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CERTAIN HYPERGEOMETRIC GENERATING FUNCTIONS INDUCED BY THE WORK OF BRAFMAN AND SHIVELY

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Abstract: In this paper, we obtain three hypergeometric generating relations associated with Kampé de Fériet double hypergeometric functions, by means of Gauss' quadratic transformation, Whipple's quadratic transformation, Kümmer's first transformation and Series rearrangement technique. Some special cases are also discussed.

Keywords and Phrases: Hypergeometric functions; Series rearrangement technique; Appell function; Jacobi polynomials; Khandekar's generalized Rice polynomials.

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1. Introduction and Preliminaries

In our investigations, we shall use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \cdots\}\,; \mathbb{N}_0 := \mathbb{N} \bigcup \{0\}\,; \mathbb{Z}_0^- := \mathbb{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}\,.$
The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

Pochhammer symbol

The Pochhammer symbol $(\alpha)_p (\alpha, p \in \mathbb{C})$ [14, p.22 eq(1), p.32 Q.N.(8) and Q.N.(9)], see also [18, p.23, eq(22) and eq(23)], is defined by

$$(\alpha)_p := \frac{\Gamma(\alpha + p)}{\Gamma(\alpha)}$$

$$(\alpha)_p = \begin{cases} 1 & ;(p=0;\alpha\in\mathbb{C}\backslash\{0\}), \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & ;(p=n\in\mathbb{N};\alpha\in\mathbb{C}), \\ \frac{(-1)^n k!}{(k-n)!} & ;(\alpha=-k;p=n;n,k\in\mathbb{N}_0;0\leq n\leq k), \\ 0 & ;(\alpha=-k;p=n;n,k\in\mathbb{N}_0;n>k), \\ \frac{(-1)^n}{(1-\alpha)_n} & ;(p=-n;n\in\mathbb{N};\alpha\in\mathbb{C}\backslash\mathbb{Z}). \end{cases}$$

It being understood conventionally that$(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$ {}_pF_q\left[\begin{array}{lc} (\alpha_p); & \\ & z \\ (\beta_q); & \end{array}\right] = {}_pF_q\left[\begin{array}{lc} \alpha_1,\alpha_2,\ldots,\alpha_p; & \\ & z \\ \beta_1,\beta_2,\ldots,\beta_q; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\,\frac{z^n}{n!}, \tag{1.1}$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that $\beta_j \neq 0,-1,-2,\ldots$; $j=1,2,\ldots,q$.

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z|<\infty$, if $p\leq q$,

(ii) converges for $|z|<1$, if $p=q+1$,

(iii) diverges for all z, $z\neq 0$, if $p>q+1$,

(iv) converges absolutely for $|z|=1$, if $p=q+1$ and $\Re(\omega)>0$,

(v) converges conditionally for $|z|=1 (z\neq 1)$, if $p=q+1$ and $-1<\Re(\omega)\leqq 0$,

(vi) diverges for $|z|=1$, if $p=q+1$ and $\Re(\omega)\leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and

$$\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j, \tag{1.2}$$

$\Re(\omega)$ being the real part of complex number ω.
Another interesting extension of the Pochhammer symbol and the associated hypergeometric functions was recently given by Srivastava et al. [[20], [21]].

Double hypergeometric function of Kampé de Fériet

Just as the Gaussian ${}_2F_1$ function was generalized to ${}_pF_q$ by increasing the number of the numerator and denominator parameters, the Appell's four double hypergeometric functions F_1, F_2, F_3, F_4 [18, p.53 (4,5,6,7)] and their seven confluent forms $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$ given by Humbert [7], [8], [9] were unified and generalized by Kampé de Fériet [10] who defined a general hypergeometric function of two variables.

The notation introduced by Kampé de Fériet for his double hypergeometric function [1, p.150, Eq.(26)] of superior order was subsequently abbreviated by Burchnall and Chaundy [4, p.112]. We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation of Srivastava and Panda [19, p.423, Eq.(26)]:

$$F^{p:q;k}_{\ell:m;n}\left[\begin{array}{l}(a_p):(b_q);(c_k);\\ \\ (\alpha_\ell):(\beta_m);(\gamma_n);\end{array} x, y\right] = \sum_{r,s=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(a_j)_{r+s}\prod\limits_{j=1}^{q}(b_j)_r\prod\limits_{j=1}^{k}(c_j)_s}{\prod\limits_{j=1}^{\ell}(\alpha_j)_{r+s}\prod\limits_{j=1}^{m}(\beta_j)_r\prod\limits_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.3}$$

