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SOME HYPERGEOMETRIC GENERATING FUNCTIONS MOTIVATED BY THE WORK OF BEDIENT AND RAINVILLE

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Abstract: In this article, some hypergeometric generating relations involving Kampé de Fériet's double hypergeometric functions are derived by using series rearrangement technique, a reduction formula and Whipple's quadratic transformation. Some special cases involving Appell's functions of second, third kind, Rainville polynomial and two Bedient's polynomials, are also obtained.

Keywords and Phrases: Hypergeometric functions, Series rearrangement technique, Whipple's quadratic transformation, Bedient polynomials.

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1. Introduction and Preliminaries

In our investigations, we shall use the following standard notations:
$\mathbb{N} := \{1,2,3,\cdots\}; \mathbb{N}_0 := \mathbb{N}\bigcup\{0\}; \mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}$.
The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively. The Pochhammer symbol $(\alpha)_p (\alpha, p \in \mathbb{C})$[12, p.22 Eq.(1), p.32, Q.N.(8) and Q.N.(9)], see also [13, p.23, Eq.(22) and Eq.(23)] is defined by

$$(\alpha)_p = \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)}$$

$$(\alpha)_p = \begin{cases} 1 & ;(p=0;\alpha\in\mathbb{C}\backslash\{0\}), \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & ;(p=n\in\mathbb{N};\alpha\in\mathbb{C}), \\ \frac{(-1)^n k!}{(k-n)!} & ;(\alpha=-k;p=n;n,k\in\mathbb{N}_0;0\leq n\leq k), \\ 0 & ;(\alpha=-k;p=n;n,k\in\mathbb{N}_0;n>k), \\ \frac{(-1)^n}{(1-\alpha)_n} & ;(p=-n;n\in\mathbb{N};\alpha\in\mathbb{C}\backslash\mathbb{Z}). \end{cases}$$

It being understood conventionally that$(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$${}_pF_q\left[\begin{array}{ll}(\alpha_p); & \\ & z \\ (\beta_q); & \end{array}\right] = {}_pF_q\left[\begin{array}{ll}\alpha_1,\alpha_2,\ldots,\alpha_p; & \\ & z \\ \beta_1,\beta_2,\ldots,\beta_q; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!}, \tag{1.1}$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\ldots\ ;\quad j=1,2,\ldots,q,$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z|<\infty$, if $p\leq q$,

(ii) converges for $|z|<1$, if $p=q+1$,

(iii) diverges for all z, $z\neq 0$, if $p>q+1$,

where by convention, a product over an empty set is interpreted as 1 and

$$\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j, \tag{1.2}$$

$\Re(\omega)$ being the real part of complex number ω.

Double hypergeometric function of Srivastava-Daoust

Just as the Gaussian ${}_2F_1$ function was generalized to ${}_pF_q$ by increasing the number of the numerator and denominator parameters, the four Appell functions F_1, F_2, F_3, F_4 [13, p. 53 Eq.(4,5,6,7)] and their seven confluent forms Φ_1, Φ_2, Φ_3, Ψ_1, Ψ_2, Ξ_1, Ξ_2 given by Humbert [6-8] were unified and generalized by Kampé de Fériet [9] who defined a general hypergeometric function of two variables.

The notation introduced by Kampé de Fériet for his double hypergeometric function [1, p. 150, Eq.(26)] of superior order was subsequently abbreviated by Burchnall and Chaundy [3, p. 112]. We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation [14, p. 423, Eq.(26)]:

$$F^{p:\ q;\ k}_{\ell:\ m;\ n}\left[\begin{array}{ccccc}(a_p) & : & (b_q) & ; & (c_k)\ ; \\ & & & & \\ (\alpha_\ell) & : & (\beta_m) & ; & (\gamma_n)\ ; \end{array} x,\ y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.2}$$

where for convergence,

$$(i) p+q<\ell+m+1, \quad p+k<\ell+n+1 \ ; \quad |x|<\infty, \quad |y|<\infty, \tag{1.3}$$

$$(ii) p+q=\ell+m+1, \quad p+k=\ell+n+1 \text{ and} \tag{1.4}$$

$$\begin{cases}|x|^{1/(p-\ell)}+|y|^{1/(p-\ell)}<1\ ; & \text{if } p>\ell, \\ \max\{|x|,|y|\}<1 \quad ; & \text{if } p\leq\ell.\end{cases} \tag{1.5}$$

For absolutely and conditionally convergence of double series (1.2), we refer to a research paper by Hài et al. [5, p. 106-107].

