^n f^{(k)} g^n g^{(k)} = 1, \tag{2.1}$$

we have

$$f(z) = e^{\alpha(z)}, \;\; g(z) = e^{\beta(z)}, \tag{2.2}$$

where $\alpha(z)$ and $\beta(z)$ are non constant entire functions.

Then $T(r, \frac{f'}{f}) = T(r, \frac{e^{\alpha}\alpha'}{e^{\alpha}}) = T(r, \alpha')$. We claim that $\alpha(z)+\beta(z) = c$, c is a constant. From (2.2), we know that either α and β are transcendental functions or both α and β are polynomials.

We deduce from (2.2) that

$$f^{(k)} = [(\alpha')^k + P_{k-1}(\alpha')]e^{\alpha}.$$

$$g^{(k)} = [(\beta')^k + Q_{k-1}(\beta')]e^{\beta}.$$

where $P_{k-1}(\alpha')$ and $Q_{k-1}(\beta')$ are differential polynomials in α' and β' of degree at most $(k-1)$ respectively. Thus by (2.1) we obtain that

$$[(\alpha')^k + P_{k-1}(\alpha')][(\beta')^k + Q_{k-1}(\beta')]e^{(n+1)(\alpha+\beta)} = 1, \tag{2.3}$$

we deduce from (2.3) that $\alpha(z) + \beta(z) = c$, c is a constant.

If $k = 1$, from (2.2) we get,

$$\alpha'\beta' e^{(n+1)(\alpha+\beta)} = 1. \tag{2.4}$$

Let $\alpha + \beta = \gamma$. If α and β are transcendental entire functions, then γ is not a constant and (2.4) implies that

$$\alpha'(\gamma' - \alpha')e^{(n+1)\gamma} = 1. \tag{2.5}$$

Since

$$\begin{aligned} T(r,\gamma') = m(r,\gamma') &= m(r, \frac{e^{(n+1)\gamma'}}{e^{(n+1)\gamma}}\gamma') \\ &= m(r, \frac{(e^{(n+1)\gamma})'}{e^(n+1)\gamma}) = S(r, e^{(n+1)\gamma}). \end{aligned}$$

Thus (2.5) implies that

$$\begin{aligned} T(r, e^{(n+1)\gamma}) &= T(r, \frac{1}{\alpha'(\gamma' - \alpha')}) \\ &\leq T(r, \alpha'(\gamma' - \alpha')) + O(1) \\ &\leq 2T(r, \alpha') + S(r, e^{(n+1)\gamma}). \end{aligned}$$

Which implies that

$$T(r, e^{(n+1)\gamma}) = O(T(r, \alpha')).$$

Thus $T(r, \gamma') = S(r, \alpha')$. In view of (2.5) and by Lemma 2.2, we get

$$T(r, \alpha') \leq \bar{N}(r, \alpha') + \bar{N}(r, \frac{1}{\alpha'}) + \bar{N}(r, \frac{1}{(\gamma' - \alpha')}) + S(r, \alpha').$$

Since α and β are transcendental entire function and in view of (2.5), we obtain $T(r, \alpha') \leq S(r, \alpha')$ and this implies that α' is a constant, which is a contradiction. Thus α and β are both polynomials and $\alpha(z) + \beta(z) = c$, for a constant c.

Hence from (2.3), we get

$$(\alpha')^{2k} = 1 + P_{2k-1}(\alpha'), \tag{2.6}$$

where $P_{2k-1}(\alpha')$ is a differential polynomial in α' of degree at most $(2k-1)$. From (2.6), we have

$$\begin{aligned} 2kT(r, \alpha') = T(r, (\alpha')^{2k}) = m(r, (\alpha')^{2k}) &\leq m(r, P_{2k-1}(\alpha')) + O(1) \\ &= m(r, \frac{P_{2k-1}(\alpha')}{(\alpha')^{2k-1}}(\alpha')^{2k-1}) + O(1) \\ &\leq m(r, \frac{P_{2k-1}(\alpha')}{(\alpha')^{2k-1}}) + m(r, (\alpha')^{2k-1}) + O(1) \\ &\leq 2k - 1T(r, \alpha') + S(r, \alpha'). \end{aligned}$$

Therefore $T(r,\alpha') \leq S(r,\alpha')$. Which implies that α' is a constant. Thus $\alpha = cz+c_1$, $\beta = -cz + c_2$. By (2.2), we represent f and g as $f = c_1e^{cz}$ and $g = c_2e^{-cz}$. Where c_1, c_2 and c are constants such that $(-1)^k(c_1c_2)^{n+1}c^{2k} = 1$.
This completes the proof of Lemma.