where, for convergence,

$$(i)\ p+q<\ell+m+1, \quad p+k<\ell+n+1, \quad |x|<\infty, \quad |y|<\infty, \text{ or} \tag{1.4}$$

$$(ii)\ p+q=\ell+m+1, \quad p+k=\ell+n+1 \text{ and} \tag{1.5}$$

$$\begin{cases}|x|^{1/(p-\ell)}+|y|^{1/(p-\ell)}<1, & \text{if } p>\ell\\ \max\{|x|,|y|\}<1, & \text{if } p\le\ell.\end{cases} \tag{1.6}$$

For absolutely and conditionally convergence of double series(1.3), we refer to a research paper by Hài et al. [6, p.106-107].

Appell function [1] see also [18, p.53, Eq.(6,7)] Appell's function of third kind is defined by

$$F_3\,[\alpha,\beta;\gamma,\delta;\lambda;x,y] = \sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_n(\gamma)_m(\delta)_n x^m y^n}{(\lambda)_{m+n} m! n!}\ , \tag{1.7}$$

where $\max\ \{\ |x|,\ |y|\ \} < 1$.

Appell's function of fourth kind is defined by

$$F_4\left[\ \alpha\ ,\ \beta\ ;\quad \gamma\ ,\quad \delta\ ;\ x,y\ \right]\ = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\delta)_n m!n!} x^m y^n\ , \tag{1.8}$$

where $\sqrt{|x|}+\sqrt{|y|}\ <\ 1.$
Series rearrangement technique is based upon certain interchanges of the order of a double (or multiple) summation. Several hypergeometric generating relations have been established using series rearrangement technique.
Here, we consider some well known results.
Srivastava identity [17, p.4, Eq.(12)]

$$\sum_{N=0}^{\infty} \Phi(N)\frac{(x+y)^N}{N!} = \sum_{m,n=0}^{\infty} \frac{\Phi(m+n)x^m y^n}{m!\ \ n!}. \tag{1.9}$$

Cauchy's double series identity [8, p.100]

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Phi(m,n) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \Phi(m-n,n)\ , \tag{1.10}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]} \Phi(m,n) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Phi(m+2n,n)\ , \tag{1.11}$$

provided that the associated double series are absolutely convergent.
Kümmer's first transformation [14, p.125, Eq.(2)]

$${}_1F_1\left[\begin{matrix} \alpha\ ; & \\ & z \\ \beta\ ; & \end{matrix}\right] = e^z\ {}_1F_1\left[\begin{matrix} \beta-\alpha\ ; & \\ & -z \\ \beta\ ; & \end{matrix}\right], \tag{1.12}$$

where $\beta \neq 0,-1,-2,\cdots$ and $|z|<\infty$.
Gauss' quadratic transformation [5, p.65, Eq.(26)] see also [12, p.251, Eq.(9.6.5)]

$${}_2F_1\left[\begin{matrix} 2a,\ 2a-b+1; & \\ & \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}} \\ b\ ; & \end{matrix}\right] = (2)^{-2a}\left(1+\sqrt{1-z}\right)^{2a}\ \ {}_2F_1\left[\begin{matrix} a,\ a+\frac{1}{2}; & \\ & z \\ b; & \end{matrix}\right], \tag{1.13}$$

where $|arg(1-z)| < \pi$ and $b \neq 0, -1, -2, \cdots$

Whipple's quadratic transformation [22, p.267, Eq.(7.1)] see also [14, p.88, Th. 31]

$$ {}_3F_2\left[\begin{array}{c} -m, \beta, \gamma; \\ \\ 1-\beta-m, 1-\gamma-m; \end{array} z\right] = (1-z)^m \; {}_3F_2\left[\begin{array}{c} \frac{-m}{2}, \frac{-m+1}{2}, 1-\beta-\gamma-m; \\ \\ 1-\beta-m, 1-\gamma-m; \end{array} \frac{-4z}{(1-z)^2}\right], \tag{1.14} $$

Here $m = 0, 1, 2, 3, \cdots$ and β, γ, $1-\beta-m$, $1-\gamma-m$, $1-\beta-\gamma-m$ are neither zero nor negative integers.