Appell functions

Appell's function of second kind [1, 13 p. 53, Eq.(5)] is defined by

$$F_2\left[\ \alpha\ ;\ \ \beta\ ,\ \ \gamma\ ;\ \ \lambda\ ,\ \ \mu\ ;\ x,\ y\ \right] = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\gamma)_n}{(\lambda)_m(\mu)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.6}$$

where $|x|+|y|<1$.

Appell's function of third kind [1, 13, p. 53, Eq.(6)] is defined by

$$F_3\left[\ \alpha\ ,\ \beta\ ;\ \gamma\ ,\ \delta\ ;\ \lambda\ ;\ x\ ,\ y\ \right] = \sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_n(\gamma)_m(\delta)_n}{(\lambda)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.7}$$

where max $\{|x|, |y|\} < 1$.
Series rearrangement technique is based upon certain interchanges of the order of double (or multiple) summation. Several hypergeometric generating relations have been established using series rearrangement technique. Here, we consider some well known results.
Cauchy's double series identity [13, p.100]:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi(m,n)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\Phi(m-n,n), \tag{1.8}$$

provided that the associated double series are absolutely convergent.
Whipple's quadratic transformation [15, p. 267, Eq.(7.1)] see also [12, p.88, Th.(31)]

$${}_3F_2\left[\begin{array}{c} -n\ ,\ \alpha\ ,\ \beta\ ;\\ 1-\alpha-n, 1-\beta-n; \end{array} z\right] = (1-z)^n\ {}_3F_2\left[\begin{array}{c} \frac{-n}{2}, \frac{-n+1}{2}, 1-\alpha-\beta-n;\\ 1-\alpha-n, 1-\beta-n; \end{array} \frac{-4z}{(1-z)^2}\right], \tag{1.9}$$

where $n\in\mathbb{N}_0$ and other parameters are neither zero nor negative integers.
Bedient's polynomials [2, p. 15, Eq.(2.5)], see also [12, p. 297, Eq.(1) and Eq.(2)]
Bedient's polynomials $R_n(\alpha,\ \beta\ ;\ y)$ are defined by

$$R_n(\alpha,\ \beta\ ;\ y)=\frac{(\alpha)_n(2y)^n}{n!}\ {}_3F_2\left[\begin{array}{c} \frac{-n}{2},\ \ \frac{-n+1}{2},\ \ \beta-\alpha;\\ \beta,\ \ 1-\alpha-n\ ; \end{array} \frac{1}{y^2}\right], \tag{1.10}$$

other Bedient's polynomials $G_n(\alpha,\ \beta\ ;\ y)$ [2, p. 44, Eq.(3.4)], see also [12, p. 297, Eq.(2)]

$$G_n\left(\alpha,\beta;y\right)=\frac{(\alpha)_n(\beta)_n(2y)^n}{(\alpha+\beta)_n\ \ n!}\ {}_3F_2\left[\begin{array}{c} \frac{-n}{2}, \frac{-n+1}{2}, 1-\alpha-\beta-n;\\ 1-\alpha-n\ , 1-\beta-n; \end{array} \frac{1}{y^2}\right]. \tag{1.11}$$

Reduction formula [12, p.70, Q.(10)], is given by

$$\frac{1}{\sqrt{(1-z)}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma-1}={}_2F_1\left[\begin{array}{c} \gamma\ ,\ \ \gamma+\frac{1}{2}\ ;\\ 2\gamma\ \ ; \end{array} z\right]. \tag{1.12}$$

Simple Bessel polynomials of Krall and Frink [10, p. 101, Eq.(3)]

$$y_n(z) = {}_2F_0\left[\begin{array}{ll} -n\ ,\ 1+n; & \\ & -\frac{z}{2} \\ - \quad ; & \end{array}\right]. \tag{1.13}$$

Rainville polynomial [12, p. 294, Eq.(3)]

$$\varphi_n(c,z) = \frac{(c)_n}{n!}\ {}_2F_0\left[\begin{array}{ll} -n\ ,\ c+n; & \\ & z \\ - \quad ; & \end{array}\right]. \tag{1.14}$$