Lemma 2.4 ([8]). *Let f be a non constant meromorphic function, k a positive integer, then*

$$N(r,\frac{1}{f^{(k)}}) \leq N(r,\frac{1}{f}) + k\bar{N}(r,f) + S(r,f).$$

Lemma 2.5 ([9]). *Let f and g be two non-constant meromorphic functions, and let k be a positive integer. If $E_k(1,f) = E_k(1,g)$, then one of the following cases must occur:*

$$\begin{aligned} T(r,f)+T(r,g) \leq\ & \overline{N}_2(r,f) + \overline{N}_2(r,\frac{1}{f}) + \overline{N}_2(r,g) + \overline{N}_2(r,\frac{1}{g}) \\ & + \overline{N}_2(r,\frac{1}{f-1}) + \overline{N}_2(r,\frac{1}{g-1}) - N_{11}(r,\frac{1}{f-1}) \\ & + N_{(k+1}(r,\frac{1}{f-1}) + N_{(k+1}(r,\frac{1}{g-1}) + S(r,f) + S(r,g). \end{aligned} \tag{2.7}$$

$$f = \frac{(b+1)g + (a-b-1)}{bg + (a-b)}, \tag{2.8}$$

where $a(\neq 0), b$ are two constants.

Lemma 2.6. *Let f and g be two non constant meromorphic functions, $n \geq 2k+5$ be a positive integer. Set*

$$F = f^n f^{(k)} \quad G = g^n g^{(k)} = 1,$$

if

$$f = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \tag{2.9}$$

where $a(\neq 0), b$ are two constants, then $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2, c are three constants, satisfying $(-1)^k(c_1c_2)^nc^{2k} = 1$,
Proof. By Lemma 2.1, we get

$$\begin{aligned} T(r,F) = T(r,f^nf^{(k)}) & \leq T(r,f^n) + T(r,f^{(k)}) \\ & \leq nT(r,f) + T(r,f) + S(r,f) \\ & \leq (n+1)T(r,f) + S(r,f), \end{aligned} \tag{2.10}$$

$$\begin{aligned} nT(r,f) &= T(r,f^n)+S(r,f) \\ &= N(r,f^n)+m(r,f^n)+S(r,f) \\ &\le N(r,f^nf^{(k)})-N(r,f^{(k)})+m(r,f^nf^{(k)})+mN(r,\frac{1}{f^{(k)}})+S(r,f) \\ &\le T(r,f^nf^{(k)})+T(r,f^{(k)})-N(r,f^{(k)})-N(r,\frac{1}{f^{(k)}})+S(r,f) \\ &\le T(r,F)+T(r,f)-N(r,f)-N(r,\frac{1}{f})-k\overline{N}(r,f)+S(r,f). \end{aligned} \tag{2.11}$$

So

$$T(r,F)\ge (n-1)T(r,f)+N(r,f)+N(r,\frac{1}{f})+k\overline{N}(r,f)+S(r,f). \tag{2.12}$$

Thus, by (2.10) and (2.11), we get

$$S(r,F)=S(r,f).$$

Similarly, we get

$$T(r,G)\ge (n-1)T(r,g)+N(r,g)+N(r,\frac{1}{g})+k\overline{N}(r,g)+S(r,g). \tag{2.13}$$

Also $S(r,G)=S(r,g)$. It is clear that the inequality $T(r,f)\le T(r,g)$ or $T(r,g)\le T(r,f)$ holds for a set of infinite measure of r. Without loss of generality, we may suppose that $T(r,f)\le T(r,g)$, holds for $r\in I$, where I is a set with infinite measure. Next we consider five cases.

Case 1. $a\ne b, b\ne 0,-1$. If $a-b-1\ne 0$, then by (2.9) we known:

$$\overline{N}(r,\frac{1}{F})=\overline{N}(r,\frac{1}{G+\frac{(a-b-1)}{b+1}}).$$

By the Nevanlinna second fundamental theorem and Lemma 2.4, we have

$$\begin{aligned} T(r,G) &\le \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{G+\frac{(a-b-1)}{b+1}})+S(r,G) \\ &= \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F})+S(r,g) \\ &\le \overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+N(r,\frac{1}{g^{(k)}})+\overline{N}(r,\frac{1}{f})+N(r,\frac{1}{f^{(k)}})+S(r,g) \end{aligned}$$

$$\le \overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+N(r,\frac{1}{g})+k\overline{N}(r,g)+\overline{N}(r,\frac{1}{f})+N(r,\frac{1}{f})+k\overline{N}(r,f)+S(r,g)$$
$$\le (k+1)\overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+N(r,\frac{1}{g})+k\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+N(r,\frac{1}{f})+S(r,g)$$
$$\le (k+3)T(r,g)+(k+2)T(r,f)+S(r,g).$$