Generalized Laguerre Function

$$ L_\nu^{(\alpha)}(x) = \frac{\Gamma(1+\alpha+\nu)}{\Gamma(1+\alpha)\,\Gamma(\nu+1)} \; {}_1F_1\left[\begin{array}{c} -\nu; \\ \\ 1+\alpha; \end{array} x\right], \tag{1.15} $$

where ν is arbitrary and valid for all finite values of x.

Classical Jacobi polynomials of first kind of order m [14, p.254, Eq. (1)]

$$ P_m^{(\alpha,\,\beta)}(x) = \frac{(1+\alpha)_m}{m!} \; {}_2F_1\left[\begin{array}{c} -m, 1+\alpha+\beta+m; \\ \\ 1+\alpha; \end{array} \frac{1-x}{2}\right]. \tag{1.16} $$

$$ P_m^{(\alpha,\,\alpha)}(x) = \frac{(1+\alpha)_m}{(1+2\alpha)_m} \; C_m^{\alpha+\frac{1}{2}}(x) $$

where polynomials $P_m^{(\alpha,\,\alpha)}(x)$ are called Ultraspherical polynomials and $C_m^{\alpha+\frac{1}{2}}(x)$ are known as Gegenbauer polynomials.

$$ P_m^{(0,0)}(x) = P_m(x) = C_m^{\frac{1}{2}}(x) $$

where $P_m(x)$ are called Legendre polynomials.

$$ P_m^{(\frac{-1}{2},\frac{-1}{2})}(x) = \frac{(\frac{1}{2})_m}{m!} \; T_m(x) $$

where polynomials $T_m(x)$ are known as Tchebycheff polynomials of first kind.

$$ P_m^{(\frac{1}{2},\frac{1}{2})}(x) = \frac{(\frac{3}{2})_m}{(m+1)!} \; U_m(x) = \frac{(\frac{3}{2})_m}{(m+1)!} \; C_m^1(x) $$

where $U_m(x)$ are called Tchebycheff polynomials of second kind.

Khandekar's generalized Rice polynomials [11, p.158 , Eq. (2.3)]

$$H_n^{(\gamma,\ \delta)}[\zeta, p, z] = \frac{(1+\gamma)_n}{n!}\ {}_3F_2\left[\begin{array}{c} -n, 1+\gamma+\delta+n, \zeta; \\ \\ 1+\gamma,\ p; \end{array} z\right], \tag{1.17}$$

which is the generalization of Rice polynomials $H_n^{(0,0)}[\zeta, p, z]$ [15, p.108].

Linear generating function

Two functions $F(x,t)$ and $G(x,t)$ of two independent variables x and t are called generating functions of the sets $\{f_n(x)\}$ and $\{g_n(x)\}$ respectively, if it is possible to represent $F(x,t)$ and $G(x,t)$ in the following series expansions of t

$$F(x,t) = \sum_{n=0}^{\infty} b_n\ f_n(x)t^n; \quad t \neq 0, \tag{1.18}$$

$$G(x,t) = \sum_{n=-\infty}^{+\infty} c_n\ g_n(x)t^n; \quad t \neq 0, \tag{1.19}$$

where the coefficients b_n and c_n are independent of x and t and may contain some parameters related with $f_n(x)$, $g_n(x)$ respectively.

Motivated by the work collected in beautiful monographs of Rainville [14, ch.(8), pp. 129-146], McBride [13, ch.(1), pp. 1-24; ch.(5),pp.72-76], Brafman [[2], [3]] and Shively [16],we obtain some generating relations in this paper.

The present article is organized as follows. In section 2, we obtain three generating relations. In section 3, we have given the proof of hypergeometric generating relations using series rearrangement technique, linear and quadratic transformation. In section 4, some special cases are also discussed.

2. Hypergeometric Generating Relations

When the values of parameters and arguments leading to the results which do not make sense are tacitly excluded. In each generating relation multiple series involved in the left hand side and the right hand side are absolutely convergent and in each hypergeometric function, denominator parameters are neither zero nor negative integers.Then

Generalization of Shively's generating relation[16]

$$F^{A:0;0}_{B:1;1}\left[\begin{array}{c} (a_A): \ - \ ; \ - \ ; \\ \\ (b_B): \ 1 \ ; \ 1 \ ; \end{array} \frac{t+\sqrt{t^2+4xt}}{2}, \frac{t-\sqrt{t^2+4xt}}{2}\right]$$

$$= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{A}(a_i)_m}{(1)_m m! \prod_{i=1}^{B}(b_i)_m}\ {}_{A+1}F_{B+2}\left[\begin{array}{c} -m,\ (a_A)+m; \\ \\ 1\ ,\ 1+m,\ (b_B)+m; \end{array} x\right] t^m, \tag{2.1}$$

where $(a_A) \equiv a_1, a_2, ..., a_A$ and $(b_B) \equiv b_1, b_2, ..., b_B$ are neither zero nor negative integers.