Linear Generating function

Two functions $F(x,t)$ and $G(x,t)$ of two independent variables x and t are called generating functions of the sets $\{f_n(x)\}$ and $\{g_n(x)\}$ respectively, if it is possible to represent $F(x,t)$ and $G(x,t)$ in the following series expansions of t

$$F(x,t) = \sum_{n=0}^{\infty} b_n\ f_n(x)t^n; \quad t \neq 0, \tag{1.15}$$

$$G(x,t) = \sum_{n=-\infty}^{+\infty} c_n\ g_n(x)t^n; \quad t \neq 0, \tag{1.15}$$

where the coefficients b_n and c_n are independent of x and t and may contain some parameters related with $f_n(x)$, $g_n(x)$ respectively.

Motivated by the work collected in beautiful monographs of Rainville [12, Ch. 8, pp. 129-146], Mcbride [11, Ch. 1, pp. 1-24, Ch. 5, pp. 72-96] and Erdlyi et al. [4, Ch. 19, pp. 245-278], we obtain some generating functions in this paper.

The present article is organized as follows: In section 2, we obtain three generating relations. In section 3, we have given the proof of these hypergeometric generating relations using series rearrangement technique and quadratic transformation. In section 4, we discuss some special cases.

2. Hypergeometric Generating Relations

When the values of parameters and variables leading to the results which do not make sense are tacitly excluded. In each generating relation multiple series involved in the left hand side and the right hand side are absolutely convergent and in each hypergeometric generating function, denominator parameters are neither zero nor negative integers. Then

First generating relation

$$F^{A:1;1}_{B:1;1}\left[\begin{array}{lll}(a_A): & \gamma\ ; & \gamma\ ; \\ & & \\ (b_B): & \delta\ ; & \delta\ ;\end{array} t\left(x-\sqrt{x^2-1}\right), t\left(x+\sqrt{x^2-1}\right)\right]$$

$$=\sum_{m=0}^{\infty}\frac{\prod_{i=1}^{A}(a_i)_m}{(\delta)_m\prod_{j=1}^{B}(b_j)_m}R_m(\gamma,\delta;x)t^m. \tag{2.1}$$

Convergence conditions

(i) When $A\leq B$ then $|t(x-\sqrt{x^2-1})|<\infty$ and $|t(x+\sqrt{x^2-1})|<\infty$,

(ii) When $A=B+1$ then $|t(x-\sqrt{x^2-1})|^{\frac{1}{A-B}}+|t(x+\sqrt{x^2-1})|^{\frac{1}{A-B}}<1$.

Second generating relation

$$F^{A:2;2}_{B:0;0}\left[\begin{array}{lll}(a_A): & \gamma\ ,\ \delta\ ; & \gamma\ ,\ \delta\ ; \\ & & \\ (b_B): & -\ ; & -\ ;\end{array} t\left(x-\sqrt{x^2-1}\right), t\left(x+\sqrt{x^2-1}\right)\right]$$

$$=\sum_{m=0}^{\infty}\frac{(\gamma+\delta)_m\prod_{i=1}^{A}(a_i)_m}{\prod_{j=1}^{B}(b_j)_m}\ G_m(\gamma,\delta;x)t^m. \tag{2.2}$$

Convergence conditions

(i) When $A\leq B-2$ then $|t(x-\sqrt{x^2-1})|<\infty$ and $|t(x+\sqrt{x^2-1})|<\infty$,

(ii) When $A=B-1$ then $\max\left\{|t(x-\sqrt{x^2-1})|\ ,\ |t(x+\sqrt{x^2-1})|\right\}<1$.

Third generating relation

$$\frac{\exp\left\{x(1-\sqrt{1-2t})\right\}}{\sqrt{1-2t}}=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ y_n\left(\frac{1}{x}\right)t^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ \varphi_n\left(1,-\frac{1}{2x}\right)t^n. \tag{2.3}$$

3. Proofs

Proof of first generating relation (2.1)

Let

$$\Psi=F^{A:1;1}_{B:1;1}\left[\begin{array}{lll}(a_A): & \gamma\ ; & \gamma\ ; \\ & & \\ (b_B): & \delta\ ; & \delta\ ;\end{array} t\left(x-\sqrt{x^2-1}\right), t\left(x+\sqrt{x^2-1}\right)\right]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n}...(a_A)_{m+n}(\gamma)_m(\gamma)_n\left(x-\sqrt{x^2-1}\right)^m\left(x+\sqrt{x^2-1}\right)^n t^{m+n}}{(b_1)_{m+n}(b_2)_{m+n}...(b_B)_{m+n}(\delta)_m(\delta)_n m!n!}. \tag{3.1}$$