By $n \ge 2k+5$ and (2.13), we get $T(r,g) \le S(r,g)$, for $r \in I$, a contradiction. If $a-b-1=0$, by (2.9) we can obtain $F = \frac{(b+1)G}{bG+1}$ we see that:

$$\overline{N}(r,F)=\overline{N}(r,\frac{1}{G+\frac{1}{b}}),$$

combining the Nevanlinna second fundamental theorem and Lemma 2.4, we have

$$\begin{aligned}
T(r,G) &\le \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{G+\frac{1}{b}})+S(r,G)\\
&= \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,F)+S(r,g)\\
&\le \overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+N(r,\frac{1}{g^{(k)}})+\overline{N}(r,f)+S(r,g)\\
&\le \overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+N(r,\frac{1}{g})+k\overline{N}(r,g)+\overline{N}(r,f)+S(r,g)\\
&\le (k+3)T(r,g)+T(r,f)+S(r,g).
\end{aligned}$$

By $n \ge (2k+5)$ and (2.13), we get $T(r,g) \le S(r,g), r \in I$ a contradiction.
Case 2. $a \ne b, b \ne -1$. So $F = \frac{a}{(a+1)-G}$. We can get $\overline{N}(r,F) = \overline{N}(r,\frac{1}{G-(a+1)})$. Similarly as Case 1, it is impossible.
Case 3. $a \ne b, b \ne 0$. So $F = \frac{G+(a-1)}{a}$. If $a-1=0$, then $F \equiv G$, So $f^n f^{(k)} = g^n g^{(k)}$.
Case 4. $a \ne b, b \ne 0, -1$, from (2.9) we can get $F = \frac{(b+1)G-1}{bG}$, $\overline{N}(r,F) = \overline{N}(r,\frac{1}{G})$. Similarly as Case 1, it is impossible. Since $a \ne 0$, now we consider the following case.
Case 5. $a = b = -1$, it yields $FG \equiv 1$, that is $f^n f^{(k)} g^n g^{(k)} = 1$. By the Lemma 2.3, we can get $f(z) = c_1 e^{(cz)}, g(z) = c_2 e^{(-cz)}$, where c_1, c_2, c are three constants satisfying $(-1)^k (c_1 c_2)^{n+1} c^{2k} = 1$. Now the proof of Lemma 2.6 is completed.

3. Proof of the Theorems

Proof of Theorem 1.1. Noticing that $k \geq 3$, we have

$$\begin{aligned}\overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) - N_{11}(r, \frac{1}{F-1}) + \overline{N}_{(k+1}(r, \frac{1}{F-1}) \\ + \overline{N}_{(k+1}(r, \frac{1}{G-1}) \leq \frac{1}{2}N(r, \frac{1}{F-1}) + \frac{1}{2}N(r, \frac{1}{G-1}) \\ \leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + O(1).\end{aligned}$$

By Lemma 2.5, we can get

$$\begin{aligned}T(r, F) + T(r, G) &< 2\left(N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G)\right) \\ &\quad + S(r, F) + S(r, G) \\ &= 2\left(N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G)\right) \\ &\quad + S(r, f) + S(r, g).\end{aligned} \tag{3.1}$$

Because

$$\begin{aligned}N_2(r, \frac{1}{F}) + N_2(r, F) &\leq N_2(r, \frac{1}{f^n f^{(k)}}) + N_2(r, f^n f^{(k)}) \\ &\leq 2\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, f),\end{aligned} \tag{3.2}$$

and

$$N_2(r, \frac{1}{G}) + N_2(r, G) \leq 2\overline{N}(r, \frac{1}{g}) + N(r, \frac{1}{g^{(k)}}) + 2\overline{N}(r, g). \tag{3.3}$$

By (3.1)-(3.3) and Lemma 2.4, we can get

$$\begin{aligned}T(r, F) + T(r, G) &\leq 2\left(2\overline{N}(r, \frac{1}{f}) + 2\overline{N}(r, f) + N(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, \frac{1}{g})\right. \\ &\quad \left. +2\overline{N}(r, g) + N(r, \frac{1}{g^{(k)}})\right) + S(r, f) + S(r, g) \\ &= 4\overline{N}(r, \frac{1}{f}) + 4\overline{N}(r, f) + 2N(r, \frac{1}{f^{(k)}}) + S(r, f) \\ &\quad + 4\overline{N}(r, \frac{1}{g}) + 4\overline{N}(r, g) + 2N(r, \frac{1}{g^{(k)}}) + S(r, g) \\ &\leq 4\overline{N}(r, \frac{1}{f}) + 4\overline{N}(r, f) + 2\left[N(r, \frac{1}{f}) + k\overline{N}(r, f)\right] + S(r, f)\end{aligned}$$