(i) When $A < B+2$ then, $\left|\frac{t+\sqrt{(t^2+4xt)}}{2}\right| < \infty$ and $\left|\frac{t-\sqrt{(t^2+4xt)}}{2}\right| < \infty$,

(ii) When $A = B+2$ then, $\sqrt{\left|\frac{t+\sqrt{(t^2+4xt)}}{2}\right|} + \sqrt{\left|\frac{t-\sqrt{(t^2+4xt)}}{2}\right|} < 1$.

Generalization of Brafman generating relation [3]

$$F^{A:2;2}_{B:0;0}\left[\begin{array}{l} (a_A) : \gamma, \delta-\gamma; \gamma, \delta-\gamma; \\ \quad \\ (b_B) : \quad - \quad ; \quad - \quad ; \end{array} \frac{t+\sqrt{(t^2-4xt)}}{2}, \frac{t-\sqrt{(t^2-4xt)}}{2}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(\gamma)_m (\delta-\gamma)_m}{m!} \frac{\prod_{i=1}^{A} (a_i)_m}{\prod_{i=1}^{B} (b_i)_m} \; {}_{A+2}F_B\left[\begin{array}{c} -m, \; (a_A)+m, \delta+m; \\ \quad \\ (b_B)+m; \end{array} x\right] t^m, \qquad (2.2)$$

where $(a_A) \equiv a_1, a_2, ..., a_A$, $(b_B) \equiv b_1, b_2, ..., b_B$ and γ, $\delta-\gamma$ are neither zero nor negative integers.

(i)When $A < B-1$ then, $\left|\frac{t+\sqrt{(t^2-4xt)}}{2}\right| < \infty$ and $\left|\frac{t-\sqrt{(t^2-4xt)}}{2}\right| < \infty$,

(ii)When $A = B-1$ then, $\max\left\{\left|\frac{t+\sqrt{(t^2-4xt)}}{2}\right|, \left|\frac{t-\sqrt{(t^2-4xt)}}{2}\right|\right\} < 1$.

Third generating relation

$$e^{-t} \; {}_1F_1\left[\begin{array}{c} -b; \\ \quad \\ 1+a; \end{array} x+t\right] = \sum_{n=0}^{\infty} \frac{(1+a+b)_n(-1)^n}{(1+a)_n} \; {}_1F_1\left[\begin{array}{c} -b; \\ \quad \\ 1+a+n; \end{array} x\right] \frac{t^n}{n!}, \qquad (2.3)$$

where $a \neq -1, -2, -3, \cdots$ and valid for all finite values of x and t.

3. Proofs. Proof of first generating relation (2.1). Let

$$\Psi = F^{A:0;0}_{B:1;1}\left[\begin{array}{l} (a_A): \; - \; ; \; - \; ; \\ \quad \\ (b_B): \; 1 \; ; \; 1 \; ; \end{array} \frac{t+\sqrt{t^2+4xt}}{2}, \frac{t-\sqrt{t^2+4xt}}{2}\right]$$

therefore,

$$\Psi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}...(a_A)_{m+n}(t+\sqrt{t^2+4xt})^m(t-\sqrt{t^2+4xt})^n}{(b_1)_{m+n}(b_2)_{m+n}...(b_B)_{m+n}(1)_m(1)_n 2^m 2^n m! n!}. \qquad (3.1)$$