Replacing m by $m-n$ in equation (3.1) and applying double series identity (1.8), we get

$$\Psi = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_{m-n}(\gamma)_n\left(x-\sqrt{x^2-1}\right)^{m-n}\left(x+\sqrt{x^2-1}\right)^n\ t^m}{(b_1)_m(b_2)_m...(b_B)_m(\delta)_{m-n}(\delta)_n(1)_m(1+m)_{-n}\ n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m\left(x-\sqrt{x^2-1}\right)^m}{(b_1)_m(b_2)_m...(b_B)_m(\delta)_m\ m!}\times$$

$$\times\left(\sum_{n=0}^{m}\frac{(-m)_n(\gamma+m)_{-n}(\gamma)_n\left(x+\sqrt{x^2-1}\right)^n}{(\delta+m)_{-n}(\delta)_n(-1)^n n!\left(x-\sqrt{x^2-1}\right)^n}\right)t^m$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m\left(x-\sqrt{x^2-1}\right)^m}{(b_1)_m(b_2)_m...(b_B)_m(\delta)_m\ m!}\times$$

$$\times\left(\sum_{n=0}^{m}\frac{(-m)_n(\gamma)_n(1-\delta-m)_n\left(x+\sqrt{x^2-1}\right)^n}{(1-\gamma-m)_n(\delta)_n(-1)^n n!\left(x-\sqrt{x^2-1}\right)^n}\right)t^m$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m\left(x-\sqrt{x^2-1}\right)^m}{(b_1)_m(b_2)_m...(b_B)_m(\delta)_m\ m!}$$

$$ {}_3F_2\left[\begin{array}{c} -m\ ,\ \gamma\ ,\ 1-\delta-m\ ; \\ \\ \delta\ ,\ 1-\gamma-m\ ; \end{array} -\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\right]t^m. \tag{3.2}$$

Applying quadratic transformation (1.9), we get

$$\Psi = \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m(2x)^m}{(b_1)_m(b_2)_m...(b_B)_m(\delta)_m\ m!}\quad {}_3F_2\left[\begin{array}{c} \frac{-m}{2},\ \frac{-m+1}{2}\ ,\ \delta-\gamma\ ; \\ \\ \delta\ ,\ 1-\gamma-m\ ; \end{array} \frac{1}{x^2}\right]t^m. \tag{3.3}$$

Now using the definition of Bedient polynomial $R_m(\gamma,\ \delta\ ;\ x)$(1.10), we get the generating relation (2.1).

Proof of second generating relation (2.2)

Let

$$\Phi = F^{A:2;2}_{B:0;0}\left[\begin{array}{c} (a_A):\ \gamma\ ,\ \delta\ ;\ \gamma\ ,\ \delta\ ; \\ \\ (b_B):\quad -\quad ;\quad -\quad ; \end{array} t\left(x-\sqrt{x^2-1}\right), t\left(x+\sqrt{x^2-1}\right)\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}...(a_A)_{m+n}(\gamma)_m(\delta)_m(\gamma)_n(\delta)_n(x-\sqrt{x^2-1})^m(x+\sqrt{x^2-1})^n t^{m+n}}{(b_1)_{m+n}(b_2)_{m+n}...(b_B)_{m+n} m!n!}. \tag{3.4}$$

Replacing m by $m-n$ in equation (3.4) and applying double series identity (1.8), we get

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_{m-n}(\delta)_{m-n}(\gamma)_n(\delta)_n(x-\sqrt{x^2-1})^{m-n}(x+\sqrt{x^2-1})^n t^m}{(b_1)_m(b_2)_m...(b_B)_m(m-n)!n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m.....(a_A)_m(\gamma)_m(\delta)_m\left(x-\sqrt{x^2-1}\right)^m}{(b_1)_m(b_2)_m.....(b_B)_m\ m!} \times$$