$$+ 4\overline{N}(r, \frac{1}{g}) + 4\overline{N}(r, g) + 2\left[N(r, \frac{1}{g}) + k\overline{N}(r, g)\right] + S(r, g),$$

we write the above equation as

$$\begin{aligned} T(r, F) + T(r, G) &\leq 4\overline{N}(r, \frac{1}{f}) + 4\overline{N}(r, f) + 2N(r, \frac{1}{f}) + 2k\overline{N}(r, f) + S(r, f) \\ &\quad + 4\overline{N}(r, \frac{1}{g}) + 4\overline{N}(r, g) + 2N(r, \frac{1}{g}) + 2k\overline{N}(r, g) + S(r, g) \\ &\leq (2k + 10)T(r, f) + (2k + 10)T(r, g) + S(r, f) + S(r, g) \\ (n + 1)[T(r, f) + T(r, g)] &\leq (2k + 10)[T(r, f) + T(r, g)] + S(r, f) + S(r, g) \\ (n - 2k - 9)(T(r, f) + T(r, g)) &\leq S(r, f) + S(r, g). \end{aligned}$$

By $n \geq 2k + 9$ and (2.12),(2.13) we obtain $T(r, f) + T(r, g) \leq S(r, f) + S(r, g)$, which is impossible. Therefore, by Lemma 2.5

$$f = \frac{(b + 1)g + (a - b - a)}{bg + (a - b)},$$

where $a(\neq 0), b$ are two constants, it follows by Lemma 2.6 that $f(z) = c_1 e^{(cz)}, g(z) = c_2 e^{(-cz)}$, where c_1, c_2, c are three constants satisfying $(-1)^k (c_1 c_2)^{n+1} c^{2k} = 1$.
The proof of Theorem 1.1 is complete.
Proof of Theorem 1.2. We can see easily:

$$\begin{aligned} &\overline{N}(r, \frac{1}{F - 1}) + \overline{N}(r, \frac{1}{G - 1}) - N_{11}(r, \frac{1}{F - 1}) + \frac{1}{2}\overline{N}_{(2}(r, \frac{1}{F - 1}) \\ &\qquad + \frac{1}{2}\overline{N}_{(2}(r, \frac{1}{G - 1}) \leq \frac{1}{2}N(r, \frac{1}{F - 1}) + \frac{1}{2}N(r, \frac{1}{G - 1}) \\ &\qquad \leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, f) + S(r, g). \end{aligned}$$

By Lemma 2.5, we can get

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2\left[N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G)\right] \\ &\quad + \overline{N}_{(3}(r, \frac{1}{F - 1}) + \overline{N}_{(3}(r, \frac{1}{G - 1}) + S(r, f) + S(r, g). \end{aligned} \tag{3.4}$$

Considering

$$\begin{aligned}\overline{N}_{(3}(r,\frac{1}{F-1}) &\le \frac{1}{2}N(r,\frac{F}{F'}) = \frac{1}{2}N(r,\frac{F'}{F}) + S(r,f)\\ &\le \frac{1}{2}\overline{N}(r,\frac{1}{F}) + \frac{1}{2}\overline{N}(r,F) + S(r,f)\\ &\le \frac{1}{2}[\overline{N}(r,\frac{1}{f}) + N(r,\frac{1}{f}) + \overline{N}(r,f)] + S(r,f)\\ &\le 2T(r,f) + S(r,f).\end{aligned} \tag{3.5}$$

Similarly, we can get

$$\overline{N}_{(3}(r,\frac{1}{F-1}) \le 2T(r,f) + S(r,f). \tag{3.6}$$

From (3.4)-(3.6) we can get

$$T(r,F)+T(r,G) \le 2\left[N_2(r,\frac{1}{f^n f^{(k)}}) + N_2(r,f^n f^{(k)}) + N_2(r,\frac{1}{g^n g^{(k)}}) + N_2(r,g^n g^{(k)})\right]$$

$$\begin{aligned}&+ 2T(r,f) + 2T(r,g) + S(r,f) + S(r,g)\\ &\le 2\left[2\overline{N}(r,\frac{1}{f}) + N(r,\frac{1}{f^{(k)}}) + 2\overline{N}(r,f) + 2\overline{N}(r,\frac{1}{g}) + N(r,\frac{1}{g^{(k)}}) + 2\overline{N}(r,g)\right]\\ &+ 2T(r,f) + 2T(r,g) + S(r,f) + S(r,g),\end{aligned}$$