Replacing m by $m-n$ in equation (3.1) and applying double series identity(1.10), we get

$$\Psi = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(a_1)_m(a_2)_m...(a_A)_m(t+\sqrt{t^2+4xt})^{m-n}(t-\sqrt{t^2+4xt})^n}{(b_1)_m(b_2)_m...(b_B)_m(1)_{m-n}(1)_n 2^m (m-n)! n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(t+\sqrt{t^2+4xt})^m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m(1)_m 2^m} \sum_{n=0}^{m} \frac{(t-\sqrt{t^2+4xt})^n}{(1+m)_{-n}(1)_n(1+m)_{-n} n!(t+\sqrt{t^2+4xt})^n}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(t+\sqrt{t^2+4xt})^m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m 2^m m!} \sum_{n=0}^{m} \frac{(-m)_n(-m)_n(t-\sqrt{t^2+4xt})^n}{(1)_n n!(t+\sqrt{t^2+4xt})^n}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(t+\sqrt{t^2+4xt})^m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m 2^m m!} \; {}_2F_1\left[\begin{matrix} -m,-m; & \\ & \frac{t-\sqrt{t^2+4xt}}{t+\sqrt{t^2+4xt}} \\ 1; & \end{matrix}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(t+\sqrt{t^2+4xt})^m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m 2^m m!} \; {}_2F_1\left[\begin{matrix} -m,-m; & \\ & \frac{1-\sqrt{1+\frac{4x}{t}}}{1+\sqrt{1+\frac{4x}{t}}} \\ 1; & \end{matrix}\right]. \qquad (3.2)$$

Applying Gauss' quadratic transformation(1.13) after simplification, we get

$$\Psi = \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m t^m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m m!} \; {}_2F_1\left[\begin{matrix} \frac{-m}{2}, \; \frac{-m+1}{2}; & \\ & \frac{-4x}{t} \\ 1\;; & \end{matrix}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m t^m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m m!} \sum_{n=0}^{[\frac{m}{2}]} \frac{(\frac{-m}{2})_n(\frac{-m+1}{2})_n(-1)^n 2^{2n} x^n}{(1)_n t^n n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m m!} \sum_{n=0}^{[\frac{m}{2}]} \frac{(-m)_{2n}(-1)^n x^n t^{m-n}}{(1)_n n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m} \sum_{n=0}^{[\frac{m}{2}]} \frac{(-1)^n x^n t^{m-n}}{(m-2n)!(1)_n n!}\;. \qquad (3.3)$$

Replacing m by $m+2n$ in equation (3.3) and applying double series identity(1.11), we get

$$\Psi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1)_{m+2n}(a_2)_{m+2n}...(a_A)_{m+2n}(-1)^n x^n t^{m+n}}{(b_1)_{m+2n}(b_2)_{m+2n}...(b_B)_{m+2n}(1)_{m+2n}(1)_n m!n!}. \qquad (3.4)$$

Replacing m by $m-n$ in equation (3.4) and applying double series identity(1.10), we get

$$\Psi = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(a_1)_{m+n}(a_2)_{m+n}...(a_A)_{m+n}(-1)^n x^n t^m}{(b_1)_{m+n}(b_2)_{m+n}...(b_B)_{m+n}(1)_{m+n}(1)_n(m-n)!n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m m!} \times$$

$$\times \left(\sum_{n=0}^{m} \frac{(a_1+m)_n(a_2+m)_n...(a_A+m)_n(-1)^n x^n}{(b_1+m)_n(b_2+m)_n...(b_B+m)_n(1+m)_n(1+m)_{-n}(1)_n n!} \right) t^m$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m m!} \left(\sum_{n=0}^{m} \frac{(a_1+m)_n(a_2+m)_n...(a_A+m)_n(-m)_n x^n}{(b_1+m)_n(b_2+m)_n...(b_B+m)_n(1+m)_n(1)_n n!} \right) t^m$$

$$= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{A}(a_i)_m}{\prod_{i=1}^{B}(b_i)_m(1)_m m!} \; {}_{A+1}F_{B+2}\left[\begin{matrix} -m,\ (a_A)+m; \\ 1\ ,\ 1+m,\ (b_B)+m; \end{matrix} \; x \right] t^m, \qquad (3.5)$$

where $(a_A) \equiv a_1, a_2, ..., a_A$ and $(b_B) \equiv b_1, b_2, ..., b_B$.