$$\times \left(\sum_{n=0}^{m} \frac{(-m)_n(\gamma+m)_{-n}(\delta+m)_{-n}(\gamma)_n(\delta)_n\left(x+\sqrt{x^2-1}\right)^n}{(-1)^n\left(x-\sqrt{x^2-1}\right)^n n!} \right) t^m$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m.....(a_A)_m(\gamma)_m(\delta)_m\left(x-\sqrt{x^2-1}\right)^m}{(b_1)_m(b_2)_m....(b_B)_m\ m!} \times$$

$$\times \left(\sum_{n=0}^{m} \frac{(-m)_n(\gamma)_n(\delta)_n(-1)^n\left(x+\sqrt{x^2-1}\right)^n}{(1-\gamma-m)_n\ (1-\delta-m)_n\left(x-\sqrt{x^2-1}\right)^n n!} \right) t^m$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m(\delta)_m\left(x-\sqrt{x^2-1}\right)^m}{(b_1)_m(b_2)_m...(b_B)_m\ m!}$$

$${}_3F_2\left[\begin{matrix} -m\ ,\ \gamma\ ,\ \delta\ ; \\ \\ 1-\gamma-m, 1-\delta-m; \end{matrix} \ -\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\right] t^m. \tag{3.5}$$

Applying quadratic transformation(1.9), we get

$$\Phi = \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m(\delta)_m(2x)^m}{(b_1)_m(b_2)_m...(b_B)_m\ m!}$$

$${}_3F_2\left[\begin{matrix} \frac{-m}{2}, \frac{-m+1}{2}, 1-\gamma-\delta-m; \\ \\ 1-\gamma-m, 1-\delta-m\ ; \end{matrix} \ \frac{1}{x^2}\right] t^m. \tag{3.6}$$

Now using the definition of Bedient polynomial $G_m(\gamma,\ \delta;\ x)$ (1.11), we get the generating relation (2.2).

Proof of third generating relation (2.3)
Let

$$\xi = \sum_{n=0}^{\infty} \frac{x^n}{n!}\ \varphi_n\left(1, -\frac{1}{2x}\right) t^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}\ {}_2F_0\left[\begin{array}{ccc} -n\ ,\ 1+n; & & \\ & & -\frac{1}{2x} \\ - & ; & \end{array}\right] t^n \tag{3.7}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{r=0}^{n} \frac{(-n)_r (1+n)_r (-1)^r t^n}{(2x)^r r!} \tag{3.8}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{r=0}^{n} \frac{(-1)^{2r} n!\ (1+n)_r\ t^n}{(n-r)!\ (2x)^r\ r!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{r=0}^{n} \frac{(n+r)!\ t^n}{(n-r)!\ (2x)^r\ r!} \tag{3.9}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(n+r)!\ x^{n-r} t^n}{(n-r)!\ 2^r n!\ r!}. \tag{3.10}$$

Replacing n by $n+r$ in equation (3.10) and applying double series identity (1.8), we get

$$\xi = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(1)_{n+2r}\ x^n\ t^{n+r}}{(1)_{n+r}\ 2^r\ n!\ r!}, \tag{3.11}$$

$$= \sum_{n=0}^{\infty} \frac{(1)_n\ x^n t^n}{(n!)^2} \sum_{r=0}^{\infty} \frac{(1+n)_{2r}}{(1+n)_r} \frac{t^r}{2^r r!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} \sum_{r=0}^{\infty} \frac{(\frac{n+1}{2})_r (\frac{n+2}{2})_r}{(n+1)_r} \frac{(2t)^r}{r!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} {}_2F_1\left[\begin{array}{ccc} \frac{n+1}{2}\ ,\ \frac{n+2}{2}; & & \\ & & 2t \\ n+1 & ; & \end{array}\right]. \tag{3.12}$$

Now applying reduction formula (1.12), we get

$$\xi = \frac{1}{\sqrt{1-2t}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2xt}{1+\sqrt{1-2t}}\right)^n \tag{3.13}$$

$$= \frac{1}{\sqrt{1-2t}}\ {}_0F_0\left[\begin{array}{ccc} - & ; & \\ & & \frac{2xt}{1+\sqrt{1-2t}} \\ - & ; & \end{array}\right]$$

$$= \frac{1}{\sqrt{1-2t}} \exp\left\{\frac{2xt}{1+\sqrt{1-2t}}\right\}$$

$$= \frac{\exp\left\{x(1-\sqrt{1-2t})\right\}}{\sqrt{1-2t}} \tag{3.14}$$

which gives third generating relation (2.3).