$$\begin{aligned}&T(r,F)+T(r,G) \le 4\overline{N}(r,\frac{1}{f}) + 2N(r,\frac{1}{f^{(k)}}) + 4\overline{N}(r,f) + 4\overline{N}(r,\frac{1}{g})\\ &+ 2N(r,\frac{1}{g^{(k)}}) + 4\overline{N}(r,g) + 2T(r,f) + 2T(r,g) + S(r,f) + S(r,g)\\ &\le 4\overline{N}(r,\frac{1}{f}) + 2N(r,\frac{1}{f}) + 2k\overline{N}(r,f) + 4\overline{N}(r,f) + 4\overline{N}(r,\frac{1}{g})\\ &+ 2N(r,\frac{1}{g}) + 2k\overline{N}(r,g) + 4\overline{N}(r,g) + 2T(r,f) + 2T(r,g) + S(r,f) + S(r,g)\\ &\le (2k+12)T(r,f) + (2k+12)T(r,g) + S(r,f) + S(r,g)\\ &(n+1)[T(r,f)+T(r,g)] \le (2k+12)[T(r,f)+T(r,g)] + S(r,f) + S(r,g)\\ &(n-2k-11)[T(r,f)+T(r,g)] \le S(r,f) + S(r,g).\end{aligned}$$

By $n \ge 2k+11$ and (2.12),(2.13), we can get $T(r,f)+T(r,g) \le S(r,f)+S(r,g)$ it is impossible.

The proof of Theorem 1.2 is complete.

Proof of Theorem 1.3. Since

$$\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1})-N_{11}(r,\frac{1}{F-1})\leq\frac{1}{2}N(r,\frac{1}{F-1})+\frac{1}{2}N(r,\frac{1}{G-1})$$
$$\leq\frac{1}{2}T(r,F)+\frac{1}{2}T(r,G)+S(r,f)+S(r,g).$$

We can see easily from Lemma 2.5 that:

$$\begin{aligned}T(r,F)+T(r,G)\leq 2\Big[&N_2(r,\frac{1}{F})+N_2(r,F)+N_2(r,\frac{1}{G})+N_2(r,G)+\\ &\overline{N}_{(2}(r,\frac{1}{F-1})+\overline{N}_{(2}(r,\frac{1}{G-1})\Big]+S(r,F)+S(r,G).\\ =2\Big[&N_2(r,\frac{1}{F})+N_2(r,F)+N_2(r,\frac{1}{G})+N_2(r,G)+\\ &\overline{N}_{(2}(r,\frac{1}{F-1})+\overline{N}_{(2}(r,\frac{1}{G-1})\Big]+S(r,f)+S(r,g).\end{aligned}\tag{3.7}$$

Considering

$$\begin{aligned}\overline{N}_{(2}(r,\frac{1}{F-1})&\leq N(r,\frac{F}{F'})=N(r,\frac{F'}{F})+S(r,f)\\ &\leq\overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+S(r,f)\\ &\leq N(r,\frac{1}{f})+N(r,\frac{1}{f^{(k)}})+\overline{N}(r,f)+S(r,f)\\ &\leq(k+3)T(r,f)+S(r,f).\end{aligned}\tag{3.8}$$

Similarly we can get

$$\overline{N}_{(2}(r,\frac{1}{G-1})\leq(k+3)T(r,g)+S(r,g),\tag{3.9}$$

from (3.7)-(3.9) we can get

$$T(r,F)+T(r,G)\leq 2[2\overline{N}(r,\frac{1}{f})+N(r,\frac{1}{f^{(k)}})+2\overline{N}(r,f)+2\overline{N}(r,\frac{1}{g})+N(r,\frac{1}{g^{(k)}})$$
$$+2\overline{N}(r,g)+(k+3)T(r,f)+(k+3)T(r,g)]+S(r,f)+S(r,g)$$
$$\leq(4k+16)T(r,f)+(4k+16)T(r,g)+S(r,f)+S(r,g)$$
$$(n+1)[T(r,f)+T(r,g)]\leq(4k+16)[T(r,f)+T(r,g)]+S(r,f)+S(r,g)$$
$$(n-4k-15)[T(r,f)+T(r,g)]\leq S(r,f)+S(r,g).$$

Since $n \geq 4k+16$ and (2.12), (2.13), we can get $T(r,f)+T(r,g) \leq S(r,f)+S(r,g)$, it is impossible.
The proof of Theorem 1.3 is complete.

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