Proof of second generating relation (2.2)

Let

$$\Phi = F^{A:2;2}_{B:0;0}\left[\begin{matrix} (a_A) : \gamma, \delta-\gamma; \gamma, \delta-\gamma; \\ (b_B) : \quad - \quad ; \quad - \quad ; \end{matrix} \; \frac{t+\sqrt{(t^2-4xt)}}{2}, \frac{t-\sqrt{(t^2-4xt)}}{2} \right]$$

Then,

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}...(a_A)_{m+n}(\gamma)_m(\delta-\gamma)_m(\gamma)_n}{(b_1)_{m+n}(b_2)_{m+n}...(b_B)_{m+n}} \times$$

$$\times \frac{(\delta-\gamma)_n(t+\sqrt{t^2-4xt})^m(t-\sqrt{t^2-4xt})^n}{2^m 2^n m! n!}. \qquad (3.6)$$

Replacing m by $m-n$ in equation (3.6) and applying double series identity(1.10), we get

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_{m-n}(\delta-\gamma)_{m-n}(\gamma)_n}{(b_1)_m(b_2)_m...(b_B)_m} \times$$

$$\times \frac{(\delta-\gamma)_n(t+\sqrt{t^2-4xt})^{m-n}(t-\sqrt{t^2-4xt})^n}{2^m(m-n)!n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m(\delta-\gamma)_m(t+\sqrt{t^2-4xt})^m}{(b_1)_m(b_2)_m...(b_B)_m(1)_m 2^m} \times$$

$$\times \sum_{n=0}^{m} \frac{(\gamma+m)_{-n}(\delta-\gamma+m)_{-n}(\gamma)_n(\delta-\gamma)_n(t-\sqrt{t^2-4xt})^n}{(1+m)_{-n} n!(t+\sqrt{t^2-4xt})^n}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m(\delta-\gamma)_m(t+\sqrt{t^2-4xt})^m}{(b_1)_m(b_2)_m...(b_B)_m m! 2^m} \times$$

$$\times\sum_{n=0}^{m}\frac{(-1)^n(\gamma)_n(\delta-\gamma)_n(-m)_n(t-\sqrt{t^2-4xt})^n}{(1-\gamma-m)_n(1-\delta+\gamma-m)_n n!(t+\sqrt{t^2-4xt})^n}$$

$$=\sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m(\delta-\gamma)_m(t+\sqrt{t^2-4xt})^m}{(b_1)_m(b_2)_m...(b_B)_m 2^m m!}\times$$

$$\times\ {}_3F_2\left[\begin{matrix}-m,\ \gamma,\ \delta-\gamma; \\ 1-\gamma-m, 1+\gamma-\delta-m;\end{matrix}\ -\left(\frac{t-\sqrt{(t^2-4xt)}}{t+\sqrt{(t^2-4xt)}}\right)\right]. \tag{3.7}$$

Applying Whipple's quadratic transformation(1.14) after simplification, we get

$$\Phi=\sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m(\delta-\gamma)_m t^m}{(b_1)_m(b_2)_m...(b_B)_m m!}\ {}_3F_2\left[\begin{matrix}\frac{-m}{2}, \frac{-m+1}{2}, 1-m-\delta; \\ 1-\gamma-m, 1+\gamma-\delta-m;\end{matrix}\ \frac{4x}{t}\right]$$

$$=\sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m(\delta-\gamma)_m t^m}{(b_1)_m(b_2)_m...(b_B)_m m!}\sum_{n=0}^{[\frac{m}{2}]}\frac{2^{2n}(\frac{-m}{2})_n(\frac{-m+1}{2})_n(1-m-\delta)_n x^n}{(1-m-\gamma)_n(1-\delta+\gamma-m)_n t^n n!}$$

$$=\sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m(\delta-\gamma)_m}{(b_1)_m(b_2)_m...(b_B)_m m!}\sum_{n=0}^{[\frac{m}{2}]}\frac{(-m)_{2n}(1-m-\delta)_n x^n t^{m-n}}{(1-m-\gamma)_n(1-\delta+\gamma-m)_n n!}$$

$$=\sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m...(a_A)_m(\gamma)_m(\delta-\gamma)_m}{(b_1)_m(b_2)_m...(b_B)_m}\sum_{n=0}^{[\frac{m}{2}]}\frac{(1-m-\delta)_n x^n t^{m-n}}{(m-2n)!(1-m-\gamma)_n(1-\delta+\gamma-m)_n n!}. \tag{3.8}$$

Replacing m by $m+2n$ in equation (3.8) and applying double series identity(1.11), we get

$$\Phi=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_{m+2n}(a_2)_{m+2n}...(a_A)_{m+2n}(\gamma)_{m+2n}(\delta-\gamma)_{m+2n}(1-m-2n-\delta)_n x^n t^{m+n}}{(b_1)_{m+2n}(b_2)_{m+2n}...(b_B)_{m+2n}m!(1-m-2n-\gamma)_n(1-\delta+\gamma-m-2n)_n n!}. \tag{3.9}$$