4. Special Cases

In first generating relation (2.1), put $A = 1, B = 0, a_1 = a$ and applying the definition of Appell's function F_2 (1.6) and Bedient polynomials $R_m(\gamma,\ \delta\ ;\ x)$ (1.10), we get a known generating relation [2. p. 15], see also [13, p. 186, Q.48(i)]

$$F_2\left[\ a;\gamma,\gamma;\delta,\delta;t(x-\sqrt{x^2-1}),t(x+\sqrt{x^2-1})\ \right] = \sum_{m=0}^{\infty} \frac{(a)_m}{(\delta)_m} R_m(\gamma,\delta;x)t^m, \tag{4.1}$$

where, $|t(x-\sqrt{x^2-1})|\ +\ |t(x+\sqrt{x^2-1})| < 1.$

In second generating relation (2.2), put $A = 0, B = 1, b_1 = b$ and applying definition of Appell's function F_3 (1.7) and Bedient polynomials $G_m(\gamma,\ \delta\ ;\ x)$ (1.11) we get a known generating relation [2]

$$F_3\left[\ \gamma,\delta;\gamma,\delta;b;t(x-\sqrt{x^2-1}),t(x+\sqrt{x^2-1})\ \right] = \sum_{m=0}^{\infty} \frac{(\gamma+\delta)_m}{(b)_m} G_m(\gamma,\delta;x)t^m, \tag{4.2}$$

where, $\max\left\{|t(x-\sqrt{x^2-1})|\ ,\ |t(x+\sqrt{x^2-1})|\right\} < 1.$

5. Conclusion

We conclude our present investigation in noting that the results deduced above are significant and can lead to derive numerous generating relations and generating functions in an analogous manner by suitable specialization of arbitrary parameters in the main findings. More importantly we observe by remarking that the results presented here are also believed to give some potentially useful contributions to non-specialists who are interested in Applied Mathematics or Mathematical Physics.

References

[1] Appell, P. and Kampé de Fériet, J., Fonctions Hypergéométriques et Hypersphériques Polynômes d' Hermite, Gauthier-Villars, Paris, 1926.

[2] Bedient, P.E., Polynomials Related to Appell Functions of Two Variables, Ph.D. Thesis, University of Michigan, 1958.

[3] Burchnall, J.L. and Chaundy, T. W., Expansions of Appell's double hypergeometric functions (II); Quart. J. Math. Oxford Ser., 12 (1941), 112-128.

[4] Erdlyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., Higher Transcendental Functions, Vol. III, McGraw-Hill Book Company, New York, Toronto and London, 1955.

[5] Hài, N.T., Marichev, O.I. and Srivastava, H.M., A note on the convergence of certain families of multiple hypergeometric series, J. Math. Anal. Appl., 164 (1992), 104-115.

[6] Humbert, P., La Fonction $W_k, \mu_1, \mu_2, ..., \mu_n(x_1, x_2, ..., xn)$, C.R. Acad. Sci. Paris, 171 (1920), 428-430.

[7] Humbert, P., The confluent hypergéométriques d'ordre supérieur á deux variables, C.R. Acad. Sci. Paris, 173 (1921), 73-96.

[8] Humbert, P., The confluent hypergeometric functions of two variables, Proc. Royal Soc. Edinburgh Sec. A., 41 (1922), 73-96.

[9] Kampé de Fériet, J., Les fonctions hypergéométriques d'ordre supérieur á deux variables, C.R. Acad. Sci. Paris, 173 (1921), 401-404.

[10] Krall, H.L. and Frink, O., A new class of orthogonal polynomials: the Bessel polynomials, Trans. Amer. Math. Soc., 65 (1949), 100-115.

[11] McBride, E.B., Obtaining Generating Functions, Springer-Verlag, New York, Heidelberg and Berlin, 1971.

[12] Rainville, E.D., Special Functions, The Macmillan Co. Inc., New York 1960; Reprinted by Chelsea publ. Co., Bronx, New York, 1971.

[13] Srivastava, H.M. and Manocha, H.L., A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester Brisbane, Toronto, 1984.

[14] Srivastava, H.M. and Panda, R., An integral representation for the product of two Jacobi polynomials, J. London Math. Soc., 12 (2) (1976), 419-425.

[15] Whipple, F.J.W., Some transformations of generalized hypergeometric series, Proc. London Math. Soc., 26 (2) (1927), 257-272.