Replacing m by $m-n$ in equation (3.9) and applying double series identity(1.10), we get

$$\Phi=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(a_1)_{m+n}(a_2)_{m+n}...(a_A)_{m+n}(\gamma)_{m+n}(\delta-\gamma)_{m+n}(1-m-n-\delta)_n x^n t^m}{(b_1)_{m+n}(b_2)_{m+n}...(b_B)_{m+n}(1-m-n-\gamma)_n(1-\delta+\gamma-m-n)_n(m-n)!n!}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(a_1)_{m+n}(a_2)_{m+n}...(a_A)_{m+n}(\gamma)_m(\delta-\gamma)_m(\delta)_{m+n}(-m)_n x^n t^m}{(b_1)_{m+n}(b_2)_{m+n}...(b_B)_{m+n}(\delta)_m m!n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m (a_2)_m ... (a_A)_m (\gamma)_m (\delta-\gamma)_m}{(b_1)_m (b_2)_m ... (b_B)_m m!} \times$$

$$\times \left(\sum_{n=0}^{m} \frac{(a_1+m)_n (a_2+m)_n ... (a_A+m)_n (\delta+m)_n (-m)_n x^n}{(b_1+m)_n (b_2+m)_n ... (b_B+m)_n n!} \right) t^m$$

$$= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{A} (a_i)_m (\gamma)_m (\delta-\gamma)_m}{\prod_{i=1}^{B} (b_i)_m m!} \ {}_{A+2}F_B \left[\begin{array}{c} -m, \ (a_A)+m, \delta+m; \\ \\ (b_B)+m; \end{array} x \right] t^m. \tag{3.10}$$

Proof of third generating relation (2.3)

Let

$$\Xi = e^{-t} \ {}_1F_1 \left[\begin{array}{c} -b \ ; \\ \\ 1+a \ ; \end{array} x+t \right] = e^{-t} \ \sum_{N=0}^{\infty} \frac{(-b)_N (x+t)^N}{(1+a)_N N!}. \tag{3.11}$$

Applying Srivastava identity (1.9) in equation (3.11),we get

$$\Xi = e^{-t} \ \sum_{m,n=0}^{\infty} \frac{(-b)_{m+n} \ x^m t^n}{(1+a)_{m+n} \ m! n!} = \sum_{m=0}^{\infty} \frac{(-b)_m x^m}{(1+a)_m m!} \ \ e^{-t} \sum_{n=0}^{\infty} \frac{(-b+m)_n \ t^n}{(1+a+m)_n \ n!}$$

$$= \sum_{m=0}^{\infty} \frac{(-b)_m x^m}{(1+a)_m m!} \ \ e^{-t} {}_1F_1 \left[\begin{array}{c} -b+m \ ; \\ \\ 1+a+m \ ; \end{array} t \right]. \tag{3.12}$$

Applying Kümmer's first transformation(1.12),we get

$$\Xi = \sum_{m=0}^{\infty} \frac{(-b)_m x^m}{(1+a)_m m!} {}_1F_1 \left[\begin{array}{c} 1+a+b \ ; \\ \\ 1+a+m \ ; \end{array} -t \right]$$

$$= \sum_{m=0}^{\infty} \frac{(-b)_m x^m}{(1+a)_m m!} \sum_{n=0}^{\infty} \frac{(1+a+b)_n (-1)^n t^n}{(1+a+m)_n \ n!}$$

$$= \sum_{n=0}^{\infty} \frac{(1+a+b)_n (-1)^n t^n}{n!} \sum_{m=0}^{\infty} \frac{(-b)_m x^m}{(1+a)_{m+n} m!}$$

$$= \sum_{n=0}^{\infty} \frac{(1+a+b)_n (-t)^n}{(1+a)_n n!} \sum_{m=0}^{\infty} \frac{(-b)_m x^m}{(1+a+n)_m m!}$$

$$= \sum_{n=0}^{\infty} \frac{(1+a+b)_n (-1)^n}{(1+a)_n} \ {}_1F_1 \left[\begin{array}{c} -b \ ; \\ \\ 1+a+n \ ; \end{array} x \right] \frac{t^n}{n!}. \tag{3.13}$$

4. Special Cases

In first generating relation (2.1), put $A = 2, B = 0$ and $a_1 = \alpha, a_2 = \beta$ and applying the definition of Appell's function F_4 (1.8), we get

$$F_4\left[\alpha, \beta; 1, 1; \frac{t+\sqrt{(t^2+4xt)}}{2}, \frac{t-\sqrt{(t^2+4xt)}}{2}\right] = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(m!)^2}\, {}_3F_2\left[\begin{array}{c} -m, \alpha+m, \beta+m; \\ \\ 1, 1+m; \end{array} x\right] t^m, \tag{4.1}$$

where $\sqrt{\left|\frac{t+\sqrt{(t^2+4xt)}}{2}\right|} + \sqrt{\left|\frac{t-\sqrt{(t^2+4xt)}}{2}\right|} < 1$ and $\alpha,\ \beta \neq 0, -1, -2, \cdots$

Applying the definition of Khandekar's polynomials (1.17) after simplification, we get

$$F_4\left[\alpha, \beta; 1, 1; \frac{t+\sqrt{(t^2+4xt)}}{2}, \frac{t-\sqrt{(t^2+4xt)}}{2}\right] = \sum_{m=0}^{\infty} \left(\frac{(\alpha)_m(\beta)_m}{2^{2m}(\frac{1}{2})_m m!}\ H_m^{(m,\alpha-m-1)}[\beta+m, 1, x]\right) t^m, \tag{4.2}$$

where $\sqrt{\left|\frac{t+\sqrt{(t^2+4xt)}}{2}\right|} + \sqrt{\left|\frac{t-\sqrt{(t^2+4xt)}}{2}\right|} < 1$ and $\alpha,\ \beta \neq 0, -1, -2, \cdots$

In second generating relation (2.2),put $A = 0, B = 1$ and $b_1 = b$ and applying the definition of Appell's function F_3 (1.7), we get

$$F_3\left[\gamma, \delta-\gamma; \gamma, \delta-\gamma; b; \frac{t+\sqrt{(t^2-4xt)}}{2}, \frac{t-\sqrt{(t^2-4xt)}}{2}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(\gamma)_m(\delta-\gamma)_m}{(b)_m m!}\ {}_2F_1\left[\begin{array}{c} -m, \delta+m; \\ \\ b+m; \end{array} x\right] t^m, \tag{4.3}$$

where $\max\left\{\left|\frac{t+\sqrt{(t^2-4xt)}}{2}\right|,\ \left|\frac{t-\sqrt{(t^2-4xt)}}{2}\right|\right\} < 1$ and $\gamma,\ \delta-\gamma,\ b \neq 0, -1, -2, \cdots$

In equation (4.3) replacing x by $\frac{(1-x)}{2}$ and applying the definition of Classical Jacobi polynomials of first kind (1.16) after simplification, we get

$$F_3\left[\gamma, \delta-\gamma; \gamma, \delta-\gamma; b; \frac{t+\sqrt{(t^2-2(1-x)t)}}{2}, \frac{t-\sqrt{(t^2-2(1-x)t)}}{2}\right]$$

$$= \sum_{m=0}^{\infty} \left(\frac{(\gamma)_m(\delta-\gamma)_m}{(b)_{2m}}\ P_m^{(b+m-1,\ \delta-b-m)}(x)\right) t^m, \tag{4.4}$$

where $\max\left\{\left|\frac{t+\sqrt{(t^2-2(1-x)t)}}{2}\right|,\ \left|\frac{t-\sqrt{(t^2-2(1-x)t)}}{2}\right|\right\} < 1$ and $\gamma,\ \delta-\gamma,\ b \neq 0, -1, -2, \cdots$

In third generating relation(2.3), applying the definition of generalized Laguerre function(1.15), we get

$$e^{-t}\ {}_1F_1\left[\begin{array}{c} -b\ ; \\ \\ 1+a\ ; \end{array} x+t\right] = \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(1+a+b)}\sum_{n=0}^{\infty}(-1)^n\ L_b^{(a+n)}(x)\ \frac{t^n}{n!}, \tag{4.5}$$

where $a \neq -1, -2, -3, \cdots$ and valid for all finite values of x and t.

5. Conclusion

In our present article, we obtained various generating relations by using Series rearrangement technique, Gauss' quadratic transformation, Whipple's quadratic transformation and Kümmer's transformation.From these generating relations readers may find their applications in various branches of Physical Sciences and may be useful to non-professionals who are interested in a wide range of problems of Mathematics and Statistics.